The congruence $\mathscr{Y}^*$ on completely regular semirings

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Abstract. We investigate the congruence generated by $\mathscr{Y}$ on completely regular semirings and get that $\mathscr{Y}^* \in [\epsilon, \nu]$ on completely regular semirings.

1. Introduction

The study of the structure of semigroups and semirings are essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semiring or a semigroup is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences $\mathscr{C}(S)$ on S. In 1999, Petrich and Reilly [8] defined a relation $\mathscr{Y}$ on a completely regular semigroup S by: for $a, b \in S$;

$$a \mathscr{Y} b \quad \text{if and only if} \quad V(a) = V(b).$$

Under certain special conditions of semigroups, $\mathscr{Y}$ was proved to be the least Clifford congruence on S. It was proposed by them as an open problem that, what can be said about $\mathscr{Y}^*$, the congruence generated by $\mathscr{Y}$ on a completely regular semigroup. Recently, in 2011, C. Guo, G. Liu and Y. Guo solved this open problem in their paper [1]. They proved that $\mathscr{Y}^* \in [\epsilon, \nu]$ on completely regular semigroups. Furthermore, they gave a description of $\mathscr{Y}^*$ on completely simple semigroups and normal cryptogroups, respectively. The main aim of this paper is to further extend these ideas on completely regular semirings.

The preliminaries and prerequisites we need for this paper are discussed in Section 2. In Section 3 we study some properties of orthodox completely regular semirings and finally in Section 4 we characterize the relation $\mathscr{Y}^*$ on completely regular semirings.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a type (2, 2)-algebra such that the semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by distributive laws, i.e., $a(b+c) = ab+ac$ and $(b+c)a =$

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$ba + ca$ for all $a, b, c \in S$. Here the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$ is not necessarily commutative. An element a in a semiring $(S, +, \cdot)$ is said to be *additively regular* if there exists an element $x \in S$ such that $a + x + a = a$.

Following [5], we say that an element a of a semiring $(S, +, \cdot)$ is *completely regular* if there exists $x \in S$ such that $a = a+x+a$, $a+x = x+a$ and $a(a+x) = a+x$. A semiring S is said to be *completely regular* if every element of S is completely regular.

Let τ be a relation on a semiring S. Define the relation τ^e on S by: for $a, b \in S$;

$$a\tau^e b \text{ if and only if } a = x + c + y,\ b = x + d + y \text{ for some } x, y \in S^0 \text{ and } c\tau d.$$

Also, we define $\tau^\natural$ by $\tau^\natural = \Big((\tau \cup \tau^{-1} \cup \epsilon)^e\Big)^t$, where ϵ is the equality congruence and η^t denotes the transitive closure of η.

Following [5], a semiring $(S, +, \cdot)$ is called a *skew-ring* if its additive reduct $(S, +)$ is a group, not necessarily an abelian group. A semiring $(S, +, \cdot)$ is said to be a *b-lattice* [5] if $(S, \cdot)$ is a band and $(S, +)$ is a semilattice. If $(S, +, \cdot)$ is a semiring, we denote Green's relations on the semigroup $(S, +)$ by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, $\mathscr{D}^+$ and $\mathscr{H}^+$. In fact, the relations $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, $\mathscr{D}^+$ and $\mathscr{H}^+$ are all congruences on the multiplicative reduct $(S, \cdot)$. Thus, if any one of these happens to be a congruence on the additive reduct $(S, +)$, it will be a congruence on the semiring $(S, +, \cdot)$. A completely regular semiring S is said to be *completely simple* [5] if $\mathscr{J}^+ = S \times S$. A congruence ξ on a semiring S is called a *b-lattice congruence (idempotent semiring congruence)* if S/ξ is a b-lattice (respectively, an idempotent semiring). A semiring S is said to be a *b-lattice* (*idempotent semiring*) Y *of semirings* $S_\alpha (\alpha \in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ξ on S such that $Y = S/\xi$ and each S_α is a ξ-class. We write $S = (Y; S_\alpha)$.

First we prove the following result.

Theorem 2.1. *The following conditions on a semiring are equivalent:*

(i) *S is completely regular;*
(ii) *every $\mathscr{H}^+$-class is a skew-ring;*
(iii) *S is union* (*disjoint*) *of skew-rings;*
(iv) *S is a b-lattice of completely simple semirings;*
(v) *S is an idempotent semiring of skew-rings.*

Proof. From [5, Theorem 3.6], it follows that first four conditions are equivalent.

