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A representation theorem for bounded distributive hyperlattices

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Abstract. A representation theorem for bounded distributive hyperlattices is given. The equivalence between the category of Priestley spaces and the dual of the category of bounded distributive hyperlattices is established.

1. Introduction

The notion of hyperstructures was introduced 80 years ago [6], it has been studied by several authors see for example [1, 4, 5, 10, 11, 12], this bibliography and the references therein is not exhaustive.

Later, Koguep *et al.* [4], Konstantinidou [5] introduced respectively the notion of hyperlattices and studied ideals and filters in these structures. Prime ideals and prime filters in hyperlattices have been examined by R. Ameri *et al.* [1]. Rasouli and Davvaz defined a fundamental relation on a hyperlattice to get a lattice from a hyperlattice. Moreover, they defined a topology on the set of prime ideals of a distributive hyperlattice [11, 12].

The Stone's representation theorems [13, 14] proved that every Boolean algebra is isomorphic to a set of $\{I_a : a \in A\}$ (where I_a denotes the set of prime ideals of A not containing a). Since then, representation theorems for distributive lattices has known a vast development.

H. A. Priestley developed another kind of duality for bounded distributive lattices [8, 9]. Such representation theorems enable a deep and a concrete comprehension of the lattices as well as their structures. Our motivation finds its place in the following opinion:

"Stone's duality and its variants are central in making the link between syntactical and semantic approaches to logic. Also in theoretical computer science, this link is central as the two sides correspond to specification languages and the space of computational states. This ability to translate faithfully between algebraic specification and spatial dynamics has often proved itself to be a powerful theoretical tool as well as a handle for making practical problems decidable" see [3].

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In this paper, we extend some results of [8, 9], where a representation theorem of bounded distributive hyperlattices is presented. In other words, the category of Priestley spaces is equivalent to the dual of the category of bounded distributive hyperlattices.

2. Preliminaries

Let X be a nonempty set and $P^*(X)$ denotes the set of all nonempty subsets of X. Maps $f : X \times X \to P^*(X)$, are called *hyperoperations* [6].

Definition 2.1. Let L be a nonempty set, $\wedge$ be a binary operation and $\sqcup$ be a hyperoperation on L. L is called a *hyperlattice* if for all $a, b, c \in L$ the following conditions hold:

(i) $a \in a \sqcup a$, and $a \wedge a = a$;

(ii) $a \sqcup b = b \sqcup a$, and $a \wedge b = b \wedge a$;

(iii) $a \in [a \wedge (a \sqcup b)] \cap [a \sqcup (a \wedge b)]$;

(iv) $a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c$, and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(v) $a \in a \sqcup b \Rightarrow a \wedge b = b$.

A hyperlattice L with the property

$$a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$$

is called *distributive*, where for all nonempty subsets A and B of L we define

$$A \sqcup B = \cup\{a \sqcup b \,|\, a \in A, b \in B\} \quad \text{and} \quad A \wedge B = \{a \wedge b \,|\, a \in A, b \in B\}.$$

The converse of condition (v) in Definition 2.1 is true. Indeed using (iii) in Definition 2.1, we obtain $a \in a \sqcup b$ by taking $b = a \wedge b$.

Hence, we can define a partial order on L by:

$$a \leq b \Leftrightarrow b \in a \sqcup b \Leftrightarrow a \wedge b = a.$$

A hyperlattice L is called *bounded* if there exist $0, 1 \in L$ such that for all $a \in L$, $0 \leq a \leq 1$.

Consider a lattice $(L, \wedge, \vee)$. We define the Nakano hyperoperation $\sqcup$ on L by $x \sqcup y = \{z \in L/z \vee x = z \vee y = x \vee y\}$, for all $x, y \in L$. To the best of our knowledge, the $\sqcup$ hyperoperation was first introduced by Nakano in [7], which is an investigation of hyperrings.

Lemma 2.2. *If $(L, \wedge, \vee)$ is a distributive lattice, then $(L, \wedge, \sqcup)$ is a distributive hyperlattice where $a \sqcup b = \{x \in L \,|\, a \vee b = a \vee x = b \vee x\}$ for all $a, b \in L$.*

Proof. Straightforward. $\square$

Lemma 2.3. *Let* $L = \{0, 1\}$*. Then,* $(\{0, 1\}, \wedge, \sqcup)$ *is a bounded distributive hyperlattice, where*

$\wedge$	0	1
0	0	0
1	0	1

and

$\sqcup$	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{0,1\}$

.

