Normal submultigroups and comultisets of a multigroup

Paul Augustine Ejegwa and Adeku Musa Ibrahim

Abstract. We study properties of normal submultigroups. It is shown that if A is a multigroup of a group X and B is a submultigroup of A, the union and intersection of comultisets of B in A are identical and equal to B.

1. Introduction

The notion of multigroup was first mentioned in [3] and defined as algebraic system that satisfied all the axioms of group except that the binary operation is multi-valued. This perspective is neither in conformity with the idea of multisets nor in alignment with other non-classical group studied in [8]. Also, the generalizations of group theory as multigroup in [5, 7, 9] are not within the framework of multiset.

The perspective of multigroups in [10, 11] seem to be better off because the notion of multiset was captured but however, do not define multigroup via count function of multiset. In [6], the concept of multigroups was introduced via count function of multiset and some properties were discussed. Further studies on the concept of multigroups via multisets can be found in [1, 2, 4].

In this paper, we study some properties of normal submultigroups, propose conjugate and normalizer in multigroups, and obtain some results. The homomorphic properties of normal submultigroups are explicated. Finally, we explore the idea of comultisets of a multigroup mentioned in [6] and deduce some results. We show that the union and intersection of comultisets of a submultigroup of a multigroup are identical and equal to the submultigroup.

2. Preliminaries

In this section, we present some existing definitions and results that are useful in the subsequent sections.

Definition 2.1. Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a set. A *multiset* A over X is a cardinal-valued function, that is, $C_A : X \to N$ such that for $x \in Dom(A)$ implies

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$A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. We denote the set of all multisets over X by $MS(X)$.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$. Different forms of representing multiset exist other than this.

Definition 2.2. Let A and B be multisets over X. Then A is called a *submultiset* of B written as $A \subseteq B$ if $C_A(x) \leqslant C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. Thus $A = B$ means that $C_A(x) = C_B(x)$ for all $x \in X$. A multiset A with the property $C_A(x) = C_B(y)$ for all $x, y \in X$, is called *regular*. Otherwise it is *irregular*.

Definition 2.3. Let A and B be multisets over X. Then the *intersection* and *union* of A and B, denoted by $A \cap B$ and $A \cup B$ respectively, are defined by the rules that for any object $x \in X$,

(*i*) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(*ii*) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ denote minimum and maximum respectively.

Definition 2.4. Let X be a group. A multiset G is called a *multigroup* of X if it satisfies the following conditions:

(*i*) $C_G(xy) \geqslant C_G(x) \wedge C_G(y) \quad \forall x, y \in X$,

(*ii*) $C_G(x^{-1}) = C_G(x) \quad \forall x \in X$,

where C_G denotes count function of G from X into a natural number $\mathbb{N}$.

For any multigroup A its inverse A^{-1} is defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \quad \forall x \in X.$$

The set of all multigroups of X is denoted by $MG(X)$. It is worthy of note that every multigroup is a multiset but the converse is not true.

Definition 2.5. Let $A \in MG(X)$. A submultiset B of A is called a *submultigroup* of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

Definition 2.6. Let $\{A_i\}_{i\in I}, I = 1, \ldots, n$ be an arbitrary family of multigroups of X. Then

$$C_{\bigcap_{i\in I} A_i}(x) = \bigwedge_{i\in I} C_{A_i}(x) \quad \forall x \in X$$

and

$$C_{\bigcup_{i\in I} A_i}(x) = \bigvee_{i\in I} C_{A_i}(x) \quad \forall x \in X.$$

The family of multigroups $\{A_i\}_{i\in I}$ of X is said to have *inf/sup assuming chain* if either $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n$ respectively.

Definition 2.7. Let $A, B \in MG(X)$. Then the *product* of A and B denoted as $A \circ B$, is defined by

$$C_{A\circ B}(x) = \bigvee \{C_A(y) \wedge C_B(z) \mid x = yz, y, z \in X\}.$$

Proposition 2.8. *Let $A \in MG(X)$. Then*

(i) $A_* = \{x \in X \mid C_A(x) > 0\}$,

(ii) $A^* = \{x \in X \mid C_A(x) = C_A(e)\}$,

where e is the identity element of X, are subgroups of X. □

Definition 2.9. Let A and B be multisets over groups X and Y and $f : X \longrightarrow Y$ be a homomorphism. Then

(i) the *image* of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$.

(ii) the *inverse image* of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by $C_{f^{-1}(B)}(x) = C_B(f(x)) \forall x \in X$.

Definition 2.10. Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a *weak homomorphism* from A to B if $f(A) \subseteq B$. If f is a weak homomorphism of A into B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a *weak isomorphism* from A to B if $f(A) \subseteq B$. If f is a weak isomorphism of A into B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a *homomorphism* from A to B if $f(A) = B$. If f is a homomorphism of A onto B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an *isomorphism* from A to B if $f(A) = B$. If f is an isomorphism of A onto B, then A is isomorphic to B denoted by $A \cong B$.

