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Implication zroupoids and identities of associative type

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Abstract. An algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is binary and 0 is a constant, is called an $\mathcal{I}$-*zroupoid* if $\mathbf{A}$ satisfies the identities: $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$ and $0'' \approx 0$, where $x' := x \to 0$, and $\mathcal{I}$ denotes the variety of all $\mathcal{I}$-zroupoids. An $\mathcal{I}$-zroupoid is *symmetric* if it satisfies $x'' \approx x$ and $(x \to y')' \approx (y \to x')'$. The variety of symmetric $\mathcal{I}$-zroupoids is denoted by $\mathcal{S}$. An identity $p \approx q$, in the groupoid language $\langle\to\rangle$, is called *an identity of associative type of length* 3 if p and q have exactly 3 (distinct) variables, say x, y, z, and are grouped according to one of the two ways of grouping: (1) $\star \to (\star \to \star)$ and (2) $(\star \to \star) \to \star$, where $\star$ is a place holder for a variable. A subvariety of $\mathcal{I}$ is said to be of *associative type of length* 3, if it is defined, relative to $\mathcal{I}$, by a single identity of associative type of length 3. In this paper we give a complete analysis of the mutual relationships of all subvarieties of $\mathcal{I}$ of associative type of length 3. We prove, in our main theorem, that there are exactly 8 such subvarieties of $\mathcal{I}$ that are distinct from each other and describe explicitly the poset formed by them under inclusion. As an application of the main theorem, we derive that there are three distinct subvarieties of the variety $\mathcal{S}$ of associative type, each defined relative to $\mathcal{S}$, by a single identity of associative type of length 3.

1. Introduction

In 1934, Bernstein gave a system of axioms for Boolean algebras in [3] using implication alone. Even though his system was not equational, it is not hard to see that one could easily convert it into an equational one by using an additional constant. In 2012, the second author extended this "modified Bernstein's theorem" to De Morgan algebras in [24] by showing that the variety of De Morgan algebras, is term-equivalent to the variety $\mathcal{DM}$ (defined below) whose defining axioms use only an implication $\to$ and a constant 0.

The primary role played by the identity (I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, where $x' := x \to 0$, in the axiomatization of each of those new varieties motivated the second author to study the identity (I) in its own right and led him

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to introduce a new equational class of algebras called *implication zroupoids* in [24] (also called *implicator groupoids* in [7]).

An algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is binary and 0 is a constant, is called a *zroupoid*. A zroupoid $\mathbf{A} = \langle A, \to, 0\rangle$ is an *implication zroupoid* ($\mathcal{I}$-*zroupoid*, for short) if $\mathbf{A}$ satisfies:

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, where $x' := x \to 0$,

(I_0) $0'' \approx 0$.

$\mathcal{I}$ denotes the variety of implication zroupoids. The varieties $\mathcal{DM}$ and $\mathcal{SL}$ are defined relative to $\mathcal{I}$, respectively, by the following identities:

(DM) $(x \to y) \to x \approx x$ (De Morgan Algebras);
(SL) $x' \approx x$ and $x \to y \approx y \to x$ (semilattices with the least element 0).

The variety $\mathcal{BA}$ of Boolean algebras is defined relative to $\mathcal{DM}$ by the following identity:

(BA) $x \to x \approx 0'$.

The variety $\mathcal{I}$ exhibits (see [24]) several interesting properties; for example, the identity $x''' \to y \approx x' \to y$ holds in $\mathcal{I}$; in particular, $\mathcal{I}$ satisfies $x'''' \approx x''$. Two of the subvarieties of $\mathcal{I}$ that are of interest in this paper are: $\mathcal{I}_{2,0}$ and $\mathcal{MC}$ which are defined relative to $\mathcal{I}$, respectively, by the following identities, where $x \wedge y := (x \to y')'$:

($\mathrm{I}_{2,0}$) $x'' \approx x$;

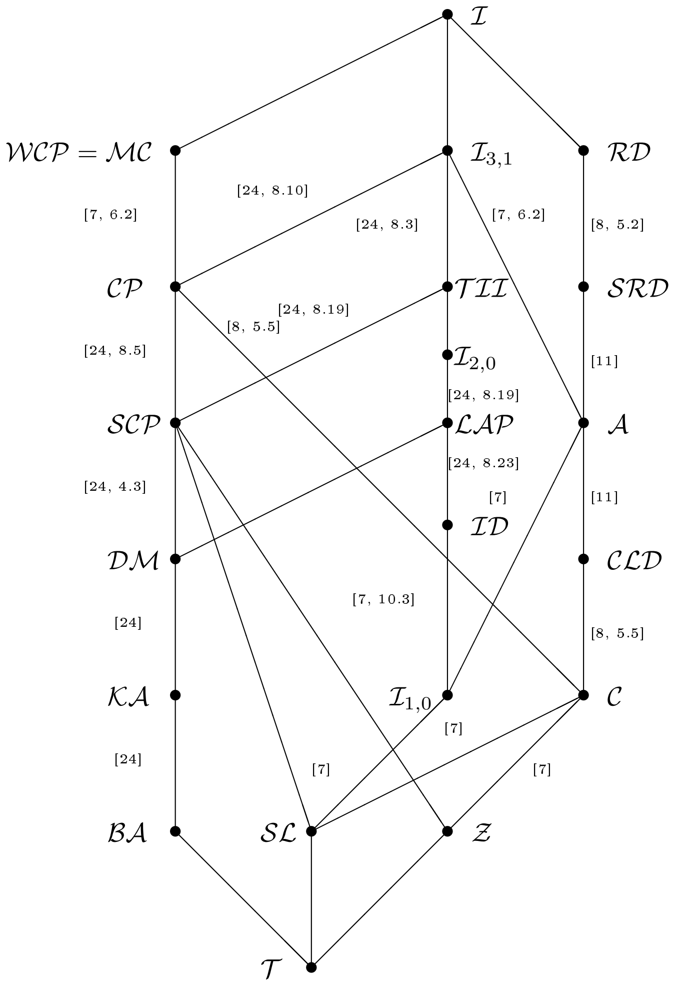
(MC) $x \wedge y \approx y \wedge x$.

The (still largely unexplored) lattice of subvarieties of $\mathcal{I}$ seems to be fairly complex. In fact, Problem 6 of [24] calls for an investigation of the structure of the lattice of subvarieties of $\mathcal{I}$.

The papers [5], [6], [7], [8], [9], [10] and [11] have addressed further the above-mentioned problem, but still partially, by introducing several new subvarieties of $\mathcal{I}$ and investigating relationships among them. The (currently known) size of the poset of subvarieties of $\mathcal{I}$ is at least 30; but it is still unknown whether the lattice of subvarieties is finite or infinite. We conjecture that its cardinality is 2^{ω}.

Motivated by the fact that not all algebras in $\mathcal{I}$ are associative with respect to the operation $\to$, the quest for finding more new subvarieties of $\mathcal{I}$ led us naturally to consider the question as to whether generalizations of the associative law would yield some new subvarieties of $\mathcal{I}$ and thereby reveal further insight into the structure of the lattice of subvarieties of $\mathcal{I}$. This quest led to the results in [9], [10] and this paper, which will show that this is indeed the case.

The poset of the (then) known varieties that appears in [8] is given below for the reader's convenience (for the definitions of the varieties in the picture, see [8]).



A look at the associative law would reveal the following characteristics:

(1) Length of the left side term = length of the right side term = 3,

(2) The number of distinct variables on the left = the number of distinct variables on right = the number of occurrences of variables on either side,

(3) The order of the variables on the left side is the same as the order of the variables on the right side,

(4) The bracketings used in the left side term and in the right side term are different from each other.

One way to generalize the associative law is to relax somewhat the restrictions (1) and (2) by choosing m distinct variables and setting the length of the left term = that of right term = n, with $n \geqslant m$, and keeping (3) and (4). But then, for $n \geqslant 4$, there will be more than two possible bracketings. So, we order all possible bracketings and assign a number to each, called bracketing number. Such identities are called "weak associative identities of length n". For a precise definition and notation of weak associative identities, we refer the reader to [10] and the references therein.