Definition 2.4. [10] A nonempty subset I of a hyperlattice L is called an *ideal* if the following conditions hold

(i) If $a, b \in I$, then $a \sqcup b \subseteq I$;

(ii) If $a \in I$, $b \leq a$, and $b \in L$, then $b \in I$.

A proper ideal I is called *prime* if $a \wedge b \in I$ implies $a \in I$ or $b \in I$ for all $a, b \in L$.

Definition 2.5. [10] A nonempty subset F of a hyperlattice L is called a *filter* if the following conditions hold

(i) If $a, b \in F$, then $a \wedge b \in F$;

(ii) If $a \in F$, $a \leq b$, and $b \in L$, then $b \in F$.

A proper filter F is called *prime* if for all $a, b \in L$ $(a \sqcup b) \cap F \neq \emptyset$ implies $a \in F$ or $b \in F$.

Theorem 2.6. [1] *If P is a prime ideal of a hyperlattice L, then $L - P$ is a prime filter of L. Similarly, if F is a prime filter of L, then $L - F$ is a prime ideal of L.*

Proposition 2.7. *If δ is a nonempty subset of a hyperlattice L, then the smallest filter containing δ has the form*

$$\langle \delta \rangle = \{x \in L \,|\, a_1 \wedge ... \wedge a_n \leq x, \ \textit{for some } a_1, .., a_n \in \delta\}.$$

Proof. First, we prove that $\langle \delta \rangle$ is nonempty. Let $a \in \delta$, since $a \leq a$, then $a \in \langle \delta \rangle$, hence $\langle \delta \rangle \neq \emptyset$. To proof that $\langle \delta \rangle$ is a filter let $x \in \langle \delta \rangle$, $y \in X$ such that $x \leq y$, then $\wedge_{i=1}^{n} a_i \leq x \leq y$, so $y \in F$.

On the other hand, for $x, y \in \langle \delta \rangle$, there exist $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m$ such that $\wedge_{i=1}^{n} a_i \leq x$ and $\wedge_{j=1}^{m} b_j \leq y$. Then, $(\wedge_{i=1}^{n} a_i) \wedge (\wedge_{j=1}^{m} b_j) \leq x \wedge y$. Therefore, $x \wedge y \in \langle \delta \rangle$.

Next, let $a \in \delta$, since $a \leq a$, we have $a \in \langle \delta \rangle$. Then $\delta \subseteq \langle \delta \rangle$.

Finally, suppose that F is a filter with $\delta \subseteq F$. Then for any $x \in \langle \delta \rangle$, then there exist $a_1, a_2, .., a_n \in \delta$ such that $\wedge_{i=1}^{n} a_i \leq x$, then $x \in F$. Therefore $\langle \delta \rangle \subseteq F$. $\square$

If $\delta = \{a\}$, we write $\langle \delta \rangle = \uparrow a = \{x \in L \,|\, a \leq x\}$.

Proposition 2.8. [10] *Let $(L, \wedge, \sqcup)$ be a distributive hyperlattice. If $a \in L$ then $\downarrow a = \{x \in L \,|\, x \leq a\}$ is an ideal.*

Theorem 2.9. *Let X be a distributive hyperlattice, F be a filter and I an ideal of X. If $F \cap I = \emptyset$, then there is a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let $\mathcal{G}$ be the family of those filters F' which satisfy $F \subseteq F'$ and $F' \cap I = \emptyset$. It follows from the Zorn's lemma that $\mathcal{G}$ has a maximal element P. Since $P \in \mathcal{G}$ it remains to prove that the filter P is prime. Since $P \cap I = \emptyset$, P is proper. Suppose P is not prime. Then there exist $a, b \in X$ such that $(a \sqcup b) \cap P \neq \emptyset$, and $a \notin P$ and $b \notin P$. Let $a_0 \in (a \sqcup b) \cap P$ and let $\delta = P \cup \{a\}$. Then $\langle \delta \rangle \cap I \neq \emptyset$, otherwise $P \subseteq \langle \delta \rangle \in G$ contradicting the maximality of P. Take $x \in \langle \delta \rangle \cap I$. This implies easily there exists $p \in P$ that $p \wedge a \leq x$, it follows that $a_0 \wedge p \wedge a \leq x$ and since $x \in I$, it follows that $a_0 \wedge p \wedge a \in I$. Similarly $a_0 \wedge q \wedge b \in I$. Then, $a_0 \wedge m \wedge a \in I$ and $a_0 \wedge m \wedge b \in I$ such that $m = p \wedge q$, it follows that $(a_0 \wedge m \wedge a) \sqcup (a_0 \wedge m \wedge b) \subseteq I$, which implies $a_0 \wedge m \in (a_0 \wedge m) \wedge (a \sqcup b) \subseteq I$, and $a_0 \wedge m \in P$, therefore $I \cap P \neq \emptyset$, which is a contradiction. □

Corollary 2.10. *Let L be a distributive hyperlattice. If I is an ideal and $a \in L - I$, then there exists a prime filter P such that $a \in P$ and $P \cap I = \emptyset$.*

Proof. Let I be an ideal, $a \in L - I$ and take $F = \langle a \rangle$, it follows $F \cap I = \emptyset$. By Theorem 2.9, there is a prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$. □

Definition 2.11. Let L and L' be two hyperlattices and $f : L \to L'$ be a mapping.