Theorem 2.11. *Let X and Y be groups and $f : X \to Y$ be an isomorphism. If $A \in MG(X)$ and $B \in MG(Y)$, then $f(A) \in MG(Y)$ and $f^{-1}(B) \in MG(X)$.* □

3. Properties of normal submultigroups

Let $A \in MG(X)$ is said to be *abelian* if $C_A(xy) = C_A(yx)$ for all $x, y \in X$. If $A, B \in MG(X)$ and $A \subseteq B$, then A is called a *normal submultigroup* of B if

$$C_A(xyx^{-1}) \geqslant C_A(y) \quad \forall x, y \in X.$$

Example 3.1. Let $X = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c,\ ac = b,\ bc = a,\ a^2 = b^2 = c^2 = e.$$

Suppose $A = [e, a, b, c]_{3,2,3,2}$ and $B = [e, a, b, c]_{5,2,4,2}$ are multigroups of X satisfying the axioms in Definition 2.4. Clearly, $A \subseteq B$. Then A is a normal submultigroup of B since

$$C_A(aba^{-1}) = C_A(b) = 3 \geqslant C_A(b), \quad C_A(bab^{-1}) = C_A(a) = 2 \geqslant C_A(a),$$
$$C_A(cbc^{-1}) = C_A(b) = 3 \geqslant C_A(b), \quad C_A(bcb^{-1}) = C_A(c) = 2 \geqslant C_A(c).$$

Definition 3.2. Let $A \in MG(X)$ and $x, y \in X$. Then x and y are called *conjugate elements* in A if

$$C_A(x) = C_A(yxy^{-1}) \quad \forall x, y \in X.$$

Two multigroups A and B of X are *conjugate* to each other if for all $x, y \in X$,

$$C_A(y) = C_B(xyx^{-1}) \ \text{ and } \ C_B(x) = C_A(yxy^{-1}), \ \text{ i.e.,}$$
$$C_A(y) = C_{B^x}(y) \ \text{ and } \ C_B(x) = C_{A^y}(x).$$

Remark 3.3. If $A, B \in MG(X)$ and A is a normal submultigroup of B. Then A_* is a normal subgroup of B_* and A^* is a normal subgroup of B^*. Moreover, A is normal if and only if A^{-1} is normal.

Proposition 3.4. *Let $A, B \in MG(X)$. Then the following statements are equivalent.*

(*i*) *A is a normal submultigroup of B,*

(*ii*) $C_A(xyx^{-1}) = C_A(y) \ \ \forall x, y \in X$,

(*iii*) $C_A(xy) = C_A(yx) \ \ \forall x, y \in X$.

Proof. Straightforward. □

Proposition 3.5. *Let $A, B \in MG(X)$ such that $A \subseteq B$ and $C_A(x) = C_A(y)$ for all $x, y \in X$. Then the following assertions are equivalent.*

(*i*) *A is a normal submultigroup of B.*

(*ii*) $C_A(yx) \geqslant C_A(xy) \wedge C_B(y) \ \ \forall x, y \in X$.

Proof. $(i) \Rightarrow (ii)$. Since A is a normal submultigroup of B and $C_A(x) = C_A(y)$, by Proposition 3.4 we have $C_A(yx) = C_A(y(xy)y^{-1}) \geqslant C_A(xy) \wedge C_B(y)$ for all $x, y \in X$.

$(ii) \Rightarrow (i)$. Since $C_A(yx) \geqslant C_A(xy) \wedge C_B(y)$, $C_A(xy) \geqslant C_A(yx) \wedge C_B(y)$, it implies $C_A(xy) = C_A(yx)$. Proposition 3.4 completes the proof. □

Proposition 3.6. *Let X be a group, A a submultigroup of $G \in MG(X)$ and B a submultiset of G. If A and B are conjugate, then B is a submultigroup of G.* □

Proposition 3.7. *Let $A, B, C \in MG(X)$ such that A and B are normal submultigroups of C. If $A \subseteq B \subseteq C$, then $A \cap B$ and $A \cup B$ are normal submultigroups of C.* □

Proposition 3.8. *Let A be a submultigroup of $B \in MG(X)$. Then A is a normal submultigroup of B if and only if $x \in X$ is constant on the conjugacy classes of A.* □

Proof. Suppose that A is a normal submultigroup of B. Then

$$C_A(y^{-1}xy) = C_A(xyy^{-1}) = C_A(x) \;\; \forall x, y \in X.$$

This implies that, $x \in X$ is constant on the conjugacy classes of A.

Conversely, let $x \in X$ be constant (that is, fixed) on each conjugacy classes of A. Then $C_A(xy) = C_A(xyxx^{-1}) = C_A(x(yx)x^{-1}) = C_A(yx)$ $\forall x, y \in X$. Hence, A is normal. □

We now give an alternative formulation of the notion of normal submultigroup in terms of commutator of a group. First, we recall that if X is a group and $x, y \in X$, then the element $x^{-1}y^{-1}xy$ is usually depicted by $[x, y]$ and is called the *commutator* of x and y.

Theorem 3.9. *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is a normal submultigroup of B if and only if*

(*i*) $C_A([x, y]) \geqslant C_A(x)$ $\forall x, y \in X$.

