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Retractive nil-extensions of completely simple semirings

Sunil Kumar Maity, Rumpa Chatterjee and Rituparna Ghosh

Abstract. A semiring S is said to be a quasi completely regular semiring if for any $a \in S$ there exists a positive integer n such that na is completely regular. The study of completely Archimedean semirings have shown that completely Archimedean semirings are nil-extensions of completely simple semirings. In this paper we introduce retractive nil-extensions of completely simple semirings and establish a relation with completely Archimedean semirings.

1. Introduction

In the year 1984, S. Bogdanovic and S. Milic [5] first studied nil-extensions of completely simple semigroups. Decomposition of completely π-regular semigroups into a semilattice of Archimedean semigroups was discussed by S. Bogdanovic in [1]. The study of nil-extensions of completely regular semigroups has been proven of great importance in semigroup theory. A completely Archimedean semigroup has been proven to be a nil-extension of a completely simple semigroup and furthermore it is Archimedean and completely π- regular. Furthermore, characterization of retractive nil-extensions of class of regular semigroups, union of groups and band of groups have been studied in the papers [2], [3] and [4]. Retractive nil-extension of a completely simple semigroup has been proved to be a rectangular band of π-groups. Therefore, retractive extension of a semigroup has been an area of great attraction in recent years of study.

In recent years, semirings have been studied by many authors, for example, by F. Pastijn, Y. Q. Guo, M. K. Sen, K. P. Shum and others (See [10], [13]). In the paper [13], completely regular semirings were introduced and it was derived that a completely regular semiring is a union of skew-rings and also a b-lattice of completely simple semirings. After this work, many interesting results on completely regular semigroups and inverse semigroups have been extended to semirings by M. K. Sen, S. K. Maity and K. P. Shum in ([12], [14]). Furthermore, extension of completely π-regular semigroups to quasi completely regular semirings in [9] have further enriched the study of analogous results. It has also been derived that quasi completely regular semiring can be described as the b-lattice of completely Archimedean semirings. In the paper [8], we had shown that a semiring

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is completely Archimedean if and only if it is nil-extension of a completely simple semiring if and only if it is Archimedean and quasi completely regular.

In this paper we further study retractive extension of a semiring. We characterize completely Archimedean semirings as retractive nil-extension of completely simple semirings. Thus we establish a relation between retractive nil-extensions of completely simple semirings and nil-extensions of completely simple semirings. The preliminaries and prerequisites for this article are discussed in section 2. In section 3 we prove some characterization theorems of completely Archimedean semirings as retractive nil-extensions of completely simple semirings. Further, we study properties on retractive nil-extension of b-lattice of skew-rings.

2. Preliminaries

A semiring $(S,+,\cdot)$ has both the additive reduct $(S,+)$ and the multiplicative reduct $(S,\cdot)$ as semigroups and multiplication distributes over addition, that is, $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in S$. We do not assume that the additive reduct $(S,+)$ is commutative. An element a of a semiring S is said to be *infinite* [7] if and only if $a+x=a=x+a$ for all $x\in S$. Infinite element in a semiring is unique and is denoted by ∞. An infinite element ∞ in a semiring S having the property that $x\cdot\infty=\infty=\infty\cdot x$ for all $x(\neq 0)\in S$ is called *strongly infinite* [7]. A semiring S is *additively regular* if for every element $a\in S$ there exists an element $x\in S$ such that $a+x+a=a$. A semiring S is said to be *additively quasi regular* if for every element $a\in S$ there exists a positive integer n such that the element $na=\underbrace{a+a+\cdots+a}_{n\text{ times}}$ is additively regular. A semiring S is *completely regular* [13] if for every element $a\in S$, there exists an element $x\in S$ such that $a=a+x+a$, $a+x=x+a$ and $a(a+x)=a+x$. A semiring $(S,+,\cdot)$ is a *quasi completely regular semiring* [9] if for every element $a\in S$, there exists a positive integer n such that na is completely regular, that is, $na=na+x+na$, $na+x=x+na$ and $na(na+x)=na+x$ for a suitable element $x\in S$. A semiring $(S,+,\cdot)$ is called a *skew-ring* if its additive reduct $(S,+)$ is a group, not necessarily an abelian group. In [13] we proved that an element a in a semiring S is completely regular if and only if it is contained in a subskew-ring of S. Let S be a semiring and R be a subskew-ring of S. If for every $a\in S$ there exists a positive integer n such that $na\in R$, then S is said to be a *quasi skew-ring*. A semiring S is said to be a *b-lattice* [13] if $(S,\cdot)$ is a band and $(S,+)$ is a semilattice. A semiring S is said to be an *idempotent semiring* if all the elements of S are additive idempotent as well as multiplicative idempotent, i.e., $a+a=a=a^2$ for all $a\in S$. An idempotent semiring satisfying the identity $a=a+x+a$ is called a *rectangular band semiring*. Throughout this paper, we let $E^+(S)$ be the set of all additive idempotents of the semiring S. We observe that the set $E^+(S)$ is non-empty and it forms an ideal of the multiplicative reduct $(S,\cdot)$ of the semiring S. If $(S,+,\cdot)$ is a semiring, we denote the Green's relations on the semigroup $(S,+)$ by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$, and