1. f is said to be a *hyperlattices homomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \sqcup y) \subseteq f(x) \sqcup f(y)$, for all $x, y \in L$.

2. f is said to be a *strong homomorphism* of a hyperlattices, if $f(x \wedge y) = f(x) \wedge f(y)$ and $f(x \sqcup y) = f(x) \sqcup f(y)$, for all $x, y \in L$. If f is a bijection, then f is said to be a *hyperlattices isomorphism* (*strong isomorphism*).

Proposition 2.12. *Let $(L, \wedge, \sqcup)$ be a hyperlattice, $(\{0,1\}, \wedge, \sqcup)$ be the hyperlattice in Lemma 2.3 and F be a subset of L. If F is a prime filter, then there is a surjective hyperlattices homomorphism $f : L \to \{0,1\}$, such that $F = f^{-1}(\{1\})$.*

Proof. Set $f(X) = 1$ if $X \subseteq F$, and $f(X) = 0$ otherwise. Since $(x \sqcup y) \cap F \neq \emptyset \Leftrightarrow (x \in F$ or $y \in F)$, then $f(x \sqcup y) = 1 \Rightarrow x \sqcup y \subseteq F \Rightarrow (x \sqcup y) \cap F \neq \emptyset \Rightarrow x \in F$ or $y \in F \Rightarrow f(x) = 1$ or $f(y) = 1$. Hence, $f(x \sqcup y) \subseteq f(x) \sqcup f(y)$.

If $f(x \sqcup y) = 0$, we have $x \sqcup y \nsubseteq F$, it follows that $(x \sqcup y) \cap F = \emptyset$, which implies $x \notin F$ and $y \notin F$, it follows that $f(x) = 0$ and $f(y) = 0$, which implies $f(x) \sqcup f(y) = 0$. Therefore, $f(x \sqcup y) \subseteq f(x) \sqcup f(y)$.

For the second homomorphism axiom, we have $f(x \wedge y) = 0 \Leftrightarrow x \wedge y \subseteq L - F \Leftrightarrow (x \in L - F$ or $y \in L - F) \Leftrightarrow (f(x) = 0$ or $f(y) = 0) \Leftrightarrow f(x) \wedge f(y) = 0$. Hence, $f(x \wedge y) = f(x) \wedge f(y)$. □

Corollary 2.13. *Let L be a distributive hyperlattice. If $a, b \in X$ are such that $a \nleq b$ there is a prime filter F such that $a \in F$ and $b \notin F$.*

Proof. Take $I = \downarrow b$ in Corollary 2.10. □

3. Priestley duality

Definition 3.1. Let $(L, \leq)$ be a poset. A subset $E \subseteq L$ is said to be *increasing* (*decreasing*) if $\forall x, y \in L$, $x \in E$ and $x \leq y \quad (y \leq x)$ implies $y \in E$.

Definition 3.2. An *ordered topological space* is a triple $(X, \tau, \leq)$ such that (X, τ) is a topological space and $(X, \leq)$ is a poset. A *clopen set* in a topological space is a set which is both open and closed. The ordered topological space is said to be *totally disconnected* if for every $x, y \in X$ such that $x \not\leq y$ there exists an increasing τ-clopen U and a decreasing τ-clopen V such that $U \cap V = \emptyset$ with $x \in U$ and $y \in V$.

Definition 3.3. *A Priestley space is a compact totally disconnected ordered topological space.*

If A is a bounded distributive hyperlattice, then its *dual space* is defined to be $T(A) = (X, \tau, \leq)$, where X is the set of homomorphisms from A onto $(\{0,1\}, \wedge, \sqcup)$, preserving 0 and 1, τ is the product topology induced from $\{0,1\}^A$, and $\leq$ is the partial order defined by $f \leq g$ in X if and only if $f(a) \leq g(a)$ for all $a \in A$. $T(A)$ is compact, and it is also totally order disconnected, i.e., a Priestley space.