(i) $\Rightarrow$ (v): Let S be a completely regular semiring. Then by [5, Theorem 3.6], it follows that each $\mathscr{H}^+$-class is a skew-ring. Let x^0 be the zero of the skew-ring H_x, where H_x is the $\mathscr{H}^+$-class containing the element $x \in S$. To complete the prove it suffices to show that $\mathscr{H}^+$ is an idempotent semiring congruence on S. For this let $a\,\mathscr{H}^+\,b$ and $c \in S$. Then $a^0 = b^0$. Now $(a + c)^0 = (a + c)(a + c)^0 = a(a+c)^0 + c(a+c)^0 = a^0(a+c)^0 + c(a+c)^0 = (a^0+c)(a+c)^0 = (a^0+c)^0(a+c)^0 = (a^0 + c)^0(a + c) = (a^0 + c)^0 a + (a^0 + c)^0 c = (a^0 + c)^0 a^0 + (a^0 + c)^0 c = (a^0 + c)^0(a^0 + c) = (a^0 + c)^0$. Similarly, we can show that $(b + c)^0 = (b^0 + c)^0$. Thus, $(a + c)^0 = (a^0 + c)^0 = (b^0 + c)^0 = (b + c)^0$. This implies $a + c\,\mathscr{H}^+\,b + c$. Dually,

$c + a\,\mathscr{H}^+\,c + b$. Hence $\mathscr{H}^+$ is a congruence on $(S, +)$. Since $\mathscr{H}^+$ is a congruence on $(S, \cdot)$, it follows that $\mathscr{H}^+$ is a congruence on the semiring S. Clearly, $2a\,\mathscr{H}^+\,a$ and $a^2\,\mathscr{H}^+\,a$. Hence $S/\mathscr{H}^+$ is an idempotent semiring. Consequently, S is an idempotent semiring of skew-rings.

$(v) \Rightarrow (i)$: This is obvious. □

Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Observe that the distributive laws imply that whenever the set $E^+(S)$ is non-empty, it forms an ideal of the multiplicative reduct $(S, \cdot)$ of S. If $a \in S$ is additively regular, we denote the set of all inverse elements of a in the semigroup $(S, +)$ by $V^+(a)$. Also we denote the least skew-ring congruence by σ and the least b-lattice of skew-ring congruence by ν on a semiring S. We always let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and S_α $(\alpha \in Y)$ is a completely simple semiring. For other notation and terminology not given in this paper, the reader is referred to the texts of Howie [3], Golan [4], and Petrich and Reilly [8].

Next we introduce some results which can be proved in a similar way as completely regular semigroup (see for example Theorem II.4.5 in [8]).

Theorem 2.2. *Let $S = (Y; S_\alpha)$ be completely regular semiring. Then $\mathscr{J}^+ = \mathscr{D}^+$.*

Lemma 2.3. *For any completely regular semiring S,*

$$\nu = \{(f, g) \mid f, g \in E^+(S) \text{ and } f\,\mathscr{D}^+\,g\}^\natural.$$

Proof. Let $\eta = \{(f, g) \mid f, g \in E^+(S),\ f\,\mathscr{D}^+\,g\}^\natural$.

Clearly, $\eta \subseteq \mathscr{D}^+$ and each $\mathscr{D}^+$- class of S/η contains a unique additive idempotent. Hence S/η is a b-lattice of skew-rings and $\nu \subseteq \eta$. On the other hand, S/ν is a b-lattice of skew-rings so that $\{(f, g) \mid f, g \in E^+(S), f\,\mathscr{D}^+\,g\} \subseteq \nu$, which implies $\{(f, g) \mid f, g \in E^+(S), f\,\mathscr{D}^+\,g\}^e \subseteq \nu$. Thus, $\nu = \{(f, g) \mid f, g \in E^+(S),\ f\,\mathscr{D}^+\,g\}^\natural$. □

Lemma 2.4. *Let $S = (Y; S_\alpha)$ be a completely regular semiring and $a \in S_\alpha$, $b \in S_\beta$, where $\beta \leqslant \alpha$. Then,*

(i) $a\,\mathscr{L}^+\,(b + a)$, $a\,\mathscr{R}^+\,(a + b)$,

(ii) $a = a + (b + a)^0 = (a + b)^0 + a$.

Proof. Follows similarly from [8, Corollary II.4.3.]. □

3. The relation $\mathscr{Y}$

We call a semiring $(S, +, \cdot)$ an *orthodox* semiring if the additive reduct $(S, +)$ is orthodox, i.e., $E^+(S)$ forms an ideal of S. We show that the relation $\mathscr{Y}$ and ν are equivalent on an orthodox completely regular semiring.

Let S be a completely regular semiring. Define a relation $\mathscr{Y}$ on S by: for $a, b \in S$;

$$a\,\mathscr{Y}\,b \text{ if and only if } V^+(a) = V^+(b).$$

We need the following result.

Lemma 3.1. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring, where Y is a b-lattice and $S_\alpha (\alpha \in Y)$ is a completely simple semiring. Then $\left(E^+(S_\alpha), +\right)$ is a rectangular band for all $\alpha \in Y$ and for any two elements $a, b \in S_\alpha$, $e \in E^+(S_\beta)$, $a + b = a + e + b$, where $\alpha, \beta \in Y$ such that $\beta \leqslant \alpha$.*

Proof. Follows similarly from [8, Lemma II.5.2]. □

Theorem 3.2. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring and $a, b \in S$. Then the following conditions are equivalent:*

(*i*) $a \mathscr{Y} b$.