A second way to generalize the associative law is to relax (3) and to keep (2), (4) and the first half of (1). So, we consider the laws of the form $p \approx q$ of length n such that (a) each of p and q contains the n (an integer $\geqslant 3$) distinct variables, say, $x_1, x_2, \ldots, x_n$, (b) p and q are terms obtained by distinct bracketings of permutations of the n variables. Let us call such laws as "identities of associative type of length n".

A third way to generalize the associative law is to relax all of four features mentioned above by allowing number of occurrences of variables on one side be different from the number on the other side. Let us refer to these as "identities of mixed type".

Specific instances of all such generalizations of the associative law have already occurred in the literature at least since late 19th century. We mention below a few such instances.

Weak associative identities of length 4 with 3 distinct variables, called "identities of Bol-Moufang type", have been investigated in the literature quite extensively for the varieties of quasigroups and loops. In fact, the first systematic analysis of the relationships among the identities of Bol-Moufang type appears to be in [12] in the context of loops. For more information about these identities in the context of quasigroups and loops, see [12], [15], [19], [20]) and the references therein.

More recently, in [9] and [10], we have made a complete analysis of relationships among weak associative identities of length $\leqslant 4$, relative to the variety $\mathcal{S}$ of symmetric I-zroupoids (i.e., satisfying $x'' \approx x$ and $(x \to y')' \approx (y \to x')'$). We have shown that 6 of the 155 subvarieties of $\mathcal{S}$, each being defined by a single weak associative law of length $m \leqslant 4$ (including the Bol-Moufang type), are distinct. Furthermore, we describe explicitly by a Hasse diagram the poset formed by them, together with the varieties $\mathcal{BA}$ and $\mathcal{SL}$.

We should mention here that such an analysis of weak associative laws of length $\leqslant 4$ relative to the variety $\mathcal{I}$ is still open.

The identities of associative type have also appeared in the literature. We mention several examples below, using $\cdot$ for the binary operation instead of $\to$.

- The identity $x \cdot (y \cdot z) \approx (z \cdot x) \cdot y$ was considered in [28] by Suschkewitsch (see also [27, Theorem 11.5]).

- Abbott [1] uses the identity $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$ as one of the defining identities in his definition of implication algebras.

- The identities $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$, $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$, and $x \cdot (y \cdot z) \approx (z \cdot x) \cdot y$ were investigated for quasigroups by Hosszú in [13].
- The identity $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ is investigated by Pushkashu in [22].
- The identities $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$ and $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$ have appeared in [14] of Kazim and Naseeruddin.

The identities of mixed type have also been considered in the literature. A few are listed below:

- left distributivity: $x \cdot (y \cdot z) \approx (x \cdot y) \cdot (x \cdot z)$, appears, according to [18], already in the late 19th century publications of logicians Peirce and Schroeder (see [17] and [25], respectively),
- right distributive: $(z \cdot y) \cdot x \approx (z \cdot x) \cdot (y \cdot x)$ (see [26]),
- distributive if it is both left and right distributive (see [26]),
- medial: $(x \cdot y) \cdot (u \cdot v) \approx (x \cdot u) \cdot (y \cdot v)$ (see [26]),
- idempotent: $x \cdot x \approx x$ (see [26]),
- left involutory (or left symmetric): $x \cdot (x \cdot y) \approx y$ (see [26]).

Several identities of associative type have appeared in the literatiure on groupoids as well. For instance,

- $(x \cdot y) \cdot z \approx (z \cdot y) \cdot x$: Abel-Grassmann's groupoid (AG-groupoid) (see [21]),
- $(x \cdot y) \cdot z \approx (z \cdot y) \cdot x$ and $x \cdot (y \cdot z) \approx y \cdot (x \cdot z)$ (AG**-groupoid),
- $x \cdot (z \cdot y) \approx (x \cdot y) \cdot z$: Hosszú-Tarski identity (see [22]),
- $(x \cdot y) \cdot z \approx (z \cdot y) \cdot x$: Left almost semigroup (LA-semigroup) (see [22]),
- $x \cdot (y \cdot z) \approx z \cdot (y \cdot x)$: Right almost semigroup (RA-semigroup) (see [22]).

Similar to the problem mentioned in [10] for weak associative identities, the following general problem presents itself naturally if we restrict our attention to identities of associative type.

Problem. *Let $\mathcal{V}$ be a given variety of algebras (whose language includes a binary operation symbol, say, '$\to$'). Investigate the mutual relationships among the subvarieties of $\mathcal{V}$, each of which is defined by a single identity of associative type of length n, for small values of the positive integer n.*

We will now consider the above problem for the variety $\mathcal{I}$. We begin a systematic analysis of the relationships among the identities of associative type of length 3 relative to the variety $\mathcal{I}$.

Definition 1.1. An identity $p \approx q$, in the groupoid language $\langle \to \rangle$, is called an *identity of associative type of length* 3 if p and q have exactly 3 (distinct) variables, say x, y, z, and these variables are grouped according to one of the following two ways of grouping:

(a) $o \to (o \to o)$, (b) $(o \to o) \to o$.

In the rest of the paper, we refer to an "identity of associative type of length 3" as simply an "identity of associative type".

We wish to determine the mutual relationships of all the subvarieties of $\mathcal{I}$ defined by the identities of associative type, which will be referred to as "subvarieties of associative type".

Our main theorem says that there are 8 of such subvarieties of $\mathcal{I}$ that are distinct from each other and describes explicitly, by a Hasse diagram, the poset formed by them, together with the varieties $\mathcal{SL}$ and $\mathcal{BA}$. As an application, we show that there are 3 distinct subvarieties of $\mathcal{S}$ of associative type.

We would like to acknowledge that the software "Prover 9/Mace 4" developed by McCune [16] has been useful to us in some of our findings presented in this paper. We have used it to find examples and to check some conjectures.

2. Preliminaries

We refer the reader to the standard references [2], [4] and [23] for concepts and results used, but not explained, in this paper.

Recall from [24] that $\mathcal{SL}$ is the variety of semilattices with a least element 0. It was shown in [7] that $\mathcal{SL} = \mathcal{C} \cap \mathcal{I}_{1,0}$, where $\mathcal{I}_{1,0}$ is defined by $x' \approx x$, and $\mathcal{C}$ is defined by $x \to y \approx y \to x$, to relative to $\mathcal{I}$.

The two-element algebras $\mathbf{2_z}$, $\mathbf{2_s}$, $\mathbf{2_b}$ were introduced in [24]. Their operations $\to$ are respectively as follows:

$\to$	0	1
0	0	0
1	0	0

$\to$	0	1
0	0	1
1	1	1

$\to$	0	1
0	1	1
1	0	1

Recall that $\mathcal{V}(\mathbf{2_b}) = \mathcal{BA}$. Recall also from [7, Corollary 10.4] that $\mathcal{V}(\mathbf{2_s}) = \mathcal{SL}$.

The following lemmas will be useful in the sequel.

Lemma 2.1. [24, 7.16] *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-zroupoid. Then* $\mathbf{A} \models x''' \to y \approx x' \to y$.

Lemma 2.2. [7, 3.4] *Let* $\mathbf{A}$ *be an I-zroupoid. Then* $\mathbf{A}$ *satisfies:*

(a) $(x \to y) \to z \approx [(x \to y) \to z]''$,

(b) $(x \to y)' \approx (x'' \to y)'$.

Lemma 2.3. [24, 8.15] *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-zroupoid. Then the following are equivalent:*

1. $0' \to x \approx x$,
2. $x'' \approx x$,
3. $(x \to x')' \approx x$,
4. $x' \to x \approx x$.

Recall that $\mathcal{I}_{2,0}$ and $\mathcal{MC}$ are the subvarieties of $\mathcal{I}$, defined, respectively, by the equations

$$x'' \approx x. \tag{$I_{2,0}$}$$

$$x \wedge y \approx y \wedge x. \tag{MC}$$

Lemma 2.4. [24] *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then*

1. $x' \to 0' \approx 0 \to x$,
2. $0 \to x' \approx x \to 0'$.