Definition 3.4. [2] Let $(X, \tau, \leq)$, and $(X', \tau', \preceq)$ be two Priestley spaces. Then $f : X \to X'$ is called

1. *increasing* if for all $x, y \in X$, $x \leq y \Rightarrow f(x) \preceq f(y)$.
2. a *Priestley spaces homomorphism* if is increasing and continuous. If it is a bijection, then it is a *Priestley spaces isomorphism*.

Lemma 3.5. *If $\delta = (X, \tau, \leq)$ is a Priestley space, then there exists a hyperoperation $\sqcup$ such that $(L(\delta), \cap, \sqcup, \emptyset, X)$ is a bounded distributive hyperlattice, where $L(\delta) = \{Y \subseteq X \mid Y$ is increasing and τ-clopen$\}$ and $\sqcup$ is defined by*

$$A \sqcup B = \{X \in L(\delta) \mid A \cup B = A \cup X = B \cup X\}$$

for all $A, B \in L(\delta)$.

Proof. By Lemma 2.2. □

Lemma 3.6. *Let A be a bounded distributive hyperlattice. Then $F_A \colon A \to L(T(A))$ defined by $F_A(a) = \{f \in X \mid f(a) = 1\}$ is a hyperlattices isomorphism.*

Proof. For all $a, b \in A$ we have

$$\begin{aligned} F_A(a \wedge b) &= \{f \in X \mid f(a \wedge b) = 1\} = \{f \in X \mid f(a) = 1\} \cap \{f \in X \mid f(b) = 1\} \\ &= F_A(a) \sqcap F_A(b), \end{aligned}$$

and

$$\begin{aligned} F_A(a \sqcup b) &= \{F_A(t) \mid t \in a \sqcup b\} = \{\{f \in X \mid f(a \sqcup b) = 1\}\} \\ &\subseteq \{\{f \in X \mid f(a) = 1\} \cup \{f \in X \mid f(b) = 1\}\} = \{F_A(a) \cup F_A(b)\} \\ &\subseteq F_A(a) \sqcup F_A(b). \end{aligned}$$

Suppose that $a \neq b$. If $a \nleq b$, there exist a prime filter F such that $a \in F$ and $b \notin F$ (Corollary 2.13). Thus, by Proposition 2.12, there is a hyperlattices homomorphism $f\colon A \to \{0,1\}$ such that $a \in f^{-1}(\{1\})$ and $b \notin f^{-1}(\{1\})$, hence $f(a) = 1$ and $f(b) = 0$, i.e., $F_A(a) \nleq F_A(b)$.

Similarly, $b \nleq a$ gives $F_A(b) \nleq F_A(a)$. Hence, $a \neq b$ implies $F_A(a) \neq F_A(b)$ i.e., F_A is injective.

To prove that F_A is surjective, let $U \in L(T(A))$. Then, for all $f \in U$ and $g \in L(T(A)) - U$, since U is increasing, we have $g < f$. Thus, $f(a_{fg}) = 1$ and $g(a_{fg}) = 0$ for some $a_{fg} \in A$. Hence, $f \in F_A(a_{fg})$ and $g \in L(T(A)) - F_A(a_{fg})$.

For fixed $f \in U$ we have $g \in L(T(A)) - U \subseteq \bigcup\limits_{i=1}^{n} (L(T(A)) - F_A(a_{fgi})) = L(T(A)) - F_A(\bigwedge\limits_{i=1}^{n} a_{fgi})$ (because $L(T(A)) - U$ is compact). For $a_f = \bigwedge\limits_{i=1}^{n} a_{fgi} =$, we have $F_A(a_f) = F_A(\bigwedge\limits_{i=1}^{n} a_{fgi}) \subset U$. On the other hand, $f(a_f) = 1$, thus $f \in F_A(a_f)$. Therefore, $U = \cup_{f \in U} F_A(a_f)$. We find again a finite covering $U = \bigcup\limits_{j=1}^{n} F_A(a_{fj})$. Hence, $\{U\} \supseteq \bigsqcup\limits_{j=1}^{n} F_A(a_{fj}) \supseteq F_A(\bigsqcup\limits_{j=1}^{n} a_{fj})$, (since $B \subset B' \Rightarrow F_A^{-1}(B) \subset F_A^{-1}(B')$ and F_A injective). Consequently, $F_A^{-1}(F_A(\bigsqcup\limits_{j=1}^{n} a_{fj})) = \bigsqcup\limits_{j=1}^{n} a_{fj}))$.