(*ii*) *There exists $e, f, g, h \in E^+(S)$ with $a = e + b + f$ and $b = g + a + h$.*

(*iii*) $a = a^0 + b + a^0$ *and* $b = b^0 + a + b^0$.

Proof. $(i) \Rightarrow (ii)$: At first we suppose that $a \mathscr{Y} b$ for $a, b \in S$. Then $V^+(a) = V^+(b)$. Let $x \in V^+(a)$. Then $x \in V^+(b)$, i.e., $a = a + x + a$, $x + a + x = x$ and $b = b + x + b$, $x + b + x = x$.

Thus, $a = (a+x)+b+(x+a) = e+b+f$, where $e = a+x$, $f = x+a \in E^+(S)$. Similarly, $b = g + a + h$, for some $g, h \in E^+(S)$.

$(ii) \Rightarrow (iii)$: We have $a^0+e+b+f+a^0 = a^0+a+a^0 = a$ for some $e, f \in E^+(S)$. Then $a \mathscr{D}^+ b$. Let $a, b \in S_\alpha$, $e \in S_\beta$ and $f \in S_\gamma$. Then $\beta, \gamma \leqslant \alpha$. Now, by Lemma 3.1, $a^0 + e + b = a^0 + b$. Similarly, $b + f + a^0 = b + a^0$. Hence, we have, $a^0 + b + a^0 = a$. Similarly, $b^0 + a + b^0 = b$.

$(iii) \Rightarrow (i)$: Let, $x \in V^+(a)$. Then, using Lemma 3.1, we have

$$\begin{aligned} b &= b^0 + a + b^0 = b^0 + a + x + a + b^0 \\ &= (b^0 + a + b^0) + x + (b^0 + a + b^0) = b + x + b. \end{aligned}$$

Similarly, $x + b + x = x$. Hence, $x \in V^+(b)$ and thus, $V^+(a) \subseteq V^+(b)$. By symmetry, it follows that $V^+(b) \subseteq V^+(a)$. Thus, $a \mathscr{Y} b$. □

Theorem 3.3. *Let $S = (Y; S_\alpha)$ be a completely regular semiring. Then $\mathscr{Y}$ is the least b-lattice of skew-rings congruence on S if and only if S is orthodox.*

Proof. By [1, Theorem 1.6], we have $\mathscr{Y}$ is the least semilattice of groups congruence on the semigroup reduct $(S, +)$ if and only if $(S, +)$ is orthodox. To complete the proof it remains to show that $\mathscr{Y}$ is a congruence on $(S, \cdot)$. For this let $a \mathscr{Y} b$ and $c \in S$. Then $a = a^0 + b + a^0$ and $b = b^0 + a + b^0$. This implies $ca = ca^0 + cb + ca^0 = (ca)^0 + cb + (ca)^0$ and $cb = cb^0 + ca + cb^0 = (cb)^0 + ca + (cb)^0$ and hence $ca \mathscr{Y} cb$. Similarly, we can show that $ac \mathscr{Y} bc$. Consequently, $\mathscr{Y}$ is a congruence on S. Since S is completely regular, it follows that $S/\mathscr{Y}$ is also completely regular. Moreover, since $(S/\mathscr{Y}, +)$ is semilattice of groups, one can easily prove that $S/\mathscr{Y}$ is a b-lattice of skew-rings, i.e., $\mathscr{Y}$ is the least b-lattice of skew-ring congruence on S. □

Theorem 3.4. *Let $S = (Y; S_\alpha)$ be an orthodox completely regular semiring. Then $\mathscr{D}^+ = \mathscr{H}^+ \mathscr{Y}$.*

Proof. Let $a\,\mathscr{D}^+\,b$ for $a, b \in S$. Now, we have, by Lemma 3.1, $b = b^0 + b + b^0 = b^0 + (a^0 + b + a^0) + b^0$. Again, $(a^0 + b + a^0)^0 + b + (a^0 + b + a^0)^0 = a^0 + b + a^0$. Hence, $(a^0 + b + a^0)\,\mathscr{Y}\,b$. Again, since $a^0 = (a^0 + b + a^0)^0$ we have $a\,\mathscr{H}^+\,(a^0 + b + a^0)$. Thus we have, $a\,(\mathscr{H}^+\,\mathscr{Y})\,b$ and hence $\mathscr{D}^+ \subseteq \mathscr{H}^+\,\mathscr{Y}$. The reverse inclusion is obvious. This completes the proof. □

We highlight a very interesting result based on the congruences that we have discussed so far.

Theorem 3.5. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and $S_\alpha\,(\alpha \in Y)$ is a completely simple semiring. Then the following conditions are equivalent:*

(i) S is orthodox,

(ii) S is a spined product of an idempotent semiring and a b-lattice of skew-rings,

(iii) S satisfies the identity $a^0 + b^0 = (a + b)^0$.