(*ii*) $C_A([x, y]) = C_A(e)$ $\forall x, y \in X$, *where e is the identity of X.*

Proof. (i). Suppose A is a normal submultigroup of B. Let $x, y \in X$, then

$$C_A(x^{-1}y^{-1}xy) \geqslant C_A(x^{-1}) \wedge C_A(y^{-1}xy) = C_A(x) \wedge C_A(x) = C_A(x).$$

Conversely, assume that A satisfies the inequality. Then for all $x, y \in X$,

$$C_A(x^{-1}yx) = C_A(yy^{-1}x^{-1}yx) \geqslant C_A(y) \wedge C_A([y, x]) = C_A(y).$$

Thus, $C_A(x^{-1}yx) \geqslant C_A(y)$ for all $x, y \in X$. Hence A is normal.

(ii). Let $x, y \in X$. Suppose A is a normal submultigroup of B. We know that A is a normal submultigroup of $B \Leftrightarrow C_A(xy) = C_A(yx) \Leftrightarrow C_A(x^{-1}y^{-1}x) =$

$C_A(y^{-1}) \Leftrightarrow C_A(x^{-1}y^{-1}xyy^{-1}) = C_A(y^{-1}) \Leftrightarrow C_A([x,y]y^{-1}) = C_A(y^{-1})$ for all $x, y \in X$. Consequently, $C_A([x,y]) = C_A(y^{-1}y) = C_A(e)$ for all $x, y \in X$.

Conversely, assume $C_A([x,y]) = C_A(e)$ for all $x, y \in X$. Then $C_A(x^{-1}y^{-1}xy) = C_A(e)$, so, $C_A((yx)^{-1}xy) = C_A(e)$. That is, $C_A(xy) = C_A(yx)$ for all $x, y \in X$. Thus, A is a normal submultigroup of B. □

Theorem 3.10. *Let A be a normal submultigroup of $G \in MG(X)$. Then $\bigcap_{x\in X} A^x$ is normal and is the largest normal submultigroup of G that is contained in A.*

Proof. Suppose $A^x \in MG(X) \forall x \in X$. Then for all $y \in X$, we observe that $\{A^x \mid x \in X\} = \{A^{xy} \mid x \in X\}$. Thus,

$$\begin{aligned}\bigwedge_{x\in X} C_{A^x}(yzy^{-1}) &= \bigwedge_{x\in X} C_A(xyzy^{-1}x^{-1}) = \bigwedge_{x\in X} C_A((xy)z(xy)^{-1}) \\ &= \bigwedge_{x\in X} C_{A^{xy}}(z) = \bigwedge_{x\in X} C_{A^x}(z) \ \ \forall y, z \in X.\end{aligned}$$

Hence, $\bigcap_{x\in X} A^x$ is a normal submultigroup of G.

Now let B be a normal submultigroup of G such that $B \subseteq A$. Then $B = B^x \subseteq A^x \forall x \in X$. Thus, $B \subseteq \bigcap_{x\in X} A^x$. Therefore, $\bigcap_{x\in X} A^x$ is the largest normal submultigroup of G that is contained in A. □

Definition 3.11. Let A be a submultigroup of $B \in MG(X)$. Then the it normalizer of A in B is the set given by

$$N(A) = \{g \in X \mid C_A(gy) = C_A(yg) \ \forall y \in X\}.$$

We now note that

$$N(A) = \{g \in X \mid C_{A^g}(y) = C_A(y) \ \forall y \in X\}.$$

It suffices to note that, $C_A(gy) = C_A(yg)$ for all $y \in X$ implies $C_A(g^{-1}yg) = C_A(y)$ for all $y \in X$. Then $C_A(g^{-1}yg) = C_A(y)$ gives $C_A(g^{-1}(gy)g) = C_A(gy)$, i.e., $C_A(yg) = C(gy)$ for all $y \in X$.

Example 3.12. Let $X = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$ such that

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g_6 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, g_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, g_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

be a group under matrix multiplication, and $A \subseteq B \in MG(X)$ such that

$$A = [g_1^{10}, g_2^5, g_3^7, g_4^5, g_5^5, g_6^5, g_7^7, g_8^8]$$

satisfying the axioms in Definition 2.4. Using Definition 3.11, $N(A) = \{g_1, g_3, g_7, g_8\}$.

Theorem 3.13. *Let A be a submultigroup of $B \in MG(X)$. Then the following assertions hold.*

(*i*) *$N(A)$ is a subgroup of X.*

(*ii*) *A is a normal submultigroup of B if and only if $N(A) = X$.*

Proof. (*i*). Let $g, h \in N(A)$. Then $C_{A^{gh}}(x) = C_{(A^h)^g}(x) = C_{A^h}(x) = C_A(x)$ for all $x \in X$ since $C_{A^g}(x) = C_A(g^{-1}xg) = C_A(x)$. Hence $gh \in N(A)$. Again, let $g \in N(A)$. We show that $g^{-1} \in N(A)$. For all $y \in X$, $C_A(gy) = C_A(yg)$ and so $C_A((gy)^{-1}) = C_A((yg)^{-1})$. Thus for all $y \in X$, $C_A(y^{-1}g^{-1}) = C_A(g^{-1}y^{-1})$ and so $C_A(yg^{-1}) = C_A(g^{-1}y)$ since $C_A(y) = C_A(y^{-1})$. Thus, $g^{-1} \in N(A)$. Hence, $N(A)$ is a subgroup of X.