We have $\bigsqcup\limits_{j=i}^{n} a_{fj} \subseteq F_A^{-1}(U)$, since $\bigsqcup\limits_{j=1}^{n} a_{fj} \in \mathcal{P}^*(A)$, i.e., $\emptyset \neq \bigsqcup\limits_{j=1}^{n} a_{fj} \subseteq A$, i.e., F_A is surjective. Since F_A is injective, there exists $a \in A$ such that $U = F_A(a)$. Therefore, F_A is a hyperlattices isomorphism. $\square$

Lemma 3.7. *If $f\colon A_1 \to A_2$ is a hyperlattices homomorphism, then the map $T(f)\colon T(A_1) \to T(A_2)$ defined by $T(f)(g) = g \circ f$ is a homomorphism of Priestley spaces.*

Proof. For all $g_1, g_2 \in T(A_1)$, from $g_1 \leq g_2$ it follows $g_1 \circ f \leq g_2 \circ f$. Hence $T(f)$ is increasing. The continuity of $T(f)$ follows from the fact that for every $a \in A_1$,

$$\begin{aligned} T(f)^{-1}(F_{A_1}(a)) &= \{g \in T(A_2) / T(f)(g) \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) / g \circ f(a) = 1\} = \{g \in T(A_2) / g(f(a)) = 1\} \\ &= F_{A_2}(f(a)). \end{aligned}$$

This completes the proof. $\square$

Lemma 3.8. *If $\delta = (X, \tau, r)$ is a Priestley space, then the map $G_\delta\colon \delta \to T(L(\delta))$ defined by*

$$G_\delta(x)(Y) = \begin{cases} 1 & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y, \end{cases}$$

for all $Y \in L(\delta)$, is an isomorphism of Priestley spaces.

Proof. To prove the surjectivity, for each $f \in T(L(\delta))$ we consider the sets $U = \{Y \in L(\delta) \mid f(Y) = 1\}$, $V = \{Z \in L(\delta) \mid f(Z) = 0\}$, $A = \cap_{Y\in U} Y$ and $B = \cup_{Z\in V} Z$. Suppose that $A - B = \emptyset$. Then $(\cap_{Y\in U} Y) \cap (\cup_{Z\in V} Z)^c = \emptyset$, consequently $(\cap_{Y\in U} Y) \cap (\cap_{Z\in V} Z^C) = \emptyset$. Since X is compact, we have $(\bigcap_{i=1}^{n} Y_i) \cap (\bigcap_{j=1}^{m} Z_j^C) = \emptyset$. Thus, $\bigcap_{i=1}^{n} Y_i \subseteq \bigcup_{j=1}^{m} Z_j$ and $f(\bigcup_{j=1}^{m} Z_j) = 1$, a contradiction because $f(\bigcup_{j=1}^{m} Z_j) = \bigvee_{j=i}^{m} f(Z_j) = 0$. Hence $A - B \neq \emptyset$. Then, there exists $x \in A - B$ such that $G_\delta(x) = f$. Therefore $G_\delta(x)(Y) = 1 \Leftrightarrow x \in Y \Leftrightarrow Y \in U \Leftrightarrow f(Y) = 1$. So, G_δ is surjective.

Let $x_1, x_2 \in \delta$, $x_1 \neq x_2$. If $x_1 \nleq x_2$, then there exists $Y_0 \in L(\delta)$ such that $x_1 \in Y_0$ and $x_2 \notin Y_0$, hence $G_\delta(x_1)(Y_0) \neq G_\delta(x_2)(Y_0)$. If $x_2 \nleq x_1$, then there exists $Y_1 \in L(\delta)$ such that $x_2 \in Y_1$ and $x_1 \notin Y_1$, hence $G_\delta(x_2)(Y_1) \neq G_\delta(x_1)(Y_1)$. Thus $x_1 \neq x_2$ implies $G_\delta(x_1)(Y) \neq G_\delta(x_2)(Y)$, so G_δ is injective.

To prove that G_δ is continuous, let Z be a τ-clopen subset of $T(L(\delta))$. Then, there exists $y \in L(\delta)$ such that $Y = F_{L(\delta)}(y)$. Thus

$$\begin{aligned} G_\delta^{-1}(Y) &= G_\delta^{-1}(F_{L(\delta)}(y)) = \{x \in X / G_\delta(x) \in F_{L(\delta)}(y)\} \\ &= \{x \in X / G_\delta(x)(y) = 1\} = \{x \in X / x \in y\} = X \cap y = y. \end{aligned}$$

Hence, G_δ is continuous.

