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Representation of monoids in the category of monoid acts

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To Bernhard Banaschewski on his 90th Birthday

Abstract. The study of monoids in the category of monoid acts leads to the notion of power action. In this paper, for a monoid T, we investigate the relationship between the category T-**Act** of all T-acts and the category T-**Pwr** of all T-power acts. For a T-power act M on a commutative monoid T, we introduce the covariant functor M^{M^-} from T-**Act** to T-**Pwr** and show that the family of assignments $(\eta_A : A \to M^{M^A})_{A\in T\text{-}\mathbf{Act}}$ constitutes a natural transformation. Moreover, the Hom-functor $(M^-)^-$ and the tensor functor $M^{-\otimes-}$ from T-**Act** $\times$ T-**Act** to T-**Pwr** are naturally equivalent.

1. Introduction and preliminaries

Representation of mathematical structures is a way for better seeing of them to study. Analyzing the internalized concepts in a topos captured the interest of some mathematicians. The general notion of a mathematical object in a topos (or a category with some properties) introduces a lot of conceptions and structures obtained from its classical versions in **Set**, the category of sets ([4]). For instance, "Algebras in a Category" are some of these structures such as groups and group actions in a topos (see [2, 8]).

For a monoid T, let T-**Act** denote the category of all T-acts and act homomorphisms between them. Considering the monoid T as a category T with one object, T-**Act** is isomorphic to the functor category $\mathbf{Set}^T$ (or $[T,\mathbf{Set}]$ in another notation), hence it is a (presheaf) topos (see [3]). Here we study the structure of monoids in the category T-**Act**, so-called *T-power acts*, or *actions over monoids* in the sense of [5] which were used to construct the hypergroups. First we verify some basic properties of the power acts. In particular, the free objects in the category T-**Pwr** of all T-power acts are constructed. For a T-power act M and a T-act A over a commutative monoid T, it is shown that the set M^A of all T-act homomorphisms from A to M is a T-power act which gives the two functors M^- (contravariant) and M^{M^-} (covariant) from T-**Act** to T-**Pwr**. Also the family of assignments $(\eta_A : A \to M^{M^A})_{A\in T\text{-}\mathbf{Act}}$ constitutes a natural transformation from

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the identity functor to UM^{M^-}, where U is the forgetful functor. Finally, we prove that $(M^A)^B$ and $M^{A\otimes B}$ are naturally isomorphic in T-**Pwr** for every T-acts A and B.

Now let us briefly recall some needed notions in the sequel.

Let T be a monoid and A be a (non-empty) set. A *right T-act* on A is a map $A \times T \to A$, $(a,t) \rightsquigarrow at$, such that for every $a \in A$ and $t, s \in T$, $(at)s = a(ts)$ and $a1 = a$. The notion of *left T-act* is defined similarly. Here by a T-act we mean a right T-act unless otherwise stated. An element θ in a T-act A is said to be a *fixed element* if $\theta t = \theta$ for each $t \in T$. Let A, B be two T-acts. A map $f : A \to B$ is called a *T-act homomorphism* or simply *act homomorphism* if $f(at) = f(a)t$, for every $a \in A$ and $t \in T$. The class of all T-acts together with the T-act homomorphisms between them forms a category which is denoted by T-**Act**. For a monoid M, $H(M)$ denotes the monoid of all endomorphisms of M with the composition of mappings as its operation. To denote the image of $x \in M$ under $\sigma \in H(M)$ we will use the postfix notation. An equivalence relation θ on a T-act A is called a *T-act congruence* if $x\theta y$ implies that $xt\theta yt$, for every $x, y \in A$ and $t \in T$. The free T-act on a non-empty set X is the set $X \times T$ with the action $(x,t)s = (x,ts)$, for every $x \in X$ and $t, s \in T$. Let A be a right T-act and B be a left T-act. The *tensor product* of A and B is the set $A \otimes B := (A \times B)/\theta$, where θ is the equivalence relation on the set $A \times B$ generated by the pairs $((at,b),(a,tb))$ for $a \in A, b \in B, t \in T$. We denote $(a,b)/\theta \in A \otimes B$ by $a \otimes b$. In the case that T is a commutative monoid, every T-act can be considered as a T-biact so that there is naturally a T-act structure on the tensor product $A \otimes B$ for any two T-acts A and B (see [6, Proposition II.5.12]). For more information on the theory of acts over monoids, see [6]. Also for some required categorical ingredients we refer to [7]. Throughout the paper T stands for a monoid unless otherwise stated.