(*ii*). Let A be a normal submultigroup of B and $g \in X$. Then for all $x \in X$, we have

$$C_{A^g}(x) \quad = \quad C_A(g^{-1}xg) = C_A((g^{-1}x)g) = C_A(g(g^{-1}x)) = C_A(x).$$

Thus, $C_{A^g}(x) = C_A(x)$ and so $g \in N(A)$. Therefore, $N(A) = X$.

Conversely, suppose $N(A) = X$. Let $x, y \in X$. To prove that A is normal, it is sufficient we show that $C_A(xy) = C_A(yx)$. Now

$$C_A(xy) = C_A(xyxx^{-1}) = C_A(x(yx)x^{-1}) = C_{A^{x^{-1}}}(yx) = C_A(yx),$$

where the last equality follows since $N(A) = X$ and so $x^{-1} \in N(A)$. Consequently, $C_{A^{x^{-1}}}(y) = C_A(y)$. Thus, A is a normal submultigroup of B. □

Remark 3.14. Let A be a submultigroup of $B \in MG(X)$. Then $S = N(A) = T$, if

$$S = \{x \in X \mid C_A(xy(yx)^{-1}) = C_A(e) \ \forall y \in X\}$$

and

$$T = \{x \in X \mid C_A(xyx^{-1}) = C_A(y) \ \forall y \in X\}.$$

Theorem 3.15. *Let A, B and C be multigroups of an abelian group X such that $A \subseteq B \subseteq C$. Then*

$$N(A) \cap N(B) \subseteq N(A \cap B).$$

Proof. Let $y \in N(A) \cap N(B)$. Then for any $x, y \in X$, we get $C_{A\cap B}(xy) = C_{A\cap B}(yx)$. thus, $C_{A\cap B}(xyx^{-1}) = C_{A\cap B}(y)$. Now

$$\begin{aligned} C_{A\cap B}(xyx^{-1}) &= C_A(xyx^{-1}) \wedge C_B(xyx^{-1}) = C_A(yxx^{-1}) \wedge C_B(yxx^{-1}) \\ &= C_A(y) \wedge C_B(y) = C_{A\cap B}(y). \end{aligned}$$

Thus, $y \in N(A \cap B)$. Hence, $N(A) \cap N(B) \subseteq N(A \cap B)$. □

Corollary 3.16. *Let $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$ and $C_A(e) = C_B(e)$. Then*

$$N(A) \cap N(B) = N(A \cap B).$$

Proof. Recall that

$$\begin{aligned} N(A) &= \{x \in X \mid C_A(xy) = C_A(yx) \ \ \forall y \in X\} \\ &= \{x \in X \mid C_A(xyx^{-1}y^{-1}) = C_A(e) \ \ \forall y \in X\}. \end{aligned}$$

Let $y \in N(A \cap B)$. Then from the definition of $N(A)$, for all $x \in X$ we get

$$C_{A\cap B}(xyx^{-1}y^{-1}) = C_A(xyx^{-1}y^{-1}) \wedge C_B(xyx^{-1}y^{-1}) = C_A(e) \wedge C_B(e),$$

implies $y \in N(A) \cap N(B)$. Since $C_A(xyx^{-1}y^{-1}) = C_A(e)$ we obtain $C_A(xy) = C_A(yx)$. Similarly in the case of B because $C_A(e) = C_B(e)$. Hence $N(A) \cap N(B) = N(A \cap B)$. □

Corollary 3.17. *Let $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$. Then*

$$N(A) \cap N(B) \subseteq N(A \circ B).$$

Proof. Let $y \in N(A) \cap N(B)$, that is $y \in N(A)$ and $y \in N(B)$. Then for all $x \in X$,

$$\begin{aligned} C_{A\circ B}(y) &= \bigvee_{y=ab} \{C_A(a) \wedge C_B(b) \mid \forall a, b \in X\} \\ &= \bigvee_{y=ab} \{C_A(x^{-1}ax) \wedge C_B(x^{-1}bx) \mid \forall a, b \in X\} \\ &\leqslant \bigvee_{x^{-1}yx=cd} \{C_A(c) \wedge C_B(d) \mid \forall c, d \in X\} \\ &= C_{A\circ B}(x^{-1}yx), \end{aligned}$$

which gives $C_{A\circ B}(y) \leqslant C_{A\circ B}(x^{-1}yx)$. The inequality holds since $y = ab \Rightarrow x^{-1}abx = cd \Rightarrow ab = xcdx^{-1} = (xcx^{-1})(xdx^{-1})$ and since $a = xcx^{-1}$ and $b = xdx^{-1}$ imply $x^{-1}ax = c$ and $x^{-1}bx = d$. Again,

$$C_{A\circ B}(x^{-1}yx) \leqslant C_{A\circ B}(x(x^{-1}yx)x^{-1}) = C_{A\circ B}(y).$$

So, $C_{A\circ B}(y) \geqslant C_{A\circ B}(x^{-1}yx)$. Thus, $C_{A\circ B}(y) = C_{A\circ B}(x^{-1}yx)$, which proves, $y \in N(A \circ B)$. Therefore, $N(A) \cap N(B) \subseteq N(A \circ B)$. □

Remark 3.18. If $A, B, C \in MG(X)$ such that $A \subseteq B \subseteq C$. Then $N(A) \subseteq N(B)$.