Lemma 2.5. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(a) $(x \to 0') \to y \approx (x \to y') \to y$,

(b) $(y \to x) \to y \approx (0 \to x) \to y$,

(c) $0 \to x \approx 0 \to (0 \to x)$,

(d) $(0 \to x) \to (0 \to y) \approx x \to (0 \to y)$,

(e) $x \to y \approx x \to (x \to y)$,

(f) $0 \to (x \to y) \approx x \to (0 \to y)$,

(g) $0 \to (x \to y')' \approx 0 \to (x' \to y)$,

(h) $x \to (y \to x') \approx y \to x'$.

Proof. For the proofs of items (a), (b), (c), (f), (g), and (h) we refer the reader to [7]. The proofs of items (d) and (e) are in [5]. □

Theorem 2.6. [8] *Let* $t_i(\overline{x}), i = 1, \ldots, 6$ *be terms and* $\mathcal{V}$ *a subvariety of* $\mathcal{I}$. *If*

$$\mathcal{V} \cap \mathcal{I}_{2,0} \models [t_1(\overline{x}) \to t_2(\overline{x})] \to t_3(\overline{x}) \approx [t_4(\overline{x}) \to t_5(\overline{x})] \to t_6(\overline{x}),$$

then

$$\mathcal{V} \models [t_1(\overline{x}) \to t_2(\overline{x})] \to t_3(\overline{x}) \approx [t_4(\overline{x}) \to t_5(\overline{x})] \to t_6(\overline{x}).$$

2.1. Identities of associative type

We now turn our attention to identities of associative type of length 3. Recall that such an identity will contain three distinct variables that occur in any order and that are grouped in one of the two (obvious) ways. The following identities play a crucial role in the sequel.

Let Σ denote the set consisting of the following 14 identities of associative type (of length 3 in the binary language $\langle \to \rangle$):

(A1) $x \to (y \to z) \approx (x \to y) \to z$ (Associative law)

(A2) $x \to (y \to z) \approx x \to (z \to y)$

(A3) $x \to (y \to z) \approx (x \to z) \to y$

(A4) $x \to (y \to z) \approx y \to (x \to z)$

(A5) $x \to (y \to z) \approx (y \to x) \to z$

(A6) $x \to (y \to z) \approx y \to (z \to x)$

(A7) $x \to (y \to z) \approx (y \to z) \to x$

(A8) $x \to (y \to z) \approx (z \to x) \to y$

(A9) $x \to (y \to z) \approx z \to (y \to x)$

(A10) $x \to (y \to z) \approx (z \to y) \to x$

(A11) $(x \to y) \to z \approx (x \to z) \to y$

(A12) $(x \to y) \to z \approx (y \to x) \to z$

(A13) $(x \to y) \to z \approx (y \to z) \to x$

(A14) $(x \to y) \to z \approx (z \to y) \to x$.

We will denote by $\mathcal{A}_i$ the subvariety of $\mathcal{I}$ defined by the identity (Ai), for $1 \leqslant i \leqslant 14$. Such varieties will be referred to as subvarieties of $\mathcal{I}$ of associative type.

The following proposition is crucial for the rest of the paper.

Proposition 2.7. *Let $\mathcal{G}$ be the variety of all groupoids of type $\{\to\}$ and Let $\mathcal{V}$ denote the subvariety of $\mathcal{G}$ defined by a single identity of associative type. Then $\mathcal{V} = \mathcal{A}_i$, for some $i \in \{1, 2, \dots, 14\}$.*

Proof. In an identity $p \approx q$ of associative type of length 3, p and q have exactly 3 (distinct) variables, say x,y,z, and these variables are grouped according to one of the two ways of bracketing mentioned above. Thus, there are six permuatations of 3 variables which give rise to the following 12 terms:

1a: $x \to (y \to z)$ 1b: $(x \to y) \to z$

2a: $x \to (z \to y)$ 2b: $(x \to z) \to y$

3a: $y \to (x \to z)$ 3b: $(y \to x) \to z$

4a: $y \to (z \to x)$ 4b: $(y \to z) \to x$

5a: $z \to (x \to y)$ 5b: $(z \to x) \to y$

6a: $z \to (y \to x)$ 6b: $(z \to y) \to x$.

It is clear that these 12 terms, in turn, will lead to 66 identities in view of the symmetric property of equality. It is routine to verify that each of the 66 identities is equivalent to one of the 14 identities of Σ in the variety of groupoids. Then the proposition follows. $\square$

Our goal, in this paper, is to determine the distinct subvarieties of $\mathcal{I}$ and to describe the poset of subvarieties of $\mathcal{I}$ of associative type. It suffices to concentrate on the varieties defined by identities (A1)-(A14), in view of the above proposition.

3. Properties of subvarieties of $\mathcal{I}$ of associative type

In this section we present properties of several subvarieties of $\mathcal{I}$ which will play a crucial role in our analysis of the identities of associative type relative to $\mathcal{I}$.

Lemma 3.1. *Let $\mathbf{A} \in \mathcal{I}$ such that $\mathbf{A} \models x' \to y \approx x \to y'$, then $\mathbf{A} \models (x \to y) \to y' \approx x \to y'$.*

Proof. Let $a, b \in A$. Then $(a \to b) \to b' \overset{2.5(a)\,\&\,2.6}{=} (a \to 0') \to b' \overset{hyp}{=} (a' \to 0) \to b' = a'' \to b' \overset{hyp}{=} a''' \to b \overset{2.1}{=} a' \to b \overset{hyp}{=} a \to b'$. $\square$

Lemma 3.2. *Let $\mathbf{A} \in \mathcal{I}_{2,0}$ such that $\mathbf{A} \models (x \to y)' \approx x \to (0 \to y)$, then $\mathbf{A} \models x \to y' \approx x \to (0 \to y)$.*

Proof. Let $a, b \in A$. Then $a \to b' \overset{2.3(1)}{=} a \to (0' \to b)' \overset{hyp}{=} a \to [0' \to (0 \to b)] \overset{2.3(1)}{=} a \to (0 \to b)$. $\square$

Lemma 3.3. *Let $\mathbf{A} \in \mathcal{I}_{2,0}$ such that $\mathbf{A} \models (x \to y)' \approx x \to (0 \to y)$. Then $\mathbf{A} \models [x \to (y \to z)']' \approx x \to (y \to (0 \to z))'$.*

Proof. Let $a, b, c \in A$. We have that $[a \to (b \to c)']' \overset{hyp}{=} a \to (0 \to (b \to c)') \overset{hyp}{=} a \to (0 \to (b \to (0 \to c))) \overset{2.5(f)}{=} a \to (b \to (0 \to (0 \to c))) \overset{hyp}{=} a \to [b \to (0 \to c)']$. $\square$

Lemma 3.4. *Let $\mathbf{A} \in \mathcal{I}$ such that $\mathbf{A}$ satisfies:*

(1) $(x \to y)' \approx x \to (0 \to y)$,

(2) $x' \to y \approx x \to y'$.

Then, $\mathbf{A} \models 0 \to [x \to (y \to z)] \approx 0 \to [(x \to y) \to z]$.

Proof. Let $a, b, c \in A$. Then, $0 \to [(a \to b) \to c] \overset{(I)}{=} 0 \to [(c' \to a) \to (b \to c)']' \overset{(I)}{=} 0 \to \{[(b \to c)'' \to c'] \to [a \to (b \to c)']\}'' \overset{2.6}{=} 0 \to \{[(b \to c) \to c'] \to [a \to (b \to c)']\}'' \overset{3.1}{=} 0 \to \{[b \to c'] \to [a \to (b \to c)']\}'' \overset{3.2\,\&\,2.6\,\&\,hyp}{=} 0 \to \{[b \to (0 \to c)] \to$

$[a \to (b \to c)']\}'' \overset{3.3\,\&\,hyp}{=} 0 \to \{[b \to (0 \to c)] \to [a \to [b \to (0 \to c)]']\}'' \overset{2.5(h)\,\&\,2.6}{=}$ $0 \to \{a \to [b \to (0 \to c)]'\}'' \overset{(3.4)}{=} 0 \to \{a \to [b \to c]''\}'' \overset{2.6}{=} 0 \to \{a \to [b \to c]\}''$ $\overset{(3.4)}{=} 0' \to \{a \to [b \to c]\}' \overset{(3.4)}{=} 0'' \to \{a \to [b \to c]\} = 0 \to \{a \to [b \to c]\}$. □

Lemma 3.5. *Let* $\mathbf{A} \in \mathcal{I}$ *such that* $\mathbf{A} \models (x \to y)' \approx (y \to x)'$. *Then* $\mathbf{A} \models (x \to y) \to z \approx (y \to x) \to z$.