2. Monoids in the category of acts: Power action

Algebra in a category is a subject for mathematicians to study algebraic structures categorically. In this theory, a base category $\mathcal{C}$ is replaced to the category **Set** and all algebraic operations are the morphisms of $\mathcal{C}$, and homomorphisms are those morphisms in $\mathcal{C}$ such that preserve the operations in the sense of commutative diagrams in $\mathcal{C}$. Note that equations in algebras are explained as commutative diagrams. For more information we refer to [2, 4, 8].

Here we study the notion of monoid in the base category T-**Act**, where T is a monoid. Let us first recall the notion of a *monoid in an arbitrary category.* Let $\mathcal{C}$ be a category with finite products. A monoid $\langle M, \cdot, 1_M \rangle$ in $\mathcal{C}$ is an object of $\mathcal{C}$ together with two morphisms $\cdot : M \times M \to M$ called *multiplication* and $1_M : \top \to M$ called *identity,* in which $\top$ is the terminal object of $\mathcal{C}$ such that the following diagrams commute:

- Association law $((x \cdot y) \cdot z = x \cdot (y \cdot z))$:

$$\begin{tikzcd} M \times M \times M \arrow[r, "\cdot \times id_M"] \arrow[d, "id_M \times \cdot"'] & M \times M \arrow[d, "\cdot"] \\ M \times M \arrow[r, "\cdot"'] & M \end{tikzcd}$$

• Identity law ($x \cdot 1_M = x = 1_M \cdot x$):

$$\begin{tikzcd} \top \times M \arrow[r, "1_M \times id_M"] \arrow[dr, "\pi_2"'] & M \times M \arrow[d, "\cdot"] & M \times \top \arrow[l, "id_M \times 1_M"'] \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd}$$

Now let M, N be two monoids in a category $\mathcal{C}$. A *homomorphism* from M to N is a morphism $f : M \to N$ in $\mathcal{C}$ such that the following diagrams commute:
• Preserving the multiplication:

$$\begin{tikzcd} M \times M \arrow[r, "\cdot"] \arrow[d, "f \times f"'] & M \arrow[d, "f"] \\ N \times N \arrow[r, "\cdot"'] & N \end{tikzcd}$$

• Preserving the identity:

$$\begin{tikzcd} \top \arrow[r, "1_M"] \arrow[dr, "1_N"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

All monoids in a category $\mathcal{C}$ with homomorphisms between them make a category denoted by $\mathbf{Mon}(\mathcal{C})$.

Here we are going to explain objects of the category $\mathbf{Mon}(T\text{-}\mathbf{Act})$ for a monoid T with identity 1. Let M be an object in this category. Then there is a T-action $M \times T \to M, (m,t) \rightsquigarrow mt$, with a T-act homomorphism $\cdot : M \times M \to M$. So for every $t, s \in T$ and $m, n \in M$ we have $(mt)s = m(ts)$, $m1 = m$ and $(m{\cdot}n)t = mt{\cdot}nt$. Since $1_M : \top \to M$ is a T-act homomorphism where $\top$ is considered as the one-element T-act, $1_M t = 1_M$. Finally, by the diagrams of associativity and identity, M is a monoid. Because of the kind of these equations, we use the notation m^t for mt and give the following definition. If no confusion arises, the identities of M and T are denoted by the same symbol 1.