$\mathscr{H}^+$. In fact, the relations $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{J}^+$ and $\mathscr{H}^+$ are all congruences of the multiplicative reduct $(S, \cdot)$. Thus, if any one of these happens to be a congruence on the additive reduct $(S, +)$, it will be a congruence on the semiring $(S, +, \cdot)$. Let $(S, +, \cdot)$ be an additively quasi regular semiring. We consider the relations $\mathscr{L}^{*+}$, $\mathscr{R}^{*+}$, and $\mathscr{J}^{*+}$ on S defined by: for $a, b \in S$,

$$a \mathscr{L}^{*+} b \text{ if and only if } pa \mathscr{L}^+ qb,$$
$$a \mathscr{R}^{*+} b \text{ if and only if } pa \mathscr{R}^+ qb,$$
$$a \mathscr{J}^{*+} b \text{ if and only if } pa \mathscr{J}^+ qb;$$

where p and q are the smallest positive integers such that pa and qb are respectively additively regular. We also let $\mathscr{H}^{*+} = \mathscr{L}^{*+} \cap \mathscr{R}^{*+}$. A quasi completely regular semiring $(S, +, \cdot)$ is said to be *completely Archimedean* if $\mathscr{J}^{*+} = S \times S$.

A nonempty subset I of a semiring $(S, +, \cdot)$ is said to be a *bi-ideal* of S if $a \in I$ and $x \in S$ imply that $a + x, x + a, ax, xa \in I$. Let I be a bi-ideal of a semiring S. We define a relation ρ_I on S by $a\rho_I b$ if and only if either $a, b \in I$ or $a = b$ where $a, b \in S$. It is easy to verify that ρ_I is a congruence on S. This congruence is said to be *Rees congruence* on S and the quotient semiring S/ρ_I contains a strongly infinite element, namely I. This quotient semiring S/ρ_I is said to be the *Rees quotient semiring* and is denoted by S/I. In this case the semiring S is said to be an *ideal extension* of I by the semiring S/I. An ideal extension S of a semiring I is a *nil-extension* of I if for any $a \in S$ there exists a positive integer n such that $na \in I$.

A subsemiring T of a semiring S is a *retract* of S if there exists a homomorphism $\phi : S \longrightarrow T$ such that $\phi(t) = t$ for all $t \in T$. Such a homomorphism is called a *retraction*. A nil-extension S of T is said to be a *retractive nil-extension* of T if T is a retract of S.

3. Retractive nil-extension

In this section we characterize completely Archimedean semirings as a retractive nil-extensions of completely simple semirings. For this first we state the following two results.

