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A characterization of almost simple groups related to $L_3(37)$

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Abstract. Let G be a finite group, and let $\Gamma(G)$ be its prime graph. The degree pattern of G is denoted by $\mathsf{D}(G) = (\deg(p_1), \dots, \deg(p_k))$, where $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $\deg(p_i)$ is the degree of vertex p_i in $\Gamma(G)$. The group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups H satisfying $|G| = |H|$ and $\mathsf{D}(G) = \mathsf{D}(H)$. In this paper, we characterize all finite groups with the same order and degree pattern as almost simple groups related to the projective special linear group $L_3(37)$.

1. Introduction

Let G be a finite group. We denote by $\omega(G)$ the set of orders of elements of G and by $\pi(G)$ the set of prime divisors of the order of G. The *spectrum* $\mu(G)$ of G is the set of elements of $\omega(G)$ that are maximal with respect to divisibility relation. Let $\pi(G) = \{p_1, \dots, p_k\}$. The prime graph $\Gamma(G)$ of a group G is the graph whose vertex set is $\pi(G)$ and two distinct primes p and q are adjacent (we write $p \sim q$) if and only if G contains an element of order pq, that is to say, $pq \in \omega(G)$. For $p \in \pi(G)$, the degree $\deg(p)$ of p is the degree of the vertex p in $\Gamma(G)$, that is to say, the number of vertices $q \in \pi(G)$ which are adjacent to p. If $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then we denote $\mathsf{D}(G) := (\deg(p_1), \deg(p_2), \dots, \deg(p_k))$, where $p_1 < p_2 < \dots < p_k$. This k-tuple is called the degree pattern of G. A group G is called *k-fold OD-characterizable* if there exist exactly k non-isomorphic finite groups having the same order and degree pattern as G. In particular, a 1-fold OD-characterizable group is simply called *OD-characterizable*. A group G is said to be an almost simple group related to L if and only if $L \trianglelefteq G \lesssim \mathsf{Aut}(L)$ for some non-abelian simple group L.

The notion of degree patterns of prime graphs and related topics has been introduced in [9]. There are natural questions mentioned in [9] about the structure of finite groups with the same degree patterns and the same orders:

Let G and M be finite groups satisfying the conditions (1) $|G| = |M|$ *and* (2) $D(G) = D(M)$.

(i) *How far do these conditions affect the structure of G?*

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(ii) *Is the number of non-isomorphic groups satisfying (1) and (2) finite?*

It is therefore important to investigate the number of non-isomorphic groups satisfying conditions (1) and (2) for important families of groups M. In a series of articles, it has been proved that some finite almost simple groups are OD-characterizable or k-fold OD-characterizable for $k \geqslant 2$, for example see [5, 11, 15]. Note in passing that a few classes of finite simple groups have been in general characterized by their degree patterns and orders, see for example [9, 13, 16]. To our knowledge, the lack of information on the spectra of almost simple groups is the main reason which makes this characterization somehow difficult in general argument but the situation for some simple groups is rather different as the spectra of these groups are known, see for example [2, 3, 8].

Motivated by [4], in this paper, we focus on groups related to $L_3(37)$ and show that $L_3(37)$ and $L_3(37) : \mathbb{Z}_2$ are OD-characterizable while $L_3(37) : \mathbb{Z}_3$ and $L_3(37) : S_3$ are 3-fold and 5-fold OD-characterizable, respectively. Indeed, we prove that

Theorem 1.1. *Let H be an almost simple group related to $L := L_3(37)$. If G is a finite group such that $\mathsf{D}(G) = \mathsf{D}(H)$ and $|G| = |H|$, then the following hold:*

(*a*) *If $H = L$, then $G \cong H$.*

(*b*) *If $H = L : \mathbb{Z}_2$, then $G \cong H$.*

(*c*) *If $H = L : \mathbb{Z}_3$, then G is isomorphic to H, $\mathbb{Z}_3 \times L$ or $\mathbb{Z}_3 \cdot L$ (non-split).*

(*d*) *If $H = L : S_3$, then G is isomorphic to H, $\mathbb{Z}_3 \times (L : \mathbb{Z}_2)$, $\mathbb{Z}_3 \cdot (L : \mathbb{Z}_2)$ (non-split), $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ or $(\mathbb{Z}_3 : L) \cdot \mathbb{Z}_2$.*

Throughout this article, all groups under consideration are finite. For $p \in \pi(G)$, we denote by G_p and $Syl_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G, respectively. All further definitions and notation are standard and can be found in [1, 7].