Definition 1. Let T be a monoid. By a (*right*) T*-power act*, we mean a monoid M equipped with a map $M \times T \to M$, $(m,t) \rightsquigarrow m^t$, in such a way that the following conditions hold for all $t, s \in T$ and $m, n \in M$:

$$(mn)^t = m^t n^t, \quad (m^t)^s = m^{ts}, \quad m^1 = m, \quad 1^t = 1.$$

If T contains a zero, then m^0 is clearly a fixed element of M where M is considered as a T-act.

Note that the notion of power act is also appeared in [5] under the name of "action over monoids".

Now we describe the morphisms of the category **Mon**(T-**Act**). Let M and N be two objects of **Mon**(T-**Act**). It is easy to see that a map $f : M \to N$ is a morphism in **Mon**(T-**Act**), so-called a *T-power act homomorphism* or simply *power act homomorphism* if and only if $f(mn) = f(m)f(n)$, $f(1) = 1$ and $f(m^t) = f(m)^t$, for all $m, n \in M$ and $t \in T$. The category of all T-power acts with T-power act homomorphisms between them is denoted by T-**Pwr** which is isomorphic to the category **Mon**(T-**Act**).

In the following, we give some examples of power acts.

Example 1.

1. Consider the monoid $(\mathbb{N}, \cdot)$. Then every commutative monoid M with m^k to be $mm \cdots m$, k-times, for every $m \in M$ and $k \in \mathbb{N}$, is an $\mathbb{N}$-power act.

2. Given a monoid M, let T be a submonoid of $H(M)$. Then we define m^σ to be $m\sigma$, for all $m \in M$ and $\sigma \in T$. Then M is a T-power act which is called the *natural power action*.

3. Given two monoids M and T with $0 \in T$, let $\phi : T \to H(M)$ be a monoid homomorphism and $u \in M$. For every $m \in M$ and $t \neq 0$ in T, define $m^t = m\phi(t)$, and $m^0 = u$. Then M is a T-power act if and only if $u\phi(t) = u$ for all $t \in T$ and $u^2 = u$. This is called the *(ϕ, u)-power action*. In particular, the $(\mathbf{id}, 1)$-power action is said to be an *identity power action* where $\mathbf{id} : T \to H(M)$ is the constant homomorphism mapping every $t \in T$ to id_M.

Proposition 1. *Let M and T be two monoids and $0 \in T$. Then each T-power act M is of the form (ϕ, u)-power act (in the sense of Example 1(3)) for a unique monoid homomorphism $\phi : T \to H(M)$ and some $u \in M$.*

Proof. Let M be a T-power act and $t \in T$. Define $\sigma_t : M \to M$ by $m\sigma_t = m^t$ for every $m \in M$. We show that the map σ_t is a monoid homomorphism. Indeed, we have $(mn)\sigma_t = (mn)^t = m^t n^t = m\sigma_t n\sigma_t$, and $1\sigma_t = 1^t = 1$ for every $m, n \in M$. Now, define $\phi : T \to H(M)$ by $\phi(t) = \sigma_t, t \in T$. The map ϕ is a monoid homomorphism. To see this, for any $t, s \in T$ and $m \in M$, $m\sigma_{ts} = m^{ts} = (m^t)^s = m\sigma_t\sigma_s$. Thus $\phi(ts) = \sigma_{ts} = \sigma_t\sigma_s = \phi(t)\phi(s)$. Also $\phi(1) = \sigma_1 = id$. Now take $u := \phi(0)$. It is clear that $u^2 = u$ and $u\phi(t) = u$ for all $t \in T$. Then M is a (ϕ, u)-power act (see Example 1(3)). For the uniqueness of ϕ, suppose that $\psi : T \to H(M)$ is a monoid homomorphism with $m^t = m\psi(t)$, for all $m \in M$ and $t \in T$. This implies that $m\psi(t) = m\phi(t)$ for all $m \in M$ and $t \in T$ which means $\psi = \phi$. $\square$

Here we define the notion of a bipower act.

Definition 2. Let T and S be monoids. By a (T,S)-*bipower act* M we mean a monoid M which is both (right) T and S-power acts simultaneously, in such a way that $(m^t)^s = (m^s)^t$, for every $m \in M$, $t \in T$ and $s \in S$.