Theorem 3.1 ([6], [9])**.** *The following conditions on a semiring $(S, +, \cdot)$ are equivalent.*

(i) *S is a quasi completely regular semiring.*

(ii) *Every $\mathscr{H}^{*+}$- class is a quasi skew-ring.*

(iii) *S is (disjoint) union of quasi skew-rings.*

(iv) *S is a b-lattice of completely Archimedean semirings.*

(v) *S is an idempotent semiring of quasi skew-rings.*

Theorem 3.2 ([8]). *The following conditions on a semiring are equivalent:*

(*i*) *S is a completely Archimedean semiring;*

(*ii*) *S is a nil-extension of a completely simple semiring;*

(*iii*) *S is Archimedean and quasi completely regular.*

Remark. Let a be a quasi completely regular element in a semiring $(S, +, \cdot)$. Then there exists a positive integer n such that na is completely regular and hence na lies in a subskew-ring of S. The zero of the subskew-ring containing na is denoted by a^0. Here it is interesting to mention that if S is a quasi completely regular semiring then S is (disjoint) union of quasi skew-rings. We know that every quasi skew-ring contains a unique additive idempotent. According to our notation a^0 is the unique additive idempotent in the quasi skew-ring containing a.

Theorem 3.3. *Let S be a completely Archimedean semiring. Then S is a retractive nil-extension of a completely simple semiring.*

Proof. Since S is a completely Archimedean semiring, hence S is a quasi completely regular semiring. Therefore, S is an idempotent semiring I of quasi skew-rings S_i, $i \in I$. Let $a, b \in S$. Then $a \in S_i$ and $b \in S_j$ for some $i, j \in I$. This implies $a + b, 2a + b, a + 2b \in S_{i+j}$. Since S_{i+j} is a quasi skew-ring, we must have a positive integer n such that $n(a + b) \in (2a + b) + S + (a + 2b)$. Hence for any two elements $a, b \in S$, there exists a positive integer n such that $n(a + b) \in (2a+b)+S+(a+2b) \subseteq 2a+S+2b$. Let $a \in S$. Then $a \in S_i$ for some $i \in I$. Let e be the unique additive idempotent in the quasi skew-ring S_i. Let $f \in E^+(S)$. We first prove that $a+f = e+a+f$ and $f+a = f+a+e$. First we prove that for every $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $u \in S$ such that $n(a+f) = ma+u+f$. Clearly, the result holds if $m = 1$. Let us assume that $n(a + f) = ma + u + f$ for some $n \in \mathbb{N}$ and $u \in S$. Now for the elements ma and $u + f$ there exists a positive integer k and $v \in S$ such that $k(ma+u+f) = 2ma+v+2(u+f) = (m+1)a+w+f$, where $w = (m-1)a + v + u + f + u \in S$. Hence for every $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ and $u \in S$ such that $n(a + f) = ma + u + f$. Let $r \in \mathbb{N}$ be such that ra is completely regular and hence ra lies in a subskew-ring R_i. Clearly, e is the zero of R_i. Then there exists $p \in \mathbb{N}$ and $x \in S$ such that $p(a + f) = ra + x + f$. Now since S is a completely Archimedean semiring, so S is a nil-extension of a completely simple semiring K. Clearly, $f \in K$ and since K is a bi-ideal of S, it follows $a + f \in K$. Hence $(a + f)\,\mathscr{H}^+\,p(a + f)$ and $(a + f) = p(a + f) + y$ for some $y \in S$. Therefore, $a + f = p(a + f) + y = ra + x + f + y = e + ra + x + f + y = e + a + f$. Similarly, we can prove that $f + a = f + a + e$. We define a mapping $\phi : S \longrightarrow K$ by $\phi(a) = a^0 + a$, for all $a \in S$. Let $a, b \in S$. Then $\phi(a + b) = (a + b)^0 + a + b = (a + b)^0 + a + a^0 + b = (a + b)^0 + a + a^0 + b + b^0 = (a + b)^0 + a + b + b^0 = a + b + b^0 = a + b^0 + b = a^0 + a + b^0 + b = \phi(a) + \phi(b)$. Again, $\phi(a)\phi(b) = (a^0 + a)(b^0 + b) = a^0b^0 + a^0b + ab^0 + ab = (ab)^0 + ab = \phi(ab)$. Therefore, ϕ is a homomorphism and since $\phi(a) = a$ for all $a \in K$, then ϕ is a retraction. Hence S is a retractive nil-extension of a completely simple semiring K. □