Proof. $(i) \Rightarrow (ii)$: Let π_1 (respectively, π_2) be the natural projection of $S/\mathscr{H}^+$ (respectively, $S/\mathscr{Y}$) onto Y. Let A be the spined product of $S/\mathscr{H}^+$ and $S/\mathscr{Y}$. Then, for any $a \in S_\alpha$, $\pi_1(a\mathscr{H}^+) = \pi_2(a\mathscr{Y}) = \alpha$.

We define a mapping, $\phi : S \to A$ by $\phi(a) = (a\mathscr{H}^+, a\mathscr{Y})$ for all $a \in S$. Clearly, ϕ is a semiring homomorphism.

Let $a, b \in S$ such that $\phi(a) = \phi(b)$. This implies $(a\mathscr{H}^+, a\mathscr{Y}) = (b\mathscr{H}^+, b\mathscr{Y})$, i.e., $a\,\mathscr{H}^+\,b$ and $a\,\mathscr{Y}\,b$, i.e., $a^0 = b^0$ and $a = a^0 + b + a^0, b = b^0 + a + b^0$. Therefore, $a = a^0 + b + a^0 = b^0 + b + b^0 = b$ and hence ϕ is injective.

To show ϕ is surjective, let $b, c \in S$ such that $(b\mathscr{H}^+, c\mathscr{Y}) \in A$. Then, $\pi_1(b\mathscr{H}^+) = \pi_2(c\mathscr{Y}) = \alpha$, say, so that $b, c \in S_\alpha$. Hence, $b\,\mathscr{D}^+\,c$. Now, by Theorem 3.4, $b\,(\mathscr{H}^+\,\mathscr{Y})\,c$. This implies $b\,\mathscr{H}^+\,a\,\mathscr{Y}\,c$ for some $a \in S$, i.e., $b\mathscr{H}^+ = a\mathscr{H}^+$ and $a\mathscr{Y} = c\mathscr{Y}$.

Hence, $\phi(a) = (a\mathscr{H}^+, a\mathscr{Y}) = (b\mathscr{H}^+, c\mathscr{Y})$, which implies that ϕ is surjective. Consequently, ϕ is an isomorphism.

$(ii) \Rightarrow (iii)$: Let S be a spined product of an idempotent semiring I and a b-lattice of skew-rings T. Since every idempotent semiring and every b-lattice of skew-rings satisfies the identity $x^0 + y^0 = (x + y)^0$ and therefore so does S.

$(iii) \Rightarrow (i)$: If S satisfies the identity $a^0 + b^0 = (a + b)^0$, then for any two elements $e, f \in E^+(S)$, we have $e^0 + f^0 = (e + f)^0$, i.e., $e + f = (e + f)^0 \in E^+(S)$. Hence S is orthodox. □

Corollary 3.6. *Let S be an orthodox completely regular semiring. Then $\mathscr{H}^+ \cap \mathscr{Y} = \epsilon$, where ϵ is the equality relation on S.*

4. The interval which $\mathscr{Y}^*$ belongs to

So far we have discussed the nature and properties of the relation $\mathscr{Y}$ on a special kind of completely regular semirings. In the following section, we try to describe

$\mathscr{Y}^*$ on completely regular semirings without any other special conditions.

Following [6, Theorem 3.1] we describe the structure of completely simple semiring.

Let R be a skew-ring, $(I,\cdot)$ and $(\Lambda,\cdot)$ are bands such that $I \cap \Lambda = \{o\}$ and $P = (p_{\lambda,i})$ be a matrix over R, $i \in I, \lambda \in \Lambda$ under the assumptions

(i) $p_{\lambda,o} = p_{o,i} = 0$,

(ii) $p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj}$,

(iii) $p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk}$,

(iv) $ap_{\lambda,i} = p_{\lambda,i}a = 0$,

(v) $ab + p_{o\mu,io} = p_{o\mu,io} + ab$,

(vi) $ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab$, for all $i,j,k \in I, \lambda,\mu,\nu \in \Lambda$ and $a,b \in R$.

On $S = I \times R \times \Lambda$, we define '+' and '$\cdot$' by

$$(i,a,\lambda) + (j,b,\mu) = (i, a + p_{\lambda,j} + b, \mu)$$

and

$$(i,a,\lambda) \cdot (j,b,\mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu).$$

Then $(S,+,\cdot)$ is a semiring which is called a *Rees matrix semiring* and is denoted by $\mathscr{M}(I,R,\Lambda;P)$. The authors in [6] proved (Theorem 3.1) that a semiring S is a completely simple semiring if and only if S is isomorphic to a Rees matrix semiring.

Next, we give a description of least skew-ring congruence to determine the interval of $\mathscr{Y}^*$ on completely regular semirings.