4. Homomorphism of normal submultigroups

In this section, we present some results on the homomorphic properties of normal submultigroups.

Theorem 4.1. *Let f be a homomorphism of an abelian group X onto an abelian group Y. Let A and B be multigroups of X such that $A \subseteq B$. Then*

$$f(N(A)) \subseteq N(f(A)).$$

Proof. Let $x \in f(N(A))$. Then $f(u) = x$ for some $u \in N(A)$. So, for all $y, z \in Y$,

$$\begin{aligned} C_{f(A)}(xyx^{-1}) &= C_A(f^{-1}(xyx^{-1})) = C_A(f^{-1}(x)f^{-1}(y)f^{-1}(x^{-1})) \\ &= C_A(f^{-1}(x)f^{-1}(y)f^{-1}(x)^{-1}) = C_A(f^{-1}(x)f^{-1}(y)(f^{-1}(x))^{-1}) \\ &= C_A(f^{-1}(f(u))f^{-1}(f(v))(f^{-1}(f(u)))^{-1}) = C_A(uvu^{-1}) \\ &= C_A(vuu^{-1}) = C_A(v) = C_A(f^{-1}(y)) = C_{f(A)}(y), \end{aligned}$$

where $v \in X$ such that $f(v) = y$. Thus, $x \in N(f(A))$, and consequently $f(N(A)) \subseteq N(f(A))$. □

Theorem 4.2. *Let $f : X \to Y$ be homomorphism of abelian groups X and Y. Let A and B be multigroups of Y such that $B \subseteq A$. Then*

$$f^{-1}(N(B)) = N(f^{-1}(B)).$$

Proof. Let $x \in f^{-1}(N(B))$. Then for all $y \in X$,

$$\begin{aligned} C_{f^{-1}(B)}(xyx^{-1}) &= C_B(f(xyx^{-1})) = C_B(f(x)f(y)f(x^{-1})) = C_B(f(x)f(y)(f(x))^{-1}) \\ &= C_B(f(y)f(x)(f(x))^{-1}) = C_B(f(y)) = C_{f^{-1}(B)}(y). \end{aligned}$$

Thus $x \in N(f^{-1}(B))$. So, $f^{-1}(N(B)) \subseteq N(f^{-1}(B))$.

Again, let $x \in N(f^{-1}(B))$ and $f(x) = u$. Then for all $v \in Y$,

$$\begin{aligned} C_B(uvu^{-1}) &= C_B(f(x)f(y)(f(x))^{-1}) = C_B(f(y)f(x)(f(x))^{-1}) \\ &= C_B(f(y)) = C_B(v), \end{aligned}$$

where $y \in X$ such that $f(y) = v$. Clearly, $u \in N(B)$, that is, $x \in f^{-1}(N(B))$. Thus, $N(f^{-1}(B)) \subseteq f^{-1}(N(B))$. Hence, $f^{-1}(N(B)) = N(f^{-1}(B))$. □

Theorem 4.3. *Let $f : X \to Y$ be an isomorphism of groups and let A be a normal submultigroup of $B \in MG(X)$. Then $f(A)$ is a normal submultigroup of $f(B) \in MG(Y)$.*

Proof. By Theorem 2.11, $f(A), f(B) \in MG(Y)$ and so, $f(A) \subseteq f(B)$. We show that $f(A)$ is a normal submultigroup of $f(B)$. Let $x, y \in Y$. Since f is an isomorphism, then for some $a \in X$ we have $f(a) = x$. Thus,

$$\begin{aligned} C_{f(A)}(xyx^{-1}) &= \bigvee_{b \in X} \{C_A(b) \mid f(b) = xyx^{-1}\} = \bigvee_{b \in X} \{C_A(a^{-1}ba) \mid f(a^{-1}ba) = y\} \\ &\geqslant \bigvee_{a^{-1}ba \in X} \{C_A(b) \mid f(b) = y\} = \bigvee_{b \in X} \{C_A(f^{-1}(y)) \mid f(b) = y\} = C_{f(A)}(y). \end{aligned}$$

Hence, $f(A)$ is a normal submultigroup of $f(B)$. □

Theorem 4.4. *Let Y be a group and $A \in MG(Y)$. If f is an isomorphism of X onto Y and B is a normal submultigroup of A, then $f^{-1}(B)$ is a normal submultigroup of $f^{-1}(A)$.*

Proof. By Theorem 2.11, $f^{-1}(A), f^{-1}(B) \in MG(X)$. Since B is a submultigroup of A, so $f^{-1}(B) \subseteq f^{-1}(A)$. Let $a, b \in X$, then we have

$$\begin{aligned} C_{f^{-1}(B)}(aba^{-1}) &= C_B(f(aba^{-1})) = C_B(f(a)f(b)(f(a))^{-1}) \\ &= C_B(f(a)(f(a))^{-1}f(b)) \\ &\geqslant C_B(e) \wedge C_B(f(b)) = C_{f^{-1}(B)}(b), \end{aligned}$$

which completes the proof. □

5. Comultisets of a multigroup

In this section, we assume that if G is a multigroup of a group X, then $G_* = X$. That is, every element of X is in G with its multiplicity or count.