Note that, since $Y \in L(\delta)$ are increasing, $x \leq y$ implies $G_\delta(x)(Y) \leq G_\delta(y)(Y)$. □

Lemma 3.9. *If $h\colon \delta_1 \to \delta_2$ is a homomorphism of Priestley spaces, then the map $L(h)\colon L(\delta_2) \to L(\delta_1)$ defined by $L(h)(y) = h^{-1}(y)$ for every $y \in L(\delta_2)$ is a hyperlattices homomorphism.*

Proof. For all $y \in L(\delta_2)$ we have $L(h)(y) \in L(\delta_1)$. For all $y, z \in L(\delta_2)$ since h^{-1} commutes with set-theoretical operations we have,

$$\begin{aligned} L(h)(y \sqcup z) &\subseteq \{h^{-1}(x) \mid h^{-1}(y \cup z) = h^{-1}(y \cup x) = h^{-1}(z \cup x)\} \\ &= \{h^{-1}(x) \mid h^{-1}(y) \cup h^{-1}(z) = h^{-1}(y) \cup h^{-1}(x) = h^{-1}(z) \cup h^{-1}(x)\} \\ &\subseteq L(h)(y) \sqcup L(h)(z). \end{aligned}$$

and $L(h)(y \cap z) = h^{-1}(y \cap z) = h^{-1}(y) \cap h^{-1}(z) = L(h)(y) \cap L(h)(z)$.

Hence, $L(h)$ is a hyperlattices homomorphism. □

Theorem 3.10. *$L(T(f)) \circ F_{A_1} = F_{A_2} \circ f$ for any hyperlattices homomorphism $f\colon A_1 \to A_2$.*

$$\begin{tikzcd} A_1 \arrow[r, "f"] \arrow[d, "F_{A_1}"'] & A_2 \arrow[d, "F_{A_2}"] \\ L(T(A_1)) \arrow[r, "L(T(f))"'] & L(T(A_2)) \end{tikzcd}$$

Proof. For all $a \in A_1$,

$$\begin{aligned}(L(T(f)) \circ F_{A_1})(a) &= L(T(f))(F_{A_1}(a)) = T^{-1}(f)(F_{A_1}(a)) \\ &= \{g \in T(A_2) \mid T(f)(g) \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) \mid g \circ f \in F_{A_1}(a)\} \\ &= \{g \in T(A_2) \mid g(f(a)) = 1\} \\ &= F_{A_2}(f(a)) = (F_{A_2} \circ f)(a),\end{aligned}$$

which completes the proof. □

Theorem 3.11. *For any homomprphism $h\colon \delta_1 \to \delta_2$ of Priestley spaces, we have $T(L(h)) \circ G_{\delta_1} = G_{\delta_2} \circ h$.*

$$\begin{tikzcd} \delta_1 \arrow[r, "h"] \arrow[d, "G_{\delta_1}"'] & \delta_2 \arrow[d, "G_{\delta_2}"] \\ L(T(\delta_1)) \arrow[r, "L(T(h))"'] & L(T(\delta_2)) \end{tikzcd}$$

Proof. $(T(L(h)) \circ G_{\delta_1})(f) = T(L(h))(G_{\delta_1}(f)) = G_{\delta_1}(f) \circ L(h)$ for all $f \in \delta_1$. Hence for all $y \in L(\delta_2)$ we have

$$\begin{aligned}(T(L(h)) \circ G_{\delta_1})(f)(y) &= (G_{\delta_1}(f) \circ L(h))(y) = G_{\delta_1}(f)\left(h^{-1}(y)\right) \\ &= \begin{cases} 1 \text{ if } f \in h^{-1}(y) \\ 0 \text{ if } f \notin h^{-1}(y) \end{cases} = \begin{cases} 1 \text{ if } h(f) \in y \\ 0 \text{ if } h(f) \notin y \end{cases} \\ &= G_{\delta_2}(h(f))(y) = (G_{\delta_2} \circ h)(f)(y).\end{aligned}$$

This completes the proof. □

Theorem 3.12. *The dual of the category of Priestley spaces is equivalent to the category of distributive hyperlattices.*

Proof. By Lemma 3.6, Lemma 3.8, Theorem 3.10 and Theorem 3.11. □

4. Examples

Example 4.1. Let $A = \{0, a, b, 1\}$. Consider the following Cayley tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sqcup$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{1\}$	$\{b,1\}$
b	$\{b\}$	$\{1\}$	$\{0,b\}$	$\{a,1\}$
1	$\{1\}$	$\{b,1\}$	$\{a,1\}$	A