Lemma 4.1. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and S_α $(\alpha \in Y)$ is a completely simple semiring and $a,b \in S_\alpha$. Then the following statements are equivalent.*

(i) $a \mathscr{Y} b$,

(ii) *$a = e+b+f$ and $b = g+a+h$ for any $e,f,g,h \in E^+(S)$ with $e \mathscr{R}^+ a \mathscr{L}^+ f$ and $g \mathscr{R}^+ b \mathscr{L}^+ h$,*

(iii) *$a = (a+x)^0 + b + (x+a)^0$ and $b = (b+x)^0 + a + (x+b)^0$ for any $x \in S_\alpha$.*

Proof. $(i) \Rightarrow (ii)$: Let $a = (i,s,\lambda)$, $b = (j,t,\mu) \in S_\alpha$ and $a \mathscr{Y} b$. For any $e = (i, -p_{\delta,i}, \delta)$, $f = (k, -p_{\lambda,k}, \lambda) \in E^+(S_\alpha)$, we have $e \mathscr{R}^+ a \mathscr{L}^+ f$.

Let $c = (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) \in S_\alpha$. Then

$$\begin{aligned} a + c + a &= (i,s,\lambda) + (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) + (i,s,\lambda) \\ &= (i, s + p_{\lambda,k} - p_{\lambda,k} - s - p_{\delta,i} + p_{\delta,i} + s, \lambda) \\ &= (i,s,\lambda) = a. \end{aligned}$$

Since, S_α is a completely simple semiring, we have $c + a + c = c$. This implies, $c \in V^+(a)$.

Since $a \mathscr{Y} b$, we have $c \in V^+(b)$. Hence $b + c + b = b$, i.e., $(j,t,\mu) + (k, -p_{\lambda,k} - s - p_{\delta,i}, \delta) + (j,t,\mu) = (j, t + p_{\mu,k} - p_{\lambda,k} - s - p_{\delta,i} + p_{\delta,j} + t, \mu) = (j,t,\mu)$.

So we get, $t = -p_{\delta,j} + p_{\delta,i} + s + p_{\lambda,k} - p_{\mu,k}$. Then

$$\begin{aligned} e+b+f &= (i,-p_{\delta,i},\delta)+(j,-p_{\delta,j}+p_{\delta,i}+s+p_{\lambda,k}-p_{\mu,k},\mu)+(k,-p_{\lambda,k},\lambda)\\ &= (i,-p_{\delta,i}+p_{\delta,j}-p_{\delta,j}+p_{\delta,i}+s+p_{\lambda,k}-p_{\mu,k}+p_{\mu,k}-p_{\lambda,k},\lambda)\\ &= (i,s,\lambda)=a. \end{aligned}$$

Similarly, we can prove for any $g,h\in E^+(S_\alpha)$ with $g\,\mathscr{R}^+\,b\,\mathscr{L}^+\,h$, $b=g+a+h$.

$(ii)\Rightarrow(iii)$: For $x,a,b\in S_\alpha$, by Lemma 2.4(i), we have $(a+x)\,\mathscr{R}^+\,a\,\mathscr{L}^+\,(x+a)$ and $(b+x)\,\mathscr{R}^+\,b\,\mathscr{L}^+\,(x+b)$. This implies $(a+x)^0\,\mathscr{R}^+\,a\,\mathscr{L}^+\,(x+a)^0$ and $(b+x)^0\,\mathscr{R}^+\,b\,\mathscr{L}^+\,(x+b)^0$. Hence by (ii), $a=(a+x)^0+b+(x+a)^0$ and $b=(b+x)^0+a+(x+b)^0$.

$(iii)\Rightarrow(i)$: Let $c\in V^+(a)$ for $a,b\in S_\alpha$. Then, $c\in S_\alpha$, $(a+c),(c+a)\in E^+(S_\alpha)$. By (iii),

$$\begin{aligned} b &= (b+c)^0+a+(c+b)^0\\ &= (b+c)^0+a+c+a+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+(c+a)^0+c+(a+c)^0+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+(c+a)+c+(a+c)+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+(a+c)^0+b+c+b+(c+a)^0+(c+b)^0\\ &= (b+c)^0+b+c+b+(c+b)^0 \quad \text{[by Lemma 2.4 and Lemma 3.1]}\\ &= b+c+b. \end{aligned}$$

This implies $c\in V^+(b)$ and hence $V^+(a)\subseteq V^+(b)$. By symmetry, we get $V^+(a)=V^+(b)$. This completes the proof. □

Following [6, Definition 5.1] a normal subgroup N of $(R,+)$ (where R is a skew-ring) is said to be a *skew-ideal* of R if $a\in N$ implies $ca, ac\in N$ for all $c\in R$.

Notation 4.2. Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a Rees matrix semiring over a skew-ring R. Let $\langle P\rangle$ denote the smallest skew-ideal of R generated by the elements of P.

Lemma 4.3. *Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a completely simple semiring. Define a relation σ on S as: for all $a,b\in S$;*

$$a\,\sigma\,b \quad \text{if and only if} \quad (g-h)\in\langle P\rangle,$$

where $a=(i,g,\lambda)$, $b=(j,h,\mu)\in S$. Then σ is the least skew-ring congruence on S.

Proof. The relation σ is obviously reflexive and symmetric.