Proof. Let $a, b, c \in A$. Then, $(a \to b) \to c \overset{2.6}{=} (a \to b)'' \to c \overset{hyp}{=} (b \to a)'' \to c$ $\overset{2.6}{=} (b \to a) \to c$. □

Definition 3.6. Let $\mathbf{A} \in \mathcal{I}$. We say that $\mathcal{A}$ is of *type* 1 if the following identities hold in $\mathbf{A}$:

$(E1)$ $(x \to y)' \approx x \to (0 \to y)$,

$(E2)$ $x' \to y \approx x \to y'$,

$(E3)$ $0 \to (x \to y) \approx 0 \to (y \to x)$,

$(E4)$ $x \to (y \to z) \approx (p(x) \to p(y)) \to p(z)$, where p is some permutation of $\{x, y, z\}$.

Theorem 3.7. *If* $\mathbf{A} \in \mathcal{I}$ *is of type 1 then* $\mathbf{A} \models (Aj)$ *for all* $1 \leqslant j \leqslant 14$.

Proof. Let $\mathbf{A} \in \mathcal{I}$ be of type 1, and $a, b, c \in A$. In view of equations $(E1)$, $(E2)$ and Lemma 3.4 we have that

$$\mathbf{A} \models 0 \to [x \to (y \to z)] \approx 0 \to [(x \to y) \to z]. \tag{3.1}$$

Then we can consider the following cases.

- Assume that $j = 1$. Then $a \to (b \to c) \overset{(E4)}{=} (p(a) \to p(b)) \to p(c) \overset{2.2}{=} [(p(a) \to p(b)) \to p(c)]'' \overset{(E1)}{=} [(p(a) \to p(b)) \to (0 \to p(c))]' \overset{2.5(f)\,\&\,2.6}{=} [0 \to [(p(a) \to p(b)) \to p(c)]]' \overset{(E3)\,\&\,(3.1)}{=} [0 \to [(a \to b) \to c]]' \overset{2.5(f)\,\&\,2.6}{=} [(a \to b) \to [0 \to c]]'$ $\overset{(E1)}{=} [(a \to b) \to c]'' \overset{2.2}{=} (a \to b) \to c$.

  The cases $j = 3, 5, 7, 8, 10$ are similar.

- Assume that $j = 2$. Then, in the same way as in the case of $j = 1$ we have that

  $$\mathbf{A} \models x \to (y \to z) \approx [0 \to [(p(x) \to p(y)) \to p(z)]]'. \tag{3.2}$$

  Then, $a \to (b \to c) \overset{(3.2)}{=} [0 \to [(p(a) \to p(b)) \to p(c)]]' \overset{(E3)\,\&\,(3.1)}{=} [0 \to [(p(a) \to p(c)) \to p(b)]]' \overset{2.5(f)\,\&\,2.6}{=} [(p(a) \to p(c)) \to [0 \to p(b)]]' \overset{(E1)}{=}$ $[(p(a) \to p(c)) \to p(b)]'' \overset{2.2(a)}{=} (p(a) \to p(c)) \to p(b) \overset{(E4)}{=} (a \to c) \to b$.

  The cases $j = 4, 6, 9$ are similar.

- Assume that $j = 11$. $(a \to b) \to c \overset{2.2(a)}{=} [(a \to b) \to c]'' \overset{(E1)}{=} [(a \to b) \to (0 \to c)]' \overset{2.5(f)\,\&\,2.6}{=} [0 \to [(a \to b) \to c]]' \overset{(E3)\,\&\,(3.1)}{=} [0 \to [(a \to c) \to b]]' \overset{2.5(f)\,\&\,2.6}{=} [(a \to c) \to [0 \to b]]' \overset{(E1)}{=} [(a \to c) \to b]'' \overset{2.2}{=} (a \to c) \to b$.

  The cases $j = 12, 13, 14$ are similar. □

To prove that the variety $\mathcal{A}_j$ is of type 1, with $j \in \{3, 5, 7, 8, 10\}$, we need the following lemmas.

Lemma 3.8. *If* $\mathbf{A} \in \mathcal{A}_5$ *then* $\mathbf{A}$ *satisfies*

(a) $x' \to y \approx x \to y'$,

(b) $(x \to y)' \approx 0 \to (x \to y)$,

(c) $x \to (0 \to y) \approx 0 \to (x \to y)$.

Proof. Let $a, b \in A$. Then

(a) $a \to b' = a \to (b \to 0) \overset{(A5)}{=} (b \to a) \to 0 \overset{2.6}{=} (b \to a'') \to 0 = (b \to (a' \to 0)) \to 0 \overset{(A5)}{=} [(a' \to b) \to 0] \to 0 = (a' \to b)'' \overset{2.6}{=} a' \to b$.

(b) Observe that $(a \to b)' = (a \to b) \to 0 \overset{(A5)}{=} b \to (a \to 0) = b \to a' \overset{3.8(a)}{=} b' \to a = (b \to 0) \to a \overset{(A5)}{=} 0 \to (b \to a)$.

(c) Notice that $0 \to (a \to b) \overset{(A5)}{=} (a \to 0) \to b \overset{2.6}{=} (a'' \to 0) \to b \overset{3.8(a)}{=} (a' \to 0') \to b \overset{2.4(2)\,\&\,2.6}{=} (0 \to a) \to b \overset{(A5)}{=} a \to (0 \to b)$.

□

Lemma 3.9. *If* $\mathbf{A} \in \mathcal{A}_8$ *then* $\mathbf{A}$ *satisfies:*

(a) $x \to y' \approx x' \to y'$,

(b) $x \to y' \approx 0 \to (y' \to x)$.

Proof. Let $a, b \in A$. Then $a \to b' = a \to (b \to 0) \overset{(A8)}{=} (0 \to a) \to b \overset{(I)}{=} [(b' \to 0) \to (a \to b)']' = [(b' \to 0) \to (a \to b)'] \to 0 \overset{(A8)}{=} (a \to b)' \to (0 \to (b' \to 0)) \overset{(A8)}{=} (a \to b)' \to ((0 \to 0) \to b') = (a \to b)' \to (0' \to b') \overset{2.3(1)\,\&\,2.6}{=} (a \to b)' \to b' = ((a \to b) \to 0) \to b' \overset{(A8)}{=} 0 \to (b' \to (a \to b)) \overset{(A8)}{=} 0 \to ((b \to b') \to a) \overset{2.6}{=} 0 \to ((b'' \to b') \to a) \overset{2.3(4)\,\&\,2.6}{=} 0 \to (b' \to a)$, implying that $\mathbf{A}$ satisfies the identity (b). Next, $0 \to (b' \to a) \overset{(A8)}{=} (a \to 0) \to b' = a' \to b'$, thus $\mathbf{A}$ satisfies the identity (a). □

Lemma 3.10. *If* $\mathbf{A} \in \mathcal{A}_{10}$ *then* $\mathbf{A}$ *satisfies:*

(a) $[0 \to (x \to y)]' \approx x \to y'$,

(b) $(y \to x)'' \approx x \to y'$,

(c) $(x \to y)' \approx x \to y'$.

Proof. Let $a, b \in A$.