2. Preliminaries

In this section, we mention some useful results to be used in proof of Theorem 1.1. Here the independence number $\alpha(\Gamma)$ of a graph Γ is the maximum cardinality of an independent set among all independent sets of Γ. Let now G be a finite group, and let $\Gamma(G)$ be its prime graph. Then we set $\alpha(G) := \alpha(\Gamma(G))$. Moreover, for a vertex $r \in \pi(G)$, let $\alpha(r, G)$ denote the maximal number of vertices in independent sets of $\Gamma(G)$ containing r.

Lemma 2.1. [12, Theorem 1] *Let G be a finite group with $\alpha(G) \geqslant 3$ and $\alpha(2, G) \geqslant 2$, and let K be the maximal normal solvable subgroup of G. Then the quotient group G/K is an almost simple group, that is, there exists a finite non-abelian simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$.*

Lemma 2.2. [10, Theorem 1] *Let S be a finite non-abelian simple group, and let p be the largest prime divisor of $|S|$ with $|S|_p = p$. Then $p \nmid |\mathsf{Out}(S)|$.*

Lemma 2.3. *The orders, spectra and degree patterns of the almost simple groups related to $L_3(37)$ are as in Table 1.*

Proof. Note that $\mathsf{Aut}(L_3(37)) \cong L_3(37) : S_3$. So if G is an almost simple group related to $L_3(37)$, then G is isomorphic to one of the groups $L_3(37)$, $L_3(37) : \mathbb{Z}_2$, $L_3(37) : \mathbb{Z}_3$, $L_3(37) : S_3$. The result for $H = L_3(37)$ can be obtained by [8, Theorem 9] and for the remaining groups we use GAP [6]. □

Table 1: The orders, spectra and degree patterns of H, where H is an almost simple group related to $L_3(37)$.

H	$\lvert H\rvert$	$\mu(H)$	$\mathsf{D}(H)$
$L_3(37)$	$2^5 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{7\cdot 67, 2^3\cdot 3\cdot 19, 2^2\cdot 3^2, 2^2\cdot 3\cdot 37\}$	$(3,3,1,2,2,1)$
$L_3(37):\mathbb{Z}_2$	$2^6 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{7\cdot 67, 2^3\cdot 3\cdot 19, 2^3\cdot 3^2, 2^2\cdot 3\cdot 37\}$	$(3,3,1,2,2,1)$
$L_3(37):\mathbb{Z}_3$	$2^5 \cdot 3^5 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{3\cdot 7\cdot 67, 2^3\cdot 3^2\cdot 19, 2^2\cdot 3^2\cdot 37\}$	$(3,5,2,2,2,2)$
$L_3(37):S_3$	$2^6 \cdot 3^5 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{3\cdot 7\cdot 67, 2^3\cdot 3^2\cdot 19, 2^2\cdot 3^2\cdot 37\}$	$(3,5,2,2,2,2)$

3. Proof of the main result

In this section, we prove Theorem 1.1 through a series of Lemmas and propositions. Observe that Theorem 1.1(a) follows from [4, Proposition 3.4]. Therefore in what follows we deal with the remaining cases.

Proposition 3.1. *Let $H := L : \mathbb{Z}_2$ where $L := L_3(37)$. If $|G| = |H|$ and $\mathsf{D}(G) = \mathsf{D}(H)$, then $G \cong H$.*

Proof. Note by Table 1 that $|G| = 2^6\cdot 3^4\cdot 7\cdot 19\cdot 37^3\cdot 67$ and $\mathsf{D}(G) = (3,3,1,2,2,1)$. Since $\deg(7) = 1$, there exists the unique prime $p_2 \in \pi(G)$ such that 7 is adjacent to p_2. Since also $|\pi(G)| = 6$, there are four more primes which are not adjacent to 7. If these four vertices, say p_3, p_4, p_5 and p_6, are pairwise adjacent, then the degrees of the vertices p_3, p_4, p_5 and p_6 are at least 3, which is impossible. Hence there exist at least two non-adjacent vertices p_3 and p_4. Let $\Delta = \{7, p_3, p_4\}$ be an independent set in $\Gamma(G)$. Then $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.1, there is a non-abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $67 \notin \pi(K)$. Assume to the contrary, that is, $67 \in \pi(K)$. We prove that p would be adjacent to 67, where $p \in \{7, 19\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 67$, and so p is adjacent to 67. If $p \notin \pi(K)$, then it follows from Frattini argument that $G = K\mathbf{N}_G(P)$, where P is a Sylow 67-subgroup of K, and so $\mathbf{N}_G(P)$ has an element x of order p. Thus $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 67$. Therefore both 7 and 19 are adjacent to 67 which contradicts the fact that the degree of 67 is 1. Therefore, $67 \notin \pi(K)$, and hence $\pi(K) \subseteq \{2, 3, 7, 19, 37\}$.