Let $a\,\sigma\,b$ and $b\,\sigma\,c$ where $a,b,c\in S$. Also, let $a=(i,g,\lambda)$, $b=(j,h,\mu)$ and $c=(k,t,\delta)\in S$. Then $(g-h)\in\langle P\rangle$ and $(h-t)\in\langle P\rangle$. This implies $(g-t)\in\langle P\rangle$. Hence $a\,\sigma\,c$. Thus, σ is transitive and hence σ is an equivalence relation on S.

Next we prove that σ is compatible with respect to the operations in S. Let $a,b\in S$ such that $a\,\sigma\,b$. Then we have, $(g-h)\in\langle P\rangle$, where $a=(i,g,\lambda)$, $b=(j,h,\mu)\in S$. Let $c=(k,t,\delta)\in S$ be arbitrary. Therefore, $a+c=(i,g,\lambda)+(k,t,\delta)=(i,g+p_{\lambda,k}+t,\delta)$. Similarly, $b+c=(j,h,\mu)+(k,t,\delta)=(j,h+p_{\mu,k}+t,\delta)$.

Now, $(g+p_{\lambda,k}+t)-(h+p_{\mu,k}+t)=g+p_{\lambda,k}-p_{\mu,k}-h$. Again, $(g-h)\in\langle P\rangle$ implies $-h+g\in\langle P\rangle$, i.e., $p_{\lambda,k}-p_{\mu,k}-h+g+p_{\mu,k}-p_{\lambda,k}\in\langle P\rangle$, i.e., $g+p_{\lambda,k}-p_{\mu,k}-h+g+p_{\mu,k}-p_{\lambda,k}-g\in\langle P\rangle$. Also, $g+p_{\mu,k}-p_{\lambda,k}-g\in\langle P\rangle$.

Thus, $g+p_{\lambda,k}-p_{\mu,k}-h \in \langle P\rangle$. Hence, $(a+c)\,\sigma\,(b+c)$. Similarly, it can be shown that $(c+a)\,\sigma\,(c+b)$.

Again, $ac = (i,g,\lambda)(k,t,\delta) = (ik, -p_{\lambda\delta,ik}+gt, \lambda\delta)$ and $bc = (jk, -p_{\mu\delta,jk}+ht, \mu\delta)$. Now, $(g-h) \in \langle P\rangle$ implies $(gt-ht) \in \langle P\rangle$, i.e., $-p_{\lambda\delta,ik}+gt-ht+p_{\lambda\delta,ik} \in \langle P\rangle$, i.e., $-p_{\lambda\delta,ik}+gt-ht+p_{\mu\delta,jk}-p_{\mu\delta,jk}+p_{\lambda\delta,ik} \in \langle P\rangle$. Since, $-p_{\mu\delta,jk}+p_{\lambda\delta,ik} \in \langle P\rangle$ it follows that $-p_{\lambda\delta,ik}+gt-ht+p_{\mu\delta,jk} \in \langle P\rangle$. Therefore, $(ac)\,\sigma\,(bc)$. Similarly, $(ca)\,\sigma\,(cb)$. Consequently, σ is a congruence on $(S,+,\cdot)$.

Next we show that σ is a skew-ring congruence on S. If we can show that there is a unique additive idempotent in S/σ, then we are done. For this it is enough to prove that all additive idempotents of S are σ related.

Let $e,f \in E^+(S)$. Then $e=(i,-p_{\lambda,i},\lambda)$ and $f=(j,-p_{\mu,j},\mu)$. Now, $-p_{\lambda,i}+p_{\mu,j} \in \langle P\rangle$ implies that $e\,\sigma\,f$. This proves that σ is a skew-ring congruence on S.

At last, we prove that σ is the least skew-ring congruence on S. For this let ξ be any skew-ring congruence on S. Then both σ and ξ are group congruences on $(S,+)$. Moreover, by [1, Lemma 2.3], it follows that σ is the least group congruence on $(S,+)$. Thus, we must have $\sigma \subseteq \xi$. Consequently, σ is the least skew-ring congruence on S. This completes the proof. □

Lemma 4.4. *Let $S=(Y;S_\alpha)$ be a completely regular semiring where Y is a b-lattice and $S_\alpha\,(\alpha \in Y)$ is a completely simple semiring. If ν is the least b-lattice of skew-rings congruence on S, then $\mathscr{Y}^* \subseteq \nu$.*

Proof. Let $a,b \in S$ and $a\mathscr{Y}b$. Then there exists some $\alpha \in Y$ such that $a,b \in S_\alpha$. Let $a=(i,g,\lambda)$, $b=(j,h,\mu)$. By Lemma 4.1, we get

$$(i,g,\lambda) = (i,-p_{\delta,i},\delta)+(j,h,\mu)+(k,-p_{\lambda.k},\lambda),$$

since $(i,-p_{\delta,i},\delta)\,\mathcal{R}^+\,(i,g,\lambda)\,\mathcal{L}^+\,(k,-p_{\lambda,k},\lambda)$.