(a) We have that $\mathbf{A} \models [0 \to (x \to y)]' \approx x \to y'$, since

$$\begin{array}{rcll}
a \to b' & = & a \to (b \to 0) & \\
& = & (0 \to b) \to a & \text{by (A10)} \\
& = & [(a' \to 0) \to (b \to a)']' & \text{by (I)} \\
& = & [((a \to 0) \to 0) \to (b \to a)']' & \\
& = & [(0 \to (0 \to a)) \to (b \to a)']' & \text{by (A10)} \\
& = & [(0 \to a) \to (b \to a)']' & \text{by 2.5 (c) and 2.6} \\
& = & [(0 \to a) \to [(b \to a) \to 0]]' & \\
& = & [(0 \to a) \to [0 \to (a \to b)]]' & \text{by (A10)} \\
& = & [(0 \to a) \to [(0 \to a) \to (0 \to b)]]' & \text{by 2.5 (f) and (d) and by 2.6} \\
& = & [(0 \to a) \to (0 \to b)]' & \text{by 2.5 (e) and 2.6} \\
& = & [0 \to (a \to b)]' & \text{by 2.5 (d), (f) and by 2.6.}
\end{array}$$

(b) Observe that $a \to b' \overset{3.10(a)}{=} [0 \to (a \to b)]' \overset{(A10)}{=} [(b \to a) \to 0]' = (b \to a)''$. Hence, $\mathbf{A} \models$ (b).

(c) Since $a \to b' \overset{3.10(a)}{=} [0 \to (a \to b)]' \overset{2.5(c)\,\&\,2.6}{=} [0 \to (0 \to (a \to b))]' \overset{(A10)}{=} [((a \to b) \to 0) \to 0]' = (a \to b)''' = (a \to b)'$, we conclude that $\mathbf{A} \models$ (c).

□

Lemma 3.11. *If* $\mathbf{A} \in \mathcal{A}_3 \cup \mathcal{A}_5 \cup \mathcal{A}_7 \cup \mathcal{A}_8 \cup \mathcal{A}_{10}$ *then* $\mathbf{A}$ *satisfies*

(1) $(x \to y)' \approx x \to (0 \to y)$,

(2) $x' \to y \approx x \to y'$ *and*

(3) $0 \to (x \to y) \approx 0 \to (y \to x)$.

Proof. Let $a, b \in A$.

- Suppose $\mathbf{A} \in \mathcal{A}_3$. Then $(a \to b)' = (a \to b) \to 0 \overset{(A3)}{=} a \to (0 \to b)$, implying $\mathbf{A} \models (1)$. Observe that $a' \to b = (a \to 0) \to b \overset{(A3)}{=} a \to (b \to 0) = a \to b'$. So, (2) holds in $\mathbf{A}$. Also, $0 \to (a \to b) \overset{(A3)}{=} (0 \to b) \to a \overset{(I)}{=} [(a' \to 0) \to (b \to a)']'$ $= [a'' \to (b \to a)']' \overset{2.6}{=} [a \to (b \to a)']' \overset{(3.11)}{=} [a' \to (b \to a)]' \overset{2.5(h)\,\&\,2.6}{=} (b \to a)' \overset{2.3(1)\,\&\,2.6}{=} 0' \to (b \to a)' \overset{3.11(2)}{=} 0'' \to (b \to a) = 0 \to (b \to a)$, proving that (3) holds in $\mathbf{A}$.

- Assume that $\mathbf{A} \in \mathcal{A}_5$. Then $(a \to b)' \overset{3.8(b)}{=} 0 \to (b \to a) = 0'' \to (b \to a) \overset{3.8(a)}{=} 0' \to (b \to a)' \overset{2.3(1)\,\&\,2.6}{=} (b \to a)' = (b \to a) \to 0 \overset{(A5)}{=} a \to (b \to 0) = a \to b' \overset{3.8(a)}{=} a' \to b \overset{2.3(1)\,\&\,2.6}{=} (0' \to a') \to b \overset{3.8(a)}{=} (0 \to a'') \to b \overset{2.6}{=} (0 \to a) \to b \overset{(A5)}{=} a \to (0 \to b)$, proving that (1) is true in $\mathbf{A}$. (2) is immediate from Lemma 3.8 (a). Next, $0 \to (a \to b) \overset{3.8(b)}{=} (a \to b)' = (a \to b) \to 0 \overset{(A5)}{=} b \to (a \to 0) = b \to a' \overset{3.8(a)}{=} b' \to a \overset{2.6}{=} (b' \to a)'' \overset{3.8(a)}{=} (b \to a')'' \overset{3.8(b)}{=} [0 \to (b \to a')]' \overset{3.8(c)}{=} [b \to (0 \to a')]' \overset{3.8(a)}{=} [b \to (0' \to a)]' \overset{2.3(1)\,\&\,2.6}{=} (b \to a)' \overset{3.8(b)}{=} 0 \to (b \to a)$, proving (3) holds in $\mathbf{A}$.

- Assume that $\mathbf{A} \in \mathcal{A}_7$. Then $(a \to b)' \overset{2.6}{=} (a \to b)''' = [(a \to b) \to 0]'' \overset{(A7)}{=} [0 \to (a \to b)]'' \overset{2.5(f)\,\&\,2.6}{=} [a \to (0 \to b)]'' \overset{(A7)}{=} [(0 \to b) \to a]'' \overset{2.6}{=} (0 \to b) \to a \overset{(A7)}{=} a \to (0 \to b)$, proving that $\mathbf{A}$ satisfies (1).

  Next, $a' \to b \overset{2.6}{=} [a' \to b]'' = [(a' \to b) \to 0]' \overset{(A7)}{=} [0 \to (a' \to b)]' \overset{2.6}{=} [0 \to (a' \to b'')]' \overset{2.5(g)\,\&\,2.6}{=} [0 \to (a \to b')']' = [0 \to ((a \to b') \to 0)]' \overset{(A7)}{=} [0 \to (0 \to (a \to b'))]' \overset{2.5(c)\,\&\,2.6}{=} [0 \to (a \to b')]' = [0 \to (a \to b')] \to 0 \overset{(A7)}{=} [(a \to b') \to 0] \to 0 = (a \to b')'' = (a \to (b \to 0))'' \overset{(A7)}{=} ((b \to 0) \to a)'' \overset{2.6}{=} (b \to 0) \to a \overset{(A7)}{=} a \to (b \to 0) = a \to b'$, proving that (2) is true in $\mathbf{A}$. Finally, observe that $0 \to (a \to b) \overset{(A7)}{=} (a \to b) \to 0 \overset{2.6}{=} (a'' \to b) \to 0 = [(a' \to 0) \to b] \to 0 \overset{(A7)}{=} [b \to (a' \to 0)] \to 0 = [b \to a''] \to 0 \overset{2.6}{=} [b \to a] \to 0 \overset{(A7)}{=} 0 \to (b \to a)$, proving that (3) holds in $\mathbf{A}$.

- Let $\mathbf{A} \in \mathcal{A}_8$. First, we will prove (2) hold in $\mathbf{A}$. Now, $a \to b' \overset{3.9(b)}{=} 0 \to (b' \to a) \overset{2.6}{=} 0 \to (b' \to a'') \overset{3.9(a)}{=} 0 \to (b'' \to a'') \overset{(A8)}{=} (a'' \to 0) \to b'' \overset{2.6}{=} (a'' \to 0) \to b = a''' \to b \overset{2.1}{=} a' \to b$, proving (2).

  Notice that $a \to (0 \to b) \overset{(A8)}{=} (b \to a) \to 0 = (b \to a)' \overset{2.6}{=} (b \to a)''' = (b \to a)'' \to 0 \overset{(2)}{=} (b \to a)' \to 0' \overset{2.3(1)\,\&\,2.6}{=} [0' \to (b \to a)'] \to 0' \overset{(2)}{=} [0'' \to (b \to a)] \to 0' = [0 \to (b \to a)] \to 0' \overset{2.5(f)\,\&\,2.6}{=} [b \to (0 \to a)] \to 0' \overset{(A8)}{=} [(a \to b) \to 0] \to 0' = (a \to b)' \to 0' \overset{(2)}{=} (a \to b) \to 0'' = (a \to b) \to 0 = (a \to b)'$, proving (1) holds in $\mathbf{A}$.