Remark 1. Every (T,S)-bipower act M for two monoids T and S can be considered as a $T \times S$-power act. To this end, we define the power action $m^{(t,s)}$ to be $(m^t)^s$ for every $m \in M$, $t \in T$ and $s \in S$. Then we have:

1. $m^{(1,1)} = (m^1)^1 = m$,
2. $1^{(t,s)} = (1^t)^s = 1$,
3. $m^{(t,s)} n^{(t,s)} = (m^t)^s (n^t)^s = (m^t n^t)^s = ((mn)^t)^s = (mn)^{(t,s)}$,
4. $(m^{(t_1,s_1)})^{(t_2,s_2)} = (((m^{t_1})^{s_1})^{t_2})^{s_2} = (((m^{t_1})^{t_2})^{s_1})^{s_2} = (m^{t_1 t_2})^{s_1 s_2} = m^{(t_1,s_1)(t_2,s_2)}$.

By a *power act congruence* on a T-power act M we mean a monoid congruence as well as a T-act congruence on M.

Suppose that M is a T and S-power act for monoids T and S. We construct a quotient of M which is a (T,S)-bipower act. To do this, let θ be the power act congruence on M generated by the set $\theta = \{((m^t)^s, (m^s)^t) : m \in M, t \in T, s \in S\}$. Define $(m/\theta)(m'/\theta) = (mm')/\theta$, $(m/\theta)^t = m^t/\theta$ and $(m/\theta)^s = m^s/\theta$ for $m, m' \in M, t \in T, s \in S$. It is easily seen that M/θ is a (T,S)-bipower act. Hence, it follows from Remark 1 that M/θ is a $T \times S$-power act.

Lastly, we show that the power act is a universal algebraic structure and verify the existence of the free power acts. The reader is referred to [1] for some required details on universal algebra.

Let M be a T-power act. Then M can be considered as an algebra of the type $\langle \cdot, (\lambda_t)_{t \in T}, 1 \rangle$, where $\cdot$ is the binary operation, λ_t is the unary operation given by $\lambda_t(m) = m^t$, for every $t \in T, m \in M$, and 1 is the nullary operation on M such that the following equations hold for every $t, s \in T$ and $x, y \in M$:

$$\lambda_t(x \cdot y) = \lambda_t(x) \cdot \lambda_t(y), \quad \lambda_s(\lambda_t(x)) = \lambda_{ts}(x), \quad \lambda_1(x) = x, \quad \lambda_t(1) = 1.$$

Therefore, the category T-**Pwr** is an equational class and then the free objects over T-acts exist in this category. We explain the construction of free T-power acts in the following.

Let A be a T-act. Consider the free monoid $Fm(A) = \{x_1 x_2 \cdots x_n : x_i \in A, n \in \mathbb{N}\} \cup \{1\}$ on the set A. Now we define a T-action on $Fm(A)$ by $(x_1 \cdots x_n)^t = x_1^t \cdots x_n^t$, $1^t = 1$ for all $t \in T$ and $x_i \in A$, then one can easily see that $Fm(A)$ is a T-power act, and the inclusion map $i : A \to Fm(A)$ is a T-act homomorphism. If M is a T-power act and $f : A \to M$ is a T-act homomorphism, we define $\overline{f} : Fm(A) \to M$ to be $\overline{f}(x_1 \cdots x_n) = f(x_1) \cdots f(x_n)$. Clearly, $\overline{f}$ is a T-power act homomorphism with $\overline{f} i = f$. Also if $g : Fm(A) \to M$ is a T-power act homomorphism with $gi = f$, then we have $g(x_1 \cdots x_n) = g(x_1) \cdots g(x_n) = f(x_1) \cdots f(x_n) = \overline{f}(x_1 \cdots x_n)$, for every $x_1, x_2, \dots, x_n \in A$, that is, $\overline{f}$ is unique. Hence, $Fm(A)$ is a free monoid in the category T-**Act** on a T-act A. Then the assignment $A \rightsquigarrow Fm(A)$ defines the free functor **Fm** : T-**Act** $\to$ T-**Pwr**. It is worth noting that the composition of **Fm** to the free functor **F** : **Set** $\to$ T-**Act**, given by $X \rightsquigarrow X \times T$, gives the free functor **Fpwr** : **Set** $\to$ T-**Pwr**, $X \rightsquigarrow Fm(X \times T)$. Consequently, $Fm(X \times T)$ is the free T-power act on a set X.