It follows that $g=-p_{\delta,i}+p_{\delta,j}+h+p_{\mu,k}-p_{\lambda,k}$ whence $g+p_{\lambda,k}-p_{\mu,k}-h-p_{\delta,j}+p_{\delta,i}=0$, where 0 is the zero of R. Taking $k=\delta=o$, we have $(g-h)=0 \in \langle P\rangle$. Then by Lemma 4.3, it follows that $a\,\sigma_\alpha\,b$, where σ_α is the least skew-ring congruence on S_α. Hence $\mathscr{Y}|_{S_\alpha} \subseteq \sigma_\alpha$ for all $\alpha \in Y$.

Let $\nu|_{S_\alpha}=\nu_\alpha$. Then $\nu=\bigcup_{\alpha\in Y}\nu_\alpha$. Since S_α/ν_α is a skew-ring, it follows that $\sigma_\alpha \subseteq \nu_\alpha$ for all $\alpha \in Y$. Therefore, $\mathscr{Y}|_{S_\alpha} \subseteq \sigma_\alpha \subseteq \nu_\alpha$ for all $\alpha \in Y$ and hence $\mathscr{Y}^* \subseteq \nu$. □

Definition 4.5. A congruence ξ on a semiring S is said to be an *additive idempotent pure congruence* if $a\,\xi\,e$ with $a \in S$ and $e \in E^+(S)$ implies that $a \in E^+(S)$.

Theorem 4.6. *Let $S=\mathscr{M}(I,R,\Lambda;P)$ be a completely simple semiring. Then $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S.*

Proof. Clearly, $\mathscr{Y}$ is an equivalence relation. Let $a,b \in S$ and $a\,\mathscr{Y}\,b$. By Lemma 4.1, for any $x,c \in S$, $a=(a+x+c)^0+b+(x+c+a)^0$ and $b=(b+x+c)^0+a+(x+c+b)^0$.

Hence, $c+a = c+(a+x+c)^0+b+(x+c+a)^0 = (c+a+x)^0+c+b+(x+c+a)^0$, by Lemma 2.4 (ii). Similarly, $c+b = (c+b+x)^0+(c+a)+(x+c+b))^0$. This implies $(c+a)\,\mathscr{Y}\,(c+b)$. Dually, it follows that $(a+c)\,\mathscr{Y}\,(b+c)$.

We now show that $(ac)\,\mathscr{Y}\,(bc)$. Let $a,b,c \in S$ and $a\,\mathscr{Y}\,b$. Then there exists some $\alpha \in Y$ such that $a,b \in S_\alpha$. Let $a=(i,x,\lambda)$, $b=(j,y,\mu)$ and $c=(k,z,\nu)$.

By Lemma 4.1, $a = e_1+b+f_1$ for all $e_1, f_1 \in E^+(S)$ with $e_1\,\mathscr{R}^+\,a\,\mathscr{L}^+\,f_1$, i.e., $(i,x,\lambda) = (i,-p_{t,i},t)+(j,y,\mu)+(s,-p_{\lambda,s},\lambda)$, for all $t\in\Lambda$ and for all $s\in I$, i.e., $(i,x,\lambda) = (i,-p_{t,i}+p_{t,j}+y+p_{\mu,s}-p_{\lambda,s},\lambda)$, for all $t\in\Lambda$ and for all $s\in I$, i.e., $x = -p_{t,i}+p_{t,j}+y+p_{\mu,s}-p_{\lambda,s}$ for all $t\in\Lambda$ and for all $s\in I$. ...(1)

We also note that $xz = yz$ for any $z\in S$. ...(2)

Again, $(i,x,\lambda)(k,z,\nu) = (i,-p_{t,i},t)(k,z,\nu)+(j,y,\mu)(k,z,\nu)+(s,-p_{\lambda,s},\lambda)(k,z,\nu)$,
i.e., $(ik,-p_{\lambda\nu,ik}+xz,\lambda\nu) = (ik,-p_{t\nu,ik},t\nu)+(jk,-p_{\mu\nu,jk}+yz,\mu\nu)+(sk,-p_{\lambda\nu,sk},\lambda\nu)$,
i.e., $(ik,-p_{\lambda\nu,ik}+xz,\lambda\nu) = (ik,-p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,sk}-p_{\lambda\nu,sk},\lambda\nu)$,
i.e., $-p_{\lambda\nu,ik}+xz = -p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,sk}-p_{\lambda\nu,sk}$,
i.e., $-p_{\lambda\nu,ik}+xz = -p_{t\nu,ik}+p_{t\nu,jk}-p_{\mu\nu,jk}+p_{\mu\nu,sk}-p_{\lambda\nu,sk}+yz$. ...(3)