Definition 5.1. Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the *left comultiset* of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the *right comultiset* of A.

Example 5.2. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group of $\{1, 2, 3\}$ such that $\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$ and $G = [\rho_0^7, \rho_1^5, \rho_2^5, \rho_3^3, \rho_4^3, \rho_5^3]$ be a multigroup of X. Then $H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2]$ is a submultigroup of G.

Now, we find the left comultisets of H by pre-multiplying each element of G by H.

$$\begin{array}{ll} \rho_0 H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] & \rho_1 H = [\rho_2^3, \rho_0^6, \rho_1^3, \rho_5^2, \rho_3^2, \rho_4^2] \\ \rho_2 H = [\rho_1^3, \rho_2^3, \rho_0^6, \rho_4^2, \rho_5^2, \rho_3^2] & \rho_3 H = [\rho_3^2, \rho_5^2, \rho_4^2, \rho_0^6, \rho_2^3, \rho_1^3] \\ \rho_4 H = [\rho_4^2, \rho_3^2, \rho_5^2, \rho_1^3, \rho_0^6, \rho_2^3] & \rho_5 H = [\rho_5^2, \rho_4^2, \rho_3^2, \rho_2^3, \rho_1^3, \rho_0^6] \end{array}$$

Similarly, the right comultisets of H are

$$\begin{array}{ll} H\rho_0 = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] & H\rho_1 = [\rho_2^3, \rho_0^6, \rho_1^3, \rho_4^2, \rho_5^2, \rho_3^2] \\ H\rho_2 = [\rho_1^3, \rho_2^3, \rho_0^6, \rho_5^2, \rho_3^2, \rho_4^2] & H\rho_3 = [\rho_3^2, \rho_4^2, \rho_5^2, \rho_0^6, \rho_1^3, \rho_2^3] \\ H\rho_4 = [\rho_4^2, \rho_5^2, \rho_3^2, \rho_2^3, \rho_0^6, \rho_1^3] & H\rho_5 = [\rho_5^2, \rho_3^2, \rho_4^2, \rho_1^3, \rho_2^3, \rho_0^6] \end{array}$$

From Example 5.2, we notice that $H = yH$ for all $y \in X$ because a multigroup is an unordered collection. Consequently, $xH = yH$ for all $x, y \in X$.

Proposition 5.3. *Let X be a group. If A is a submultigroup of a multigroup G of X, then $yA = Ay$ for all $y \in X$.*

Proof. Assume A is a submultigroup of G. Then $\forall x \in A_*$ we have

$$C_{yA}(x) = C_A(y^{-1}x) \geqslant C_A(y) \wedge C_A(x) = C_A(x) \wedge C_A(y) = C_A(x) \wedge C_A(y^{-1}).$$

Suppose by hypothesis, $C_A(x) \wedge C_A(y) = C_A(xy)$. Then $C_{yA}(x) \geqslant C_{Ay}(x)$. Again,

$$C_{Ay}(x) = C_A(xy^{-1}) \geqslant C_A(x) \wedge C_A(y) = C_A(y) \wedge C_A(x) = C_A(y^{-1}) \wedge C_A(x).$$

By the same hypothesis, we get $C_{Ay}(x) \geqslant C_{yA}(x)$. Hence, $C_{yA}(x) = C_{Ay}(x)$, that is, $yA = Ay$. $\square$

Remark 5.4. If A is a submultigroup of a multigroup G of a group X, then each yA (and Ay) are submultigroups of G.

Proposition 5.5. *If H is a submultigroup of $A \in MG(X)$, then the number of comultisets of H equals the cardinality of H_*.*

Proof. Recall that $H_* = \{x \in X \mid C_H(x) > 0\}$, that is, H_* is a set. Since comultisets of H is formed by pre-multiplying each element of X (since $A_* = X$) by H and $C_{yH}(x) = C_H(y^{-1}x) \forall y \in X$ must exist, hence the result follows. $\square$

Proposition 5.6. *Let H be a submultigroup of $A \in MG(X)$. The union and intersection of the comultisets of H are comparable to H.*

Proof. $H = yH$ for all $y \in X$. Hence, the union and intersection of yH for all $y \in X$ are equal to H. $\square$

Proposition 5.7. *Let X be a group. Any submultigroup A of a multigroup G and for any $z \in X$, the submultiset zAz^{-1}, where $C_{zAz^{-1}}(x) = C_A(z^{-1}xz)$ for each $x \in X$ is a submultigroup of G.*