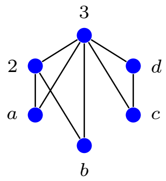
Now we prove that S is isomorphic to L. By Lemma 2.2, $67 \notin \pi(\mathsf{Out}(S))$, then $67 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $67 \in \pi(S)$. Therefore by [14, Table 1], S is isomorphic to L as claimed. Moreover, since $|G| = |L : \mathbb{Z}_2| = 2|L|$, we deduce that K is isomorphic to 1 or $\mathbb{Z}_2$.

If K is isomorphic to $\mathbb{Z}_2$, then $G = \mathbf{C}_G(K)$ as $G/\mathbf{C}_G(K)$ isomorphic to a subgroup of $\mathsf{Aut}(K) = 1$. Therefore $K \leqslant Z(G)$ which implies that $\deg(2) = 5$, which is a contradiction. Thus $K = 1$, and so $G \cong L : \mathbb{Z}_2$. □

Proposition 3.2. *Let $H := L : \mathbb{Z}_3$ where $L := L_3(37)$. If $|G| = |H|$ and $\mathsf{D}(G) = \mathsf{D}(H)$, then G is isomorphic to one of the groups H, $\mathbb{Z}_3 \times L$ and $\mathbb{Z}_3 \cdot L$ (non-split).*

Proof. Note by Table 1 that $|G| = 2^5 \cdot 3^5 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$ and $\mathsf{D}(G) = (3, 5, 2, 2, 2, 2)$. Therefore, $\Gamma(G)$ must be the graph as in Figure 1 in which $\{a, b, c, d\} = \{7, 19, 37, 67\}$.

Figure 1: The prime graph $\Gamma(G)$ of G in Propositions 3.2 and 3.3.



We observe by Figure 1 that $\{a, b, c\}$ is an independent set in $\Gamma(G)$, and so $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.1, there is a finite non-abelian simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. By the same argument as in Proposition 3.1, we can show that $67 \notin \pi(K)$, and so $\pi(K) \subseteq \{2, 3, 7, 19, 37\}$. It follows from Lemma 2.2, $67 \notin \pi(\mathsf{Out}(S))$, then $67 \in \pi(S)$. Therefore by [14, Table 1], S is isomorphic to L. Thus $L \leqslant G/K \leqslant \mathsf{Aut}(L)$, and so $|K| = 1$ or 3, which implies that K is isomorphic to 1 or $\mathbb{Z}_3$.

If $K = 1$, then since $L \leqslant G/K \leqslant \mathsf{Aut}(L)$ and $|G| = |L : \mathbb{Z}_3|$, we conclude that G is isomorphic to $L : \mathbb{Z}_3$.

If K is isomorphic to $\mathbb{Z}_3$, then $G/K \cong L$. In this case, we have that $G/\mathbf{C}_G(K) \leqslant \mathsf{Aut}(K) = \mathbb{Z}_2$. Thus $|G/\mathbf{C}_G(K)|$ is 1 or 2. If $|G/\mathbf{C}_G(K)| = 2$, then K is a proper subgroup of $\mathbf{C}_G(K)$, and so $1 \neq \mathbf{C}_G(K)/K \trianglelefteq G/K \cong L$. This implies that $G = \mathbf{C}_G(K)$, which is a contradiction. Therefore, $|G/\mathbf{C}_G(K)| = 1$. Then $K \leqslant Z(G)$, that is to say, G is a central extension of $\mathbb{Z}_3$ by L. If G splits over K, then G is isomorphic to $\mathbb{Z}_3 \times L$, otherwise, G is isomorphic to $\mathbb{Z}_3 \cdot L$ (non-split). □

Proposition 3.3. *Let $H := L : S_3$ where $L := L_3(37)$. If $|G| = |H|$ and $\mathsf{D}(G) = \mathsf{D}(H)$, then G is isomorphic to one of the groups H, $\mathbb{Z}_3 \times (L : \mathbb{Z}_2)$, $\mathbb{Z}_3 \cdot (L : \mathbb{Z}_2)$ (non-split), $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ and $(\mathbb{Z}_3 : L) \cdot \mathbb{Z}_2$.*