3. Power acts over commutative monoids

This section is devoted to study T-power acts for which T is a commutative monoid. This kind of power acts displays a close relationship between Hom-functors and tensor functors.

For a T-power act M and a T-act A, let us denote $M^A := Hom_{T\text{-}\mathbf{Act}}(A, M)$, the set of all T-act homomorphisms from A to M where M is considered as a T-act. It is easily seen that the set M^A is a monoid under the operation $(f \cdot g)(a) := f(a)g(a)$, for every $f, g \in M^A, a \in A$. Note that the identity element of M^A is $1 : A \to M$ mapping every $a \in A$ to $1 \in M$. Now we get the following:

Lemma 1. *Let M be a T-power act and A be a T-act, where T is a commutative monoid. Then the monoid M^A is a T-power act together with the action $f^t(a) := (f(a))^t$, for every $f \in M^A, t \in T, a \in A$.*

Proof. Take any $f \in M^A$ and $t \in T$. First note that $f^t \in M^A$. Indeed, for every $t, s \in T, a \in A$, the commutativity of T implies that

$$f^t(as) = (f(as))^t = ((f(a))^s)^t = (f(a))^{st} = (f(a))^{ts} = ((f(a))^t)^s = (f^t(a))^s.$$

Moreover, for every $f, g \in M^A, t, s \in T$ and $a \in A$, we have:

1. $(f \cdot g)^t(a) = ((f \cdot g)(a))^t = (f(a)g(a))^t = (f(a))^t(g(a))^t = f^t(a)g^t(a) = (f^t \cdot g^t)(a)$.
2. $(f^t)^s(a) = (f^t(a))^s = ((f(a))^t)^s = f(a)^{ts} = f^{ts}(a)$.
3. $f^1(a) = (f(a))^1 = f(a)$.
4. $1^t(a) = (1(a))^t = 1^t = 1$.

This means that M^A is a T-power act. □

We carry on this section with studying of the connections between the categories T-**Act** and T-**Pwr** for which T is a commutative monoid.

Proposition 2. *Let M be a T-power act on a commutative monoid T. The following assertions hold:*

(i) *There is a contravariant Hom-functor $M^- = Hom_{T\text{-}\mathbf{Act}}(-, M) : T$-**Act** $\to$ T-**Pwr** assigning each T-act A to M^A, and each T-act homomorphism $h : A \to B$*

to $M^h : M^B \to M^A$ *mapping each* $f \in M^B$ *to* $f \circ h$. *Moreover, this yields a covariant Hom-functor* $M^{M^-} = Hom_{T\text{-}\mathbf{Pwr}}(M^-, M) : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$ *in a natural way.*

(ii) *The family of assignments* $(\eta_A : A \to M^{M^A})_{A \in T\text{-}\mathbf{Act}}$ *each of them assigning* $a \mapsto \hat{a} : M^A \to M$, $\hat{a}(f) = f(a)$ *for every* $a \in A, f \in M^A$, *constitutes a natural transformation from the identity functor* $Id_{T\text{-}\mathbf{Act}}$ *to the functor* UM^{M^-} *where* $U : T\text{-}\mathbf{Act} \to T\text{-}\mathbf{Pwr}$ *is the forgetful functor.*