Proof. Let $x, y \in X$, then we have $C_{zAz^{-1}}(e) = C_A(e)$ and

$$\begin{aligned} C_{zAz^{-1}}(xy^{-1}) &= C_A(z^{-1}xy^{-1}z) = C_A(z^{-1}xzz^{-1}y^{-1}z) \\ &\geqslant C_A(z^{-1}xz) \wedge C_A(z^{-1}y^{-1}z) = C_{zAz^{-1}}(x) \wedge C_{zAz^{-1}}(y^{-1}) \\ &= C_{zAz^{-1}}(x) \wedge C_{zAz^{-1}}(y) \end{aligned}$$

for all $z \in X$. Hence zAz^{-1} is a submultigroup of G. $\square$

Corollary 5.8. *Let $\{A_i\}_{i \in I} \in MG(X)$, then*

(*i*) $\bigcap_{i \in I} zA_iz^{-1} \in MG(X)$ *for all* $z \in X$,

(*ii*) $\bigcup_{i \in I} zA_iz^{-1} \in MG(X)$ *for all* $z \in X$ *provided* $\{A_i\}_{i \in I}$ *have sup/inf assuming chain.* $\square$

Proposition 5.9. *Let $A \in MG(X)$ and for all $g, h \in X$, then the following statements hold:*

(i) $Ag \circ Ag = Ag$,

(ii) $Ag \circ Ah = Ah \circ Ag$,

(iii) $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$,

(iv) $(Ag \circ Ah)^{-1} = Ag \circ Ah$.

Proof. Let $g, h \in X$.

(i). From Definition 2.7, we have

$$\begin{aligned} C_{Ag \circ Ag}(x) &= \bigvee \{C_{Ag}(y) \wedge C_{Ag}(z) \mid x = yz, \forall y, z \in X\} \\ &= \bigvee_{y \in X} \{C_{Ag}(xy^{-1}) \wedge C_{Ag}(y) \mid x \in X\} = C_{Ag}(x). \end{aligned}$$

Hence, $Ag \circ Ag = Ag$.

(ii). $$\begin{aligned} C_{Ag \circ Ah}(x) &= \bigvee \{C_{Ag}(y) \wedge C_{Ah}(z) \mid x = yz, \forall y, z \in X\} \\ &= \bigvee \{C_{Ah}(z) \wedge C_{Ag}(y) \mid x = yz, y, z \in X\} = C_{Ah \circ Ag}(x). \end{aligned}$$

Hence, $Ag \circ Ah = Ah \circ Ag$.

(iii). We show that, the left and right hand sides are equal. By Definition 2.4

$$C_{(Ag \circ Ah)^{-1}}(x) = C_{Ag \circ Ah}(x^{-1}) = C_{Ag \circ Ah}(x).$$

Again, from the right hand side we get

$$\begin{aligned} C_{(Ah)^{-1} \circ (Ag)^{-1}}(x) &= \bigvee_{y \in X} \{C_{(Ah)^{-1}}(y^{-1}) \wedge C_{(Ag)^{-1}}(yx) \mid x \in X\} \\ &= \bigvee_{y \in X} \{C_{Ah}(y^{-1}) \wedge C_{Ag}(yx) \mid x \in X\} \\ &= C_{Ah \circ Ag}(x) = C_{Ag \circ Ah}(x). \end{aligned}$$

Hence, $(Ag \circ Ah)^{-1} = (Ah)^{-1} \circ (Ag)^{-1}$.

(iv). Straightforward from (iii). □

Proposition 5.10. *Let A be a commutative multigroup of a group X. Then*

(i) $Ay \circ Az = Ayz$ *for all $y, z \in X$,*

(ii) $yA \circ zA = yzA$ *for all $y, z \in X$.*

Proof. (i). Let $A \in MG(X)$ and $x \in X$, then we have

$$\begin{aligned} C_{Ay\circ Az}(x) &= \bigvee_{x=zy} \{C_{Ay}(z) \wedge C_{Az}(y) \mid \forall y, z \in X\} \\ &= \bigvee_{x=zy} \{C_A(zy^{-1}) \wedge C_A(yz^{-1}) \mid \forall y, z \in X\} \\ &= \{C_{A\cap A}((zy^{-1})(yz^{-1})) \mid \forall y, z \in X\} \\ &= \{C_A(xz^{-1}y^{-1}) \mid x = yz, \forall y, z \in X\} = \{C_{Ayz}(x) \mid x = yz, \forall y, z \in X\}. \end{aligned}$$

Hence, $Ay \circ Az = Ayz$.

(ii). Similar to (i). □

Corollary 5.11. *Let A be a multigroup of a group X and $y, z \in X$. The following statements are equivalent.*

(i) $(Ay \circ Az)^{-1} = Ay \circ Az$,

(ii) $Ay \circ Az = Ayz$.