Then $(A, \wedge, \sqcup, 0, 1)$ is a bounded distributive hyperlattice. $T(A)$ is the set of homomorphisms from A onto $\{0,1\} = \{f_1, f_2\}$ and its bidual is: $L(T(A)) = \{\emptyset, \{f_1\}, \{f_2\}, \{f_1, f_2\}\}$, where

	0	a	b	1
f_1	0	1	0	1
f_2	0	0	1	1

$\sqcup$	$\emptyset$	$\{f_1\}$	$\{f_2\}$	$\{f_1, f_2\}$
$\emptyset$	$\emptyset$	$\{f_1\}$	$\{f_2\}$	$\{f_1, f_2\}$
$\{f_1\}$	$\{f_1\}$	$\{\emptyset, \{f_1\}\}$	$\{f_1, f_2\}$	$\{\{f_2\}, \{f_1, f_2\}\}$
$\{f_2\}$	$\{f_2\}$	$\{f_1, f_2\}$	$\{\emptyset, \{f_2\}\}$	$\{\{f_1\}, \{f_1, f_2\}\}$
$\{f_1, f_2\}$	$\{f_1, f_2\}$	$\{\{f_2\}, \{f_1, f_2\}\}$	$\{\{f_1\}, \{f_1, f_2\}\}$	$\{\emptyset, \{f_1\}, \{f_2\}, \{f_1, f_2\}\}$

Then $(L(T(A)), \cap, \sqcup, \emptyset, X)$ is a bounded distributive hyperlattice with $X = \{f_1, f_2\}$. $F_A \colon A \to L(T(A))$ is given by $F_A(0) = \emptyset$, $F_A(a) = \{f_1\}$, $F_A(b) = \{f_2\}$, $F_A(1) = \{f_1, f_2\}$.

Example 4.2. Let $D(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the set of positive divisors of 30 and $(D(30), \wedge, \vee)$ the lattice where $a \wedge b$ and $a \vee b$ are respectively the greatest common divisor and the least common multiplier of a and b. Define on $D\,(30)$ the hyperoperation by: $a \sqcup b = \{x \in D(30) \mid a \vee b = a \vee x = b \vee x\}$, for all $a, b \in L$. Then $(D\,(30)\,, \wedge, \sqcup, 1, 30)$ is a bounded distributive hyperlattice. $T\,(D\,(30))$ is the set of homomorphisms from $D(30)$ onto $\{0, 1\} = \{f_1, f_2, f_3\}$.

$D\,(30)$	1	2	3	5	6	10	15	30
f_1	0	1	0	0	1	1	0	1
f_2	0	0	1	0	1	0	1	1
f_3	0	0	0	1	0	1	1	1

Its bidual is: $L(T(D\,(30))) = \{\emptyset, \{f_1\}, \{f_2\}, \{f_3\}, \{f_1, f_2\}, \{f_1, f_3\}, \{f_2, f_3\}, X\}$, where $X = \{f_1, f_2, f_3\}$.

$\sqcup$	$\emptyset$	$\{f_1\}$	$\{f_2\}$	$\{f_3\}$
$\emptyset$	$\emptyset$	$\{f_1\}$	$\{f_2\}$	$\{f_3\}$
$\{f_1\}$	$\{f_1\}$	$\{\emptyset, \{f_1\}\}$	$\{f_1, f_2\}$	$\{f_1, f_3\}$
$\{f_2\}$	$\{f_2\}$	$\{f_1, f_2\}$	$\{\emptyset, \{f_2\}\}$	$\{f_2, f_3\}$
$\{f_3\}$	$\{f_3\}$	$\{f_1, f_3\}$	$\{f_2, f_3\}$	$\{\emptyset, \{f_3\}\}$
$\{f_1, f_2\}$	$\{f_1, f_2\}$	$\{\{f_2\}\,, \{f_1, f_2\}\}$	$\{\{f_1\}\,, \{f_1, f_2\}\}$	$\{X$
$\{f_1, f_3\}$	$\{f_1, f_3\}$	$\{\{f_3\}\,, \{f_1, f_3\}\}$	X	$\{\{f_1\}\,, \{f_1, f_3\}\}$
$\{f_2, f_3\}$	$\{f_2, f_3\}$	$\{X\}$	$\{\{f_3\}\,, \{f_2, f_3\}\}$	$\{\{f_2\}\,, \{f_2, f_3\}\}$
X	X	$\{\{f_2, f_3\}, X\}$	$\{\{f_1, f_3\}\,, X\}$	$\{\{f_1, f_2\}\,, X\}$