Proof. (i) For every T-act A, $M^A \in T\text{-}\mathbf{Pwr}$ by Lemma 1. Considering a T-act homomorphism $h : A \to B$, we claim that M^h is a T-power act homomorphism. Clearly, M^h is a monoid homomorphism. Let $t \in T$ and $f \in M^B$. Then $M^h(f^t)(a) = (f^t \circ h)(a) = f^t(h(a)) = f(h(a)t) = f(h(at)) = (M^h(f))(at) = (M^h(f))^t(a)$, for every $a \in A$. So $M^h(f^t) = (M^h(f))^t$, as desired. Assume that $h : A \to B$ and $k : B \to C$ are homomorphisms in T-**Act** and $f \in M^C$. It follows that $M^{k \circ h}(f) = f \circ (k \circ h) = (f \circ k) \circ h = M^h(M^k(f)) = (M^h \circ M^k)(f)$. That is, $M^{k \circ h} = M^h \circ M^k$. Also clearly $M^{id_A} = id_{M^A}$. Therefore, M^- is a contravariant functor. For the second part, it suffices to note that $M^{M^-} = M^- \circ U \circ M^-$ where $U : T\text{-}\mathbf{Pwr} \to T\text{-}\mathbf{Act}$ is the forgetful functor.

(ii) First we show that the map $\hat{a} : M^A \to M$ is a morphism in T-**Pwr**, for each a in a T-act A. Let $f, g \in M^A$ and $t \in T$. Then $\hat{a}(f \cdot g) = (f \cdot g)(a) = f(a)g(a) = \hat{a}(f)\hat{a}(g)$, and $\hat{a}(f^t) = f^t(a) = (f(a))^t = (\hat{a}(f))^t$. Moreover, each η_A is a morphism in T-**Act** because $\widehat{at}(f) = f(at) = f^t(a) = \widehat{a}(f^t) = (\widehat{a})^t(f)$ for all $a \in A, t \in T, f \in M^A$. Hence, $\eta_A(at) = (\eta_A(a))^t$. It remains to prove the commutativity of the following diagram:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & M^{M^A} \arrow[d, "M^{M^f}"] \\ B \arrow[r, "\eta_B"'] & M^{M^B} \end{tikzcd}$$

Let $a \in A$, $\beta \in M^B$. We have $M^{M^f} \circ \eta_A(a)(\beta) = (\hat{a} \circ M^f)(\beta) = \hat{a}(\beta \circ f) = (\beta \circ f)(a) = \beta(f(a)) = \widehat{f(a)}(\beta) = \eta_B \circ f(a)(\beta)$, as required. □

Remark 2. (i) Let Γ be a subclass of morphisms in T-**Act** and M be a T-power act for a commutative monoid T. Then one can easily check that M is a Γ-injective object in T-**Act**, i.e. injective with respect to all Γ-morphisms, if and only if the contravariant functor M^- maps every Γ-morphism to an onto morphism in T-**Pwr**.

(ii) Let $\mathcal{C}$ be the category of all contravariant functors from T-**Act** to T-**Pwr** for a commutative monoid T, and natural transformations between them. Then the assignment $M \rightsquigarrow M^-$ gives a covariant functor $T\text{-}\mathbf{Pwr} \to \mathcal{C}$. More explicitly, for every morphism $\alpha : M \to N$ in T-**Pwr**, one can define a natural transformation $\widehat{\alpha} = (\widehat{\alpha}_A)_{A \in T\text{-}\mathbf{Act}} : M^- \to N^-$ to be $\widehat{\alpha}_A(f) = \alpha \circ f$, for all $f \in M^A$. That is, for every T-act homomorphism $h : A \to B$, the following diagram commutes:

$$\begin{tikzcd} M^A \arrow[r, "\widehat{\alpha}_A"] & N^A \\ M^B \arrow[u, "M^h"] \arrow[r, "\widehat{\alpha}_B"'] & N^B \arrow[u, "N^h"'] \end{tikzcd}$$

Indeed, $N^h \circ \widehat{\alpha}_B(f) = N^h(\alpha \circ f) = (\alpha \circ f) \circ h = \alpha \circ (f \circ h) = \widehat{\alpha}_A(f \circ h) = \widehat{\alpha}_A \circ M^h(f)$, for every $f \in M^B$.