Proof. According to Table 1, we have that $|G| = 2^6 \cdot 3^5 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$ and $\mathsf{D}(G) = (3, 5, 2, 2, 2, 2)$. Therefore, the prime graph $\Gamma(G)$ is the graph as in Figure 1, where $\{a, b, c\}$ forms an independent set in $\Gamma(G)$, and so $\alpha(G) \geqslant 3$. Moreover, $\alpha(2, G) \geqslant 2$ as $\deg(2) = 3$ and $|\pi(G)| = 6$. Now we apply Lemma 2.1 and conclude that there is a finite non-abelian simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. Again, by the same manner as in Proposition 3.1, we have that $67 \notin \pi(K)$, and hence $\pi(K) \subseteq \{2, 3, 7, 19, 37\}$. By Lemma 2.2, $67 \notin \pi(\mathsf{Out}(S))$, and since $67 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, it follows that $67 \in \pi(S)$, and so by [14, Table 1], S is isomorphic to L. Thus $L \leqslant G/K \leqslant \mathsf{Aut}(L)$ implying that $|K| \in \{1, 2, 3, 6\}$. Hence K is isomorphic to one of the groups 1, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$ and S_3.

If $K = 1$, then since $L \leqslant G/K \leqslant \mathsf{Aut}(L)$ and $|G| = 6|L|$, we conclude that G is isomorphic to $L : S_3$.

If K is isomorphic to $\mathbb{Z}_2$, then K is central in G, and so $\deg(2) = 5$ in $\Gamma(G)$, which is a contradiction.

If K is isomorphic to $\mathbb{Z}_3$, then $K \leqslant \mathbf{C}_G(K)$ and $G/K \cong L : \mathbb{Z}_2$, and so $G/\mathbf{C}_G(K)$ is isomorphic to a subgroup of $\mathsf{Aut}(K) \cong \mathbb{Z}_2$. Thus, $|G/\mathbf{C}_G(K)| = 1$ or 2. If $|G/\mathbf{C}_G(K)| = 1$, then $K \leqslant Z(G)$, that is to say, G is a central extension of $\mathbb{Z}_3$ by $L : \mathbb{Z}_2$. This implies that G is isomorphic to $\mathbb{Z}_3 \times (L : \mathbb{Z}_2)$ or $\mathbb{Z}_3 \cdot (L : \mathbb{Z}_2)$ (non-split). If $|G/\mathbf{C}_G(K)| = 2$, then K is a proper subgroup of $\mathbf{C}_G(K)$, and so $\mathbf{C}_G(K)/K$ is a nontrivial normal subgroup of $G/K \cong L : \mathbb{Z}_2$. Thus $\mathbf{C}_G(K)/K \cong L$. Since $K \leqslant Z(\mathbf{C}_G(K))$, it follows that $\mathbf{C}_G(K)$ is a central extension of K by L, and hence $\mathbf{C}_G(K)$ is isomorphic to $\mathbb{Z}_3 \times L$ or $\mathbb{Z}_3 \cdot L$ (non-split). Therefore G is isomorphic to $(\mathbb{Z}_3 \times L) \cdot \mathbb{Z}_2$ or $(\mathbb{Z}_3 \cdot L) \cdot \mathbb{Z}_2$.

If K is isomorphic to $\mathbb{Z}_6$, then $K \leqslant \mathbf{C}_G(K)$ and $G/K \cong L$. Since $G/\mathbf{C}_G(K)$ is isomorphic to a subgroup of $\mathbb{Z}_2$, it follows that $|G/\mathbf{C}_G(K)| = 1$ or 2. If $|G/\mathbf{C}_G(K)| = 1$, then $K \leqslant Z(G)$, and so $\deg(2) = 5$, which is a contradiction. Thus $|G/\mathbf{C}_G(K)| = 2$. Since K is a proper subgroup of $\mathbf{C}_G(K)$, the group $\mathbf{C}_G(K)/K$ is a nontrivial normal subgroup of $G/K \cong L$, which is a contradiction.

If K is isomorphic to S_3, then $K \cap \mathbf{C}_G(K) = 1$ and $G/K \cong L$. Note that $G/\mathbf{C}_G(K)$ is isomorphic to a subgroup of $\mathsf{Aut}(K) \cong S_3$. Then $\mathbf{C}_G(K) \neq 1$. Since $\mathbf{C}_G(K) \cong \mathbf{C}_G(K)K/K$ is a non-identity normal subgroup of $G/K \cong L$, we conclude that $G = \mathbf{C}_G(K)K$, where $\mathbf{C}_G(K) \cong L$ and $K \cap \mathbf{C}_G(K) = 1$. This implies that G is isomorphic to $K \times \mathbf{C}_G(K) \cong S_3 \times L$, however this case can be ruled out as $\deg(2) = 3$. $\square$

Proof of Theorem 1.1. The proof of Theorem 1.1 follows immediately from Proposition 3.4 in [4] and Propositions 3.1–3.3.

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