$\sqcup$	$\{f_1,f_2\}$	$\{f_1,f_3\}$
$\emptyset$	$\{f_1,f_2\}$	$\{f_1,f_3\}$
$\{f_1\}$	$\{\{f_2\},\{f_1,f_2\}\}$	$\{\{f_3\},\{f_1,f_3\}\}$
$\{f_2\}$	$\{\{f_1\},\{f_1,f_2\}\}$	X
$\{f_3\}$	X	$\{\{f_1\},\{f_1,f_3\}\}$
$\{f_1,f_2\}$	$\{\emptyset,\{f_1\},\{f_2\},\{f_1,f_2\}\}$	$\{\{f_2,f_3\},X\}$
$\{f_1,f_3\}$	$\{\{f_2,f_3\},X\}$	$\{\emptyset,\{f_1\},\{f_3\},\{f_1,f_3\}\}$
$\{f_2,f_3\}$	$\{\{f_1,f_3\},X\}$	$\{\{f_1,f_2\},X\}$
X	$\{\{f_3\},\{f_1,f_3\},\{f_2,f_3\},X\}$	$\{\{f_2\},\{f_1,f_2\},\{f_2,f_3\},X\}$

$\sqcup$	$\{f_2,f_3\}$	X
$\emptyset$	$\{f_2,f_3\}$	X
$\{f_1\}$	X	$\{\{f_2,f_3\},X\}$
$\{f_2\}$	$\{\{f_3\},\{f_2,f_3\}\}$	$\{\{f_1,f_3\},X\}$
$\{f_3\}$	$\{\{f_2\},\{f_2,f_3\}\}$	$\{\{f_1,f_2\},X\}$
$\{f_1,f_2\}$	$\{\{f_1,f_3\},X\}$	$\{\{f_3\},\{f_1,f_3\},\{f_2,f_3\},X\}$
$\{f_1,f_3\}$	$\{\{f_1,f_2\},X\}$	$\{\{f_2\},\{f_1,f_2\},\{f_2,f_3\},X\}$
$\{f_2,f_3\}$	$\{\emptyset,\{f_2\},\{f_3\},\{f_2,f_3\}\}$	$\{\{f_1\},\{f_1,f_2\},\{f_1,f_3\},X\}$
X	$\{\{f_1\},\{f_1,f_2\},\{f_1,f_3\},X\}$	$L(T(A))$

Then $(L(T(A)),\cap,\sqcup,\emptyset,X)$ is a bounded distributive hyperlattice. $F_A(1)=\emptyset$, $F_A(2)=\{f_1\}$, $F_A(3)=\{f_2\}$, $F_A(5)=\{f_3\}$, $F_A(6)=\{f_1,f_2\}$, $F_A(10)=\{f_1,f_3\}$, $F_A(15)=\{f_2,f_3\}$, $F_A(30)=X$.

Example 4.3. Let $(X,\tau,\leq)$ be a Priestley space, where $X=\{a,b,c\}$ and $\leq$ is given by

$\leq$	a	b	c
a	1	0	1
b	0	1	1
c	0	0	1

and $L(X)=\{\emptyset,\{c\},\{a,c\},\{b,c\},X\}$, where

$\sqcup$	$\emptyset$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	X
$\emptyset$	$\emptyset$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	X
$\{c\}$	$\{c\}$	$\{\emptyset,\{c\}\}$	$\{a,c\}$	$\{b,c\}$	X
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{\emptyset,\{c\},\{a,c\}\}$	X	$\{\{b,c\},X\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	X	$\{\emptyset,\{c\},\{b,c\}\}$	$\{\{a,c\},X\}$
X	X	X	$\{\{b,c\},X\}$	$\{\{a,c\},X\}$	$L(X)$

$\leq$	$\emptyset$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	X
$\emptyset$	1	1	1	1	1
$\{c\}$	0	1	1	1	1
$\{a,c\}$	0	0	1	0	1
$\{b,c\}$	0	0	0	1	1
X	0	0	0	0	1

and $T(L(X)) = \{f_1, f_2, f_3\}$ such that

$L(X)$	f_1	f_2	f_3
$\emptyset$	0	0	0
$\{c\}$	0	0	1
$\{a,c\}$	1	0	1
$\{b,c\}$	0	1	1
X	1	1	1

The isomorphism $G_X : X \to T(L(X))$ is defined by $G_X(a) = f_1$, $G_X(b) = f_2$, $G_X(c) = f_3$.

Conclusion

In this paper, we propose a new way to represent distributive hyperlattices. It is shown that the dual of the category of Priestley spaces is equivalent to the category of bounded distributive hyperlattices.

For further investigations, we give the following open question.

Question. *Is there a relation between the category of bounded distributive hyperlattices and the category of bounded distributive lattices?*

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