Proof. Combining Proposition 5.9 and Proposition 5.10, the result follows. □

Theorem 5.12. *Let A be a commutative multigroup of a group X and $g, h \in X$, then $Ag \circ Ah = Agh$ if and only if $gA \circ hA = ghA$. Consequently, $Agh = ghA$.*

Proof. Let $A \in MG(X)$ and $g, h \in X$. Suppose $Ag \circ Ah = Agh$. Then

$$\begin{aligned} C_{Agh}(x) = C_{Ag\circ Ah}(x) &= \bigvee_{y\in X}(C_{Ag}(y) \wedge C_{Ah}(y^{-1}x)) = \bigvee_{y\in X}(C_A(yg^{-1}) \wedge C_A(y^{-1}xh^{-1})) \\ &= \bigvee_{y\in X}(C_A(g^{-1}y) \wedge C_A(h^{-1}y^{-1}x)) = \bigvee_{y\in X}(C_{gA}(y) \wedge C_{hA}(y^{-1}x)) \\ &= C_{gA\circ hA}(x) = C_{ghA}(x). \end{aligned}$$

So, $gA \circ hA = ghA$.

Conversely, let $gA \circ hA = ghA$. Then

$$\begin{aligned} C_{ghA}(x) = C_{gA\circ hA}(x) &= \bigvee_{y\in X}(C_{gA}(y) \wedge C_{hA}(y^{-1}x)) = \bigvee_{y\in X}(C_A(g^{-1}y) \wedge C_A(h^{-1}y^{-1}x)) \\ &= \bigvee_{y\in X}(C_A(yg^{-1}) \wedge C_A(y^{-1}xh^{-1})) = \bigvee_{y\in X}(C_{Ag}(y) \wedge C_{Ah}(y^{-1}x)) \\ &= C_{Ag\circ Ah}(x) = C_{Agh}(x). \end{aligned}$$

Thus, $Ag \circ Ah = Agh$. Hence, $Ag \circ Ah = Agh \Leftrightarrow gA \circ hA = ghA$. It follows that $Agh = ghA$. □

Proposition 5.13. *Let A be a normal submultigroup of $B \in MG(X)$. Then $C_{xA}(xz) = C_{xA}(zx) = C_A(z)$ for all $x, z \in X$.*

Proof. Let $x, z \in X$. Suppose A is a normal submultigroup of B, then by Proposition 3.4 and the fact that $C_A(xz) = C_A(zx)$, we get $C_{xA}(xz) = C_{xA}(zx) = C_A(x^{-1}zx) = C_A(z)$. Hence, $C_{xA}(xz) = C_{xA}(zx) = C_A(z)$ for all $z \in X$. □

Theorem 5.14. *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is a normal submultigroup of B if and only if for all $x \in X$, $Ax = xA$.*

Proof. Suppose A is a normal submultigroup of B. Then for all $x \in X$, we have $C_{Ax}(y) = C_A(yx^{-1}) = C_A(x^{-1}y) = C_{xA}(y)$ for all $y \in X$. Thus, $Ax = xA$.

Conversely, let $Ax = xA$ for all $x \in X$. Then, $C_A(xy) = C_{x^{-1}A}(y) = C_{Ax^{-1}}(y) = C_A(yx)$ for all $y \in X$. Hence A is a normal submultigroup of B by Proposition 3.4. □

References

[1] **J.A. Awolola and P.A. Ejegwa**, *On some algebraic properties of order of an element of a multigroup*, Quasigroups and Related Systems **25** (2017), 21 − 26.

[2] **J.A. Awolola and A.M. Ibrahim**, *Some results on multigroups*, Quasigroups and Related Systems **24** (2016), 169 − 177.

[3] **M. Dresher and O. Oré**, *Theory of multigroups*, American J. Math. **60** (1938), 705 − 733.

[4] **A.M. Ibrahim and P.A. Ejegwa**, *A survey on the concept of multigroup theory*, J. Nigerian Asso. Mathl. Physics **38** (2016), 75 − 89.

[5] **L. Mao**, *Topological multigroups and multifields*, Int. J. Math. Combinatorics **1** (2009), 8 − 17.

[6] **Sk. Nazmul, P. Majumdar and S.K. Samanta**, *On multisets and multigroups*, Annals Fuzzy Math. Inform. **6** (2013), 643 − 656.

[7] **W. Prenowitz**, *Projective geometries as multigroups*, American J. Math. **65** (1943), 235 − 256.

[8] **A. Rosenfeld**, *Fuzzy subgroups*, J. Math. Anal. Appl. **35** (1971), 512 − 517.

[9] **B.M. Schein**, *Multigroups*, J. Algebra **111** (1987), 114 − 132.

[10] **Y. Tella and S. Daniel**, *A study of group theory in the context of multiset theory*, Int. J. Sci. Tech. **2** (2013), 609 − 615.

[11] **Y. Tella and S. Daniel**, *Symmetric groups under multiset perspective*, IOSR J. Math. **7** (2013), no. 5, 47 − 52.

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P.A. Ejegwa
Department of Mathematics/Statistics/Computer Science, University of Agriculture P.M.B. 2373, Makurdi-Nigeria
E-mail: ejegwa.augustine@uam.edu.ng

A.M. Ibrahim
Department of Mathematics, Ahmadu Bello University, Zaria-Nigeria
E-mail: amibrahim@abu.edu.ng