Now, let $e=(ik,-p_{\delta,ik},\delta)$, $f=(l,-p_{\lambda\nu,l},\lambda\nu)\in E^+(S)$. Then $e\,\mathscr{R}^+\,(ac)\,\mathscr{L}^+\,f$.
Now,

$$\begin{aligned} e+bc+f &= (ik,-p_{\delta,ik},\delta)+(jk,-p_{\mu\nu,jk}+yz,\mu\nu)+(l,-p_{\lambda\nu,l},\lambda\nu)\\ &= ik,-p_{\delta,ik}+p_{\delta,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,l}-p_{\lambda\nu,l},\lambda\nu)\\ &= (ik,-p_{\delta\nu,ik}+p_{\delta\nu,i}-p_{\delta,i}+p_{\delta,j}-p_{\delta\nu,j}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+\\ &\qquad yz+p_{\mu\nu,lk}-p_{\mu,lk}+p_{\mu,l}-p_{\lambda,l}+p_{\lambda,lk}-p_{\lambda\nu,lk},\lambda\nu) \end{aligned}$$

i.e., $e+bc+f = (ik,-p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk},\lambda\nu)$, ...(4)

[By putting once $t=\delta$ and $t=\delta\nu$ and equating in (1) and again by putting $s=l$ and $s=lk$ and equating in (1) we obatin (4)]

Now, by substituting $t=\delta$ and $s=l$ in (3) we can obtain

$$\begin{aligned} -p_{\lambda\nu,ik}+xz &= -p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+p_{\mu\nu,lk}-p_{\lambda\nu,lk}+yz\\ &= -p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk}. \end{aligned}$$

Therefore,

$$\begin{aligned} e+bc+f &= (ik,-p_{\delta\nu,ik}+p_{\delta\nu,jk}-p_{\mu\nu,jk}+yz+p_{\mu\nu,lk}-p_{\lambda\nu,lk},\lambda\nu)\\ &= (ik,-p_{\lambda\nu,ik}+xz,\lambda\nu)\\ &= ac. \end{aligned}$$

Thus, we see that $ac = e+bc+f$ for any $e,f\in E^+(S)$ with $e\,\mathscr{R}^+\,(ac)\,\mathscr{L}^+\,f$. Similarly, we can show that $bc = g+ac+h$ for any $g,h\in E^+(S)$ with $g\,\mathscr{R}^+\,(bc)\,\mathscr{L}^+\,h$. Consequently, $\mathscr{Y}$ is a congruence on the semiring S.

Next we show that $\mathscr{Y}$ is an additive idempotent pure congruence on S. Let $a\in S$ with $a=(i,g,\lambda)\in S$, $e=(k,-p_{\lambda,k},\lambda)\in E^+(S)$ and $a\,\mathscr{Y}\,e$. Then $V^+(a) = V^+(e)$. By Lemma 4.1, for $f=(i,-p_{\lambda,i},\lambda)$, $h=(k,-p_{\lambda,k},\lambda)\in E^+(S)$ with $f\,\mathcal{R}^+\,a\,\mathcal{L}^+\,h$, we have $a = f+e+h = (i,-p_{\lambda,i},\lambda)+(k,-p_{\lambda,k},\lambda)+(k,-p_{\lambda,k},\lambda) = (i,-p_{\lambda,i},\lambda)\in E^+(S)$. Thus $\mathscr{Y}$ is an additive idempotent pure congruence on S.

Let η be any additive idempotent pure congruence on S. Let $a,b\in S$ such that $a\,\eta\,b$. Then by [1, Theorem 2.5], it follows that $a\,\mathscr{Y}\,b$. Hence, $\eta\subseteq\mathscr{Y}$, which proves that $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S. $\square$

Theorem 4.7. *Let $S = (Y; S_\alpha)$ be a completely regular semiring, where Y is a b-lattice and $S_\alpha\,(\alpha \in Y)$ is a completely simple semiring. Then $\mathscr{Y}^* = \epsilon$ on S if and only if for each $\alpha \in Y$, ϵ_α is the unique additive idempotent pure congruence on S_α, where ϵ is the trivial congruence.*

Proof. First suppose that for each $\alpha \in Y$, ϵ_α is the unique additive idempotent pure congruence on S_α. Since $\mathscr{Y}$ is the greatest additive idempotent pure congruence on S, it follows that $\mathscr{Y}|_{S_\alpha} = \epsilon_\alpha$ on S_α. Hence $\mathscr{Y}^* = \epsilon$.

Conversely, let $\mathscr{Y}^* = \epsilon$. Now since $\mathscr{Y} \subseteq \mathscr{Y}^* = \epsilon$ and $\mathscr{Y}$ is reflexive on S, it follows that $\mathscr{Y} = \epsilon$ on S. This implies $\mathscr{Y}|_{S_\alpha} = \epsilon_\alpha$ and hence by Theorem 4.6, it follows that ϵ_α is the unique additive idempotent pure congruence on S_α for each $\alpha \in Y$. □

Combining Theorem 3.4, Lemma 4.4 and Theorem 4.7 we get the following result.

Theorem 4.8. *Let S be a completely regular semiring. Then $\mathscr{Y}^* \in [\epsilon, \nu]$, where ϵ is the equality congruence and ν is the least b-lattice of skew-ring congruence on S.*

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