  The identity
  $$0 \to (x \to y) \approx (x \to y)' \tag{3.3}$$
  holds in $\mathbf{A}$, since $0 \to (a \to b) = 0'' \to (a \to b) \overset{(2)}{=} 0' \to (a \to b)' \overset{2.3(1)\,\&\,2.6}{=} (a \to b)'$.

Then $0 \to (a \to b) \overset{(3.3)}{=} (a \to b)' = (a \to b) \to 0 \overset{(A8)}{=} b \to (0 \to a) \overset{(1)}{=} (b \to a)'$ $\overset{(3.3)}{=} 0 \to (b \to a)$, proving (3) is true in $\mathbf{A}$.

- Assume that $\mathbf{A} \in \mathcal{A}_{10}$. Hence

$$\begin{array}{rcll}
a' \to b & = & a' \to b'' & \text{by 2.6} \\
& = & (a' \to b')' & \text{by 3.10 (c)} \\
& = & (a' \to b') \to 0 & \\
& = & 0 \to (b' \to a') & \text{by (A10)} \\
& = & 0 \to (b' \to (a \to 0)) & \\
& = & 0 \to ((0 \to a) \to b') & \text{by (A10)} \\
& = & (b' \to (0 \to a)) \to 0 & \text{by (A10)} \\
& = & (b' \to (0 \to a))' & \\
& = & ((b \to 0) \to (0 \to a))' & \\
& = & [(0 \to a) \to (0 \to b)]' & \text{by (A10)} \\
& = & [0 \to (a \to b)]' & \text{by 2.5 (f) \ \& \ (d) and by 2.6} \\
& = & [(b \to a) \to 0]' & \text{by (A10)} \\
& = & (b \to a)'' & \\
& = & a \to b' & \text{by 3.10 (b),}
\end{array}$$

proving (2) holds in $\mathbf{A}$.

Consider $a \to (0 \to b) \overset{(A10)}{=} (b \to 0) \to a \overset{(I)}{=} [(a' \to b) \to (0 \to a)']' \overset{3.10(c)}{=}$ $[(a' \to b) \to (0 \to a')]' \overset{(2)}{=} [(a' \to b) \to (0' \to a)]' \overset{2.3(1)\,\&\,2.6}{=} [(a' \to b) \to a]'$ $\overset{(2)}{=} [(a \to b') \to a]' \overset{2.5(b)\,\&\,2.6}{=} [(0 \to b') \to a]' \overset{(A10)}{=} [a \to (b' \to 0)]' = [a \to b'']'$ $\overset{2.6}{=} (a \to b)'$, proving (1).

To finish off the proof, $0 \to (a \to b) \overset{(A10)}{=} (b \to a) \to 0 \overset{3.10(c)}{=} b \to a'$ $\overset{(2)}{=} b' \to a = (b \to 0) \to a \overset{(A10)}{=} a \to (0 \to b) \overset{(1)}{=} (a \to b)' = (a \to b) \to 0$ $\overset{(A10)}{=} 0 \to (b \to a)$.

□

Theorem 3.12. $\mathcal{A}_3 = \mathcal{A}_5 = \mathcal{A}_7 = \mathcal{A}_8 = \mathcal{A}_{10}$.

Proof. Let $\mathbf{A} \in \mathcal{A}_3 \cup \mathcal{A}_5 \cup \mathcal{A}_7 \cup \mathcal{A}_8 \cup \mathcal{A}_{10}$. By Lemma 3.11 we have that $\mathbf{A}$ is of type 1. Then, using Theorem 3.7, $\mathbf{A} \in \mathcal{A}_j$ for all $j \in \{3, 5, 7, 8, 10\}$. □

Lemma 3.13. *If* $\mathbf{A} \in \mathcal{A}_{13}$ *then* $\mathbf{A}$ *satisfies*

(a) $(x \to y)' \approx (0 \to x) \to y$,

(b) $(x \to y)' \approx x' \to y'$,

(c) $(x \to y)' \approx (0 \to y) \to x'$,

(d) $(x \to y)' \approx (x \to y)''$,

(e) $(x \to y)' \approx (y \to x)'$.

Proof. Let us consider $a, b \in A$.

(a) $(a \to b)' = (a \to b) \to 0 \overset{(A13)}{=} (b \to 0) \to a \overset{(A13)}{=} (0 \to a) \to b$. Hence $\mathbf{A} \models$ (a).

(b) Observe that $(a \to b)' \overset{2.6}{=} 0' \to (a \to b)' = (0 \to 0) \to (a \to b)' \overset{(A13)}{=} [0 \to (a \to b)'] \to 0 \overset{(I20)\,\&\,2.6}{=} [0 \to (a \to b'')'] \to 0 \overset{2.5(g)\,\&\,2.6}{=} [0 \to (a' \to b')] \to 0 \overset{(A13)}{=} [(a' \to b') \to 0] \to 0 = (a' \to b')'' \overset{2.6}{=} a' \to b'$

(c) Observe $(a \to b)' \overset{(a)}{=} (0 \to a) \to b \overset{2.6}{=} (0 \to a)'' \to b \overset{(b)}{=} (0' \to a')' \to b \overset{2.6}{=} a'' \to b \overset{(A13)}{=} (0 \to b) \to a'$.

(d) Note that $(a \to b)' \overset{(c)}{=} (0 \to b) \to a' \overset{(I)}{=} [(a'' \to 0) \to (b \to a')']' = [a''' \to (b \to a')']' \overset{2.1}{=} [a' \to (b \to a')']' \overset{(b)}{=} [a' \to (b' \to a'')]' \overset{2.6}{=} [a' \to (b' \to a)]' \overset{2.5(h)\,\&\,2.6}{=} (b' \to a)' \overset{(A13)}{=} ((0 \to a) \to b)' \overset{(A13)}{=} ((a \to b) \to 0)' = (a \to b)''$.

(e) We have $(b \to a)' \overset{(c)}{=} (0 \to a) \to b' \overset{2.6}{=} [(0 \to a) \to b']'' \overset{(b)}{=} [(0 \to a)' \to b'']' \overset{(d)}{=} [(0 \to a)'' \to b'']' \overset{2.6}{=} [(0 \to a) \to b]' \overset{(d)}{=} [(0 \to a) \to b]'' \overset{2.6}{=} (0 \to a) \to b \overset{(a)}{=} (a \to b)'$.

□

Theorem 3.14. $\mathcal{A}_{11} = \mathcal{A}_{12} = \mathcal{A}_{13}$.

Proof. Let us consider $\mathbf{A} \in \mathcal{A}_{11}$ and $a, b, c \in A$. Hence $(a \to b) \to c \overset{2.3(1)\,\&\,2.6}{=} ((0' \to a) \to b) \to c \overset{(A11)}{=} ((0' \to b) \to a) \to c \overset{2.3(1)\,\&\,2.6.}{=} (b \to a) \to c$. Hence, $\mathbf{A} \in \mathcal{A}_{12}$, implying $A_{11} \subseteq A_{12}$.

Now assume that $\mathbf{A} \in \mathcal{A}_{12}$ and $a, b, c \in A$. Then $(a \to b) \to c \overset{(I)}{=} [(c' \to a) \to (b \to c)']' \overset{(I)}{=} \{[(b \to c)'' \to c'] \to [a \to (b \to c)']'\}'' \overset{2.6}{=} \{[(b \to c) \to c'] \to [a \to (b \to c)']'\}'' \overset{(A12)}{=} \{[c' \to (b \to c)] \to [a \to (b \to c)']'\}'' \overset{2.5(h)\,\&\,2.6}{=} \{(b \to c) \to [a \to (b \to c)']'\}'' \overset{(A12)}{=} \{[a \to (b \to c)']' \to (b \to c)\}'' = \{[[a \to (b \to c)'] \to 0] \to (b \to c)\}'' \overset{(A12)}{=} \{[0 \to [a \to (b \to c)']] \to (b \to c)\}'' \overset{2.5(b)\,\&\,2.6}{=} \{[(b \to c) \to [a \to (b \to c)']] \to (b \to c)\}'' \overset{2.5(h)\,\&\,2.6}{=} \{[a \to (b \to c)'] \to (b \to c)\}'' \overset{2.5(a)\,\&\,2.6}{=} \{[a \to 0'] \to (b \to c)\}'' \overset{(A12)}{=} \{[0' \to a] \to (b \to c)\}'' \overset{2.3(1)\,\&\,2.6}{=} \{a \to (b \to c)\}'' \overset{(A12)}{=} \{(b \to c) \to a\}'' \overset{2.6}{=} (b \to c) \to a$, which implies that $A_{12} \subseteq A_{13}$.