At the end, we give the following theorem which shows the relationship between Hom-functors and tensor functors.

Theorem 1. *For a T-power act M on a commutative monoid T, the Hom-functor $(M^-)^- : T$-**Act** $\times$ T-**Act** $\to T$-**Pwr** is naturally equivalent to the tensor functor $M^{-\otimes -} : T$-**Act** $\times$ T-**Act** $\to T$-**Pwr**.*

Proof. For every T-acts A and B, we define $\phi = \phi_{A,B} : M^{A \otimes B} \to (M^A)^B$ mapping each T-power act homomorphism $f : A \otimes B \to M$ to $\phi(f) : B \to M^A$, where $\phi(f)(b) : A \to M$, for every $b \in B$, maps every $a \in A$ to $f(a \otimes b)$. It follows from [6, Corollary II.5.20] that ϕ is a T-act isomorphism. Moreover, it is clear that ϕ is a monoid homomorphism. Hence, ϕ is an isomorphism in T-**Pwr**. It remains to prove the naturality of $(\phi_{A,B})_{A,B} : M^{-\otimes -} \to (M^-)^-$. Consider any T-act homomorphisms $f : A \to A'$ and $g : B \to B'$. We show that the following diagram commutes:

$$\begin{tikzcd} M^{A\otimes B} \arrow[r, "\phi_{A,B}"] & (M^A)^B \\ M^{A'\otimes B'} \arrow[u, "M^{f\otimes g}"] \arrow[r, "\phi_{A',B'}"] & (M^{A'})^{B'} \arrow[u, "(M^f)^g"'] \end{tikzcd}$$

Indeed, for every $a \in A$ and $b \in B$, we have

$$\begin{aligned} ((\phi_{A,B} \circ M^{f\otimes g})(\alpha))(b)(a) &= \phi_{A,B}(M^{f\otimes g}(\alpha))(b)(a) = M^{f\otimes g}(\alpha)(a \otimes b) \\ &= (\alpha \circ (f \otimes g))(a \otimes b) \\ &= \alpha(f(a) \otimes g(b)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (((M^f)^g \circ \phi_{A',B'})(\alpha))(b)(a) &= (M^f)^g(\phi_{A',B'}(\alpha))(b)(a) \\ &= (M^f \circ \phi_{A',B'}(\alpha) \circ g)(b)(a) \\ &= M^f(\phi_{A',B'}(\alpha)(g(b)))(a) \\ &= (\phi_{A',B'}(\alpha)(g(b)) \circ f)(a) \\ &= \phi_{A',B'}(\alpha)(g(b))(f(a)) \\ &= \alpha(f(a) \otimes g(b)). \end{aligned}$$

Hence, $\phi_{A,B} \circ M^{f\otimes g} = (M^f)^g \circ \phi_{A',B'}$. $\square$

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References

[1] **S. Burris, H.P. Sankappanavar**, *A Course in Universal Algebra*, Springer-Verlag, 1981.

[2] **M.M. Ebrahimi**, *Algebra in a Grothendieck topos: injectivity in quasi-equational classes*, J. Pure Appl. Algebra **26** (1982), no. 3, 269 − 280.

[3] **M.M. Ebrahimi, M. Mahmoudi**, *The category of M-sets*, Italian J. Pure Appl. Math. **9** (2001), 123 − 132.

[4] **P.T. Johnstone**, *Sketches of an Elephant: A Topos Theory Compendium*, Vol. 1,2, Oxford: Clarendon Press, 2002.

[5] **A. Karimi Feizabadi, H. Rasouli**, *Actions over monoids and hypergroups*, Quasigroups Related Systems **23** (2015), 257 − 262.

[6] **M. Kilp, U. Knauer, A.V. Mikhalev**, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.

[7] **S. MacLane**, *Categories for the working mathematicians*, Graduate Texts in Mathematics 5, Springer-Verlag, 1971.

[8] **S. Mac Lane, I. Moerdijk**, *Sheaves in Geometry and Logic*, Springer-Verlag, 1992.

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