If $\mathbf{A} \in \mathcal{A}_{13}$ and $a, b, c \in A$, then $(a \to b) \to c \overset{(A13)}{=} (b \to c) \to a \overset{(A13)}{=} (c \to a) \to b \overset{2.6}{=} (c \to a)'' \to b \overset{3.13(e)}{=} (a \to c)'' \to b \overset{2.6}{=} (a \to c) \to b$, concluding that $A_{13} \subseteq A_{11}$. □

4. Main theorem

In this section we will prove our main theorem. But first we need one more lemma.

Lemma 4.1. *If* $\mathbf{A} \in \mathcal{A}_2 \cup \mathcal{A}_6 \cup \mathcal{A}_9$ *then* $\mathbf{A} \in \mathcal{A}_{11}$.

Proof. We will see that $\mathbf{A} \models (x \to y)' \approx (y \to x)'$.

Let $a, b \in A$.

- If $\mathbf{A} \in \mathcal{A}_2$, $(a \to b) \to 0 \overset{2.3(1)\,\&\,2.6}{=} (0' \to (a \to b)) \to 0 \overset{(A2)}{=} (0' \to (b \to a)) \to 0 \overset{2.3(1)\,\&\,2.6}{=} (b \to a) \to 0$.
- If $\mathbf{A} \in \mathcal{A}_6$, $(a \to b) \to 0 \overset{2.3(1)\,\&\,2.6}{=} (0' \to (a \to b)) \to 0 \overset{(A6)}{=} (a \to (b \to 0')) \to 0 \overset{(A6)}{=} (b \to (0' \to a)) \to 0 \overset{2.3(1)\,\&\,2.6}{=} (b \to a) \to 0$.
- If $\mathbf{A} \in \mathcal{A}_9$, then

$$
\begin{array}{rcll}
(a \to b)' & = & (a \to b) \to 0 & \\
 & = & (a \to (0' \to b)) \to 0 & \text{by 2.3 (1) and 2.6} \\
 & = & (b \to (0' \to a)) \to 0 & \text{by (A9)} \\
 & = & (b \to a) \to 0 & \text{by 2.3 (1) and 2.6} \\
 & = & (b \to a)' &
\end{array}
$$

Now, apply Lemma 3.5, to get $\mathbf{A} \in A_{12}$. Therefore, using Theorem 3.14, we conclude $\mathbf{A} \in \mathcal{A}_{11}$. ☐

We are now ready to present the main theorem of this paper.

Theorem 4.2. *We have*

(a) *The following are the* 8 *subvarieties of* $\mathcal{I}$ *of associative type that are distinct from each other.*

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_{11} \text{ and } \mathcal{A}_{14}.$$

(b) *They satisfy the following relationships:*

1. $\mathcal{SL} \subset \mathcal{A}_3 \subset \mathcal{A}_4$,
2. $\mathcal{BA} \subset \mathcal{A}_4 \subset \mathcal{I}$,
3. $\mathcal{A}_3 \subset \mathcal{A}_1 \subset \mathcal{I}$,
4. $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_{11}$, $\mathcal{A}_3 \subset \mathcal{A}_6 \subset \mathcal{A}_{11}$ *and* $\mathcal{A}_3 \subset \mathcal{A}_9 \subset \mathcal{A}_{11}$,
5. $\mathcal{A}_{11} \subset \mathcal{A}_{14} \subset \mathcal{I}$.

Proof. Observe that, in view of Theorem 3.12 and Theorem 3.14 we can conclude that each of the 14 subvarieties of associative type of $\mathcal{I}$ is equal to one of the following varieties:

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_{11}, \mathcal{A}_{14}.$$

We first wish to prove (b). Notice that by Lemma 3.11 we have that $\mathbf{A} \in \mathcal{A}_3$ is of type 1. Then, using Theorem 3.7, $\mathbf{A} \in \mathcal{A}_j$ for all $1 \leqslant j \leqslant 14$. Hence

$$\mathcal{A}_3 \subseteq \mathcal{A}_j \text{ for all } 1 \leqslant j \leqslant 14. \tag{4.4}$$

1. Recall that $\mathcal{SL} = \mathcal{C} \cap I_{1,0}$. Then, we get $\mathcal{C} \subseteq \mathcal{A}_1$ and $\mathcal{I}_{1,0} \subseteq \mathcal{A}_1$ by [7, Theorem 8.2] and [7, Theorem 9.3], respectively, implying $\mathcal{SL} \subseteq \mathcal{A}_3$, and $\mathcal{A}_3 \subseteq \mathcal{A}_4$ by (4.4).

   The algebras $\mathbf{2_z}$ and $\mathbf{2_b}$ show that $\mathcal{SL} \neq \mathcal{A}_3$ and $\mathcal{A}_3 \neq \mathcal{A}_4$, respectively.

2. In view of [10] we have that $\mathcal{BA} \subset \mathcal{S}$. By [9, Lemma 3.1], $\mathcal{S} \models x \to (y \to z) \approx y \to (x \to z)$. Thus, $\mathcal{BA} \subseteq \mathcal{A}_4$.

   The algebra $\mathbf{2_s}$ shows that $\mathcal{BA} \neq \mathcal{A}_4$ and the following algebra shows that $\mathcal{A}_4 \neq \mathcal{I}$, respectively.

$\to$	0	1	2
0	0	0	0
1	2	0	0
2	0	0	0

3. The algebra $\mathbf{2_b}$ shows that $\mathcal{A}_1 \neq \mathcal{I}$ and the following algebra witnesses that $\mathcal{A}_3 \neq \mathcal{A}_1$.

$\to$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

4. Using (4.4) and Lemma 4.1 we can conclude that $\mathcal{A}_3 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_{11}$, $\mathcal{A}_3 \subseteq \mathcal{A}_6 \subseteq \mathcal{A}_{11}$ and $\mathcal{A}_3 \subseteq \mathcal{A}_9 \subseteq \mathcal{A}_{11}$.

   The following algebras show that $\mathcal{A}_3 \neq \mathcal{A}_2$ and $\mathcal{A}_2 \neq \mathcal{A}_{11}$, respectively.

$\to$	0	1	2
0	0	0	0
1	2	0	2
2	0	0	0

$\to$	0	1	2
0	0	0	0
1	2	0	0
2	0	0	0

   The following algebras show that $\mathcal{A}_3 \neq \mathcal{A}_6$ and $\mathcal{A}_6 \neq \mathcal{A}_{11}$, respectively.

$\to$	0	1	2	3
0	0	0	0	0
1	0	2	3	0
2	0	0	0	0
3	0	0	0	0

$\to$	0	1	2
0	0	0	0
1	2	0	0
2	0	0	0

   The following algebras show that $\mathcal{A}_3 \neq \mathcal{A}_9$ and $\mathcal{A}_9 \neq \mathcal{A}_{11}$, respectively.

$\to$	0	1	2	3
0	0	0	0	0
1	0	2	3	0
2	0	0	0	0
3	0	0	0	0

$\to$	0	1	2
0	0	0	0
1	2	0	0
2	0	0	0

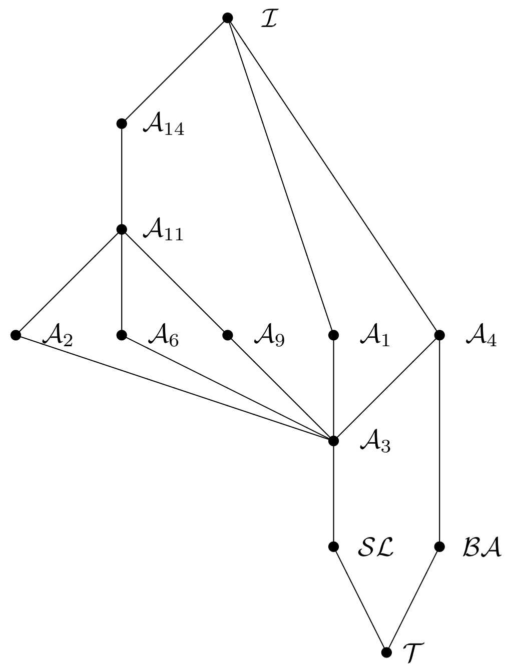
5. Let $\mathbf{A} \in \mathcal{A}_{11}$ and $a, b, c \in A$. By Theorem 3.14, $\mathcal{A}_{11} = \mathcal{A}_{12} = \mathcal{A}_{13}$. Hence $(a \to b) \to c \overset{(A13)}{=} (b \to c) \to a \overset{(A12)}{=} (c \to b) \to a$. Therefore $\mathcal{A}_{11} \subseteq \mathcal{A}_{14}$.

   The algebra $\mathbf{2_b}$ shows that $\mathcal{A}_{14} \neq \mathcal{I}$ and the following algebra shows that $\mathcal{A}_{11} \neq \mathcal{A}_{14}$.

$\to$	0	1	2	3
0	0	1	2	3
1	2	3	2	3
2	1	1	3	3
3	3	3	3	3

The proof of the theorem is now complete since (a) is an immediate consequence of (b). $\square$

The Hasse diagram of the poset of subvarieties of $\mathcal{I}$ of associative type, together with $\mathcal{SL}$ and $\mathcal{BA}$, is:



5. Identities in symmetric implication zroupoids

Let $\mathbf{A} \in \mathcal{I}$. $\mathbf{A}$ is involutive if $\mathbf{A} \in \mathcal{I}_{2,0}$. $\mathbf{A}$ is meet-commutative if $\mathbf{A} \in \mathcal{MC}$. $\mathbf{A}$ is symmetric if $\mathbf{A}$ is both involutive and meet-commutative. Let $\mathcal{S}$ denote the variety of symmetric $\mathcal{I}$-zroupoids. In other words, $\mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{MC}$. The variety $\mathcal{S}$ was investigated in [7], [9] and [10] and has some interesting properties.

In this section we give an application of the main theorem, Theorem 4.2, to describe the poset of the subvarieties of the variety $\mathcal{S}$.

Lemma 5.1. [9, Lemma 3.1 (a)] *Let* $\mathbf{A} \in \mathcal{S}$*. Then* $\mathbf{A}$ *satisfies* $x \to (y \to z) \approx y \to (x \to z)$.

Lemma 5.2. [9, Lemma 2.1] $\mathcal{MC} \cap \mathcal{I}_{1,0} \subseteq \mathcal{C} \cap \mathcal{I}_{1,0} = \mathcal{SL}$.

Lemma 5.3. [9, Lemma 3.2] *Let* $\mathbf{A} \in \mathcal{S}$ *such that* $\mathbf{A} \models x \to x \approx x$*. Then* $\mathbf{A} \models x' \approx x$.

Lemma 5.4. $\mathcal{A}_{11} \cap \mathcal{S} = \mathcal{SL}$.

Proof. Let $\mathbf{A} \in \mathcal{A}_{11} \cap \mathcal{S}$ and $a \in A$. Since $\mathcal{S} \subseteq \mathcal{I}_{2,0}$, we have

$$\begin{array}{rcll} a & = & a' \to a & \text{by Lemma 2.3 (4)} \\ & = & (0' \to a') \to a & \text{by Lemma 2.3 (1)} \\ & = & (0' \to a) \to a' & \text{by (A11)} \\ & = & a \to a' & \text{by Lemma 2.3 (1)} \\ & = & a'' \to a' & \\ & = & a' & \text{by Lemma 2.3 (4).} \end{array}$$

Therefore, $\mathbf{A} \models x \approx x'$. Then, by Lemma 5.2, $\mathbf{A} \in \mathcal{SL}$. □

Lemma 5.5. $\mathcal{A}_1 \cap \mathcal{S} \subseteq \mathcal{SL}$.

Proof. Let $\mathbf{A} \in \mathcal{A}_1 \cap \mathcal{S}$ and $a \in A$. Then

$$\begin{array}{rcll} a & = & 0' \to a & \text{by Lemma 2.3 (1)} \\ & = & (0 \to 0) \to a & \\ & = & 0 \to (0 \to a) & \text{by (A1)} \\ & = & 0 \to a & \text{by Lemma 2.5 (c).} \end{array}$$

Consequently,

$$\mathbf{A} \models x \approx 0 \to x. \tag{5.5}$$

Therefore,

$$\begin{array}{rcll} a & = & a' \to a & \text{by Lemma 2.3 (4)} \\ & = & (a \to 0) \to a & \\ & = & a \to (0 \to a) & \text{by (A1)} \\ & = & a \to a & \text{by equation (5.5)} \end{array}$$

Thus, by Lemma 5.3, $\mathbf{A} \models x' \approx x$. Using Lemma 5.2 we can conclude the proof. □

We will denote by $\mathcal{S}_i$ the variety $\mathcal{A}_i \cap \mathcal{S}$ with $1 \leqslant i \leqslant 14$.

Proposition 5.6. *Each of the* 14 *subvarieties of associative type of* $\mathcal{S}$ *is equal to one of the following varieties:*

$$\mathcal{SL},\ \mathcal{S}_{14},\ \mathcal{S}.$$

Proof. From Theorem 3.12 and Theorem 3.14 we know that each of the 14 subvarieties of associative type of $\mathcal{I}$ is equal to one of the following varieties:

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_{11}, \mathcal{A}_{14}.$$

Using Theorem 4.2, Lemma 5.4 and Lemma 5.5 we have that

$$\mathcal{SL} \subseteq \mathcal{S}_3 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_{11} \subseteq \mathcal{SL},$$

$$\mathcal{SL} \subseteq \mathcal{S}_6 \subseteq \mathcal{S}_{11} \subseteq \mathcal{SL},$$

$$\mathcal{SL} \subseteq \mathcal{S}_9 \subseteq \mathcal{S}_{11} \subseteq \mathcal{SL}$$

and

$$\mathcal{SL} \subseteq \mathcal{S}_1 \subseteq \mathcal{SL}$$

By Lemma 5.1, $\mathcal{A}_4 = \mathcal{S}$, So, $\mathcal{S}_4 = \mathcal{S}$. □

We are now ready to present the main theorem of this section.

Theorem 5.7. *We have*

(a) *The following are the* 3 *subvarieties of* $\mathcal{S}$ *of associative type that are distinct from each other.*

$$\mathcal{SL},\ \mathcal{S}_{14},\ \mathcal{S}.$$

(b) *They satisfy the following relationships*

1. $\mathcal{SL} \subset \mathcal{S}_{14} \subset \mathcal{S}$,
2. $\mathcal{BA} \not\subset \mathcal{S}_{14}$.

Proof. We first prove (b).

1. By Theorem 4.2, $\mathcal{SL} \subseteq \mathcal{S}_{14}$.

   The following algebras show that $\mathcal{SL} \neq \mathcal{S}_{14}$ and $\mathcal{S}_{14} \neq \mathcal{S}$, respectively.

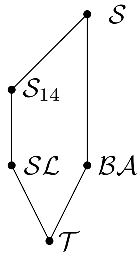
$\to$	0	1	2	3
0	0	1	2	3
1	2	3	2	3
2	1	1	3	3
3	3	3	3	3

$\to$	0	1
0	1	1
1	0	1

2. Since $\mathbf{2_b} \not\models (S_{14})$, it follows that $\mathcal{BA} \not\subseteq \mathcal{S}_{14}$.

The proof of the theorem is now complete since (a) is an immediate consequence of Proposition 5.6 and (b). □

The Hasse diagram of the poset of subvarieties of $\mathcal{S}$ of associative type, together with $\mathcal{BA}$, is:



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