Corollary 3.4. *The following conditions on a semiring S are equivalent:*

(i) *S is a completely Archimedean semiring.*

(ii) *S is a retractive nil extension of a completely simple semiring.*

(iii) *S is a nil-extension of a completely simple semiring.*

Theorem 3.5. *A semiring S is a completely Archimedean semiring if and only if S is a rectangular band semiring of quasi skew-rings.*

Proof. Let S be a completely Archimedean semiring. Hence S is a quasi completely regular semiring and hence S is an idempotent semiring $I(= S/\mathscr{H}^{*+})$ of quasi skew-rings S_i $(i \in I)$. To show I is a rectangular band semiring, let $x = a\mathscr{H}^{*+}$, $y = b\mathscr{H}^{*+} \in I$, where $a, b \in S$. Since S is completely Archimedean, we must have $a^0 = (a + b + a)^0$. Then $x = a\mathscr{H}^{*+} = a^0\mathscr{H}^{*+} = (a + b + a)^0\mathscr{H}^{*+} = (a + b + a)\mathscr{H}^{*+} = a\mathscr{H}^{*+} + b\mathscr{H}^{*+} + a\mathscr{H}^{*+} = x + y + x$. Thus I is a rectangular band semiring and consequently, S is a rectangular band semiring of quasi skew-rings.

Conversely, let S be a rectangular band semiring Y of quasi skew-rings, T_i $(i \in Y)$. Then clearly S is a quasi completely regular semiring and $Y = S/\mathscr{H}^{*+}$. To show S is completely Archimedean, we only show that $\mathscr{J}^{*+} = S \times S$. Let $a, b \in S$. Then $a\mathscr{H}^{*+} = a\mathscr{H}^{*+} + b\mathscr{H}^{*+} + a\mathscr{H}^{*+} = (a + b + a)\mathscr{H}^{*+}$ implies $a^0 = (a+b+a)^0$. Since $\mathscr{J}^{*+}$ is b-lattice congruence, we have $a\mathscr{J}^{*+} = a^0\mathscr{J}^{*+} = (a + b + a)^0\mathscr{J}^{*+} = (a + b + a)\mathscr{J}^{*+} = (a + b)\mathscr{J}^{*+} = (b + a + b)\mathscr{J}^{*+} = (b + a + b)^0\mathscr{J}^{*+} = b^0\mathscr{J}^{*+} = b\mathscr{J}^{*+}$. Therefore, $\mathscr{J}^{*+} = S \times S$ and hence S is a completely Archimedean semiring. □

Corollary 3.6. *A semiring S is a completely simple semiring if and only if S is a rectangular band semiring of skew-rings.*

Theorem 3.7. *The following conditions on a semiring S are equivalent:*

(i) *S is a nil-extension of a b-lattice of skew-rings.*

(ii) *S is a retractive nil-extension of b-lattice of skew-rings.*

Proof. (i) $\Rightarrow$ (ii): Let S be a nil-extension of K, where K is a b-lattice Y of skew-rings $R_\alpha (\alpha \in Y)$. Then $(K, +)$ is a semilattice $(Y, +)$ of groups $(R_\alpha, +)$ and hence $(K, +)$ is a Clifford semigroup. Thus for any $k \in K$ and $e \in E^+(K)$ we must have $e + k = k + e$. Now we define a mapping $\phi : S \longrightarrow K$ by: for $a \in S$,

$$\phi(a) = a^0 + a.$$

Let $a, b \in S$. Then $\phi(a + b) = (a + b)^0 + (a + b) = (a^0 + b^0) + (a + b) = (b^0 + a^0) + (a + b) = b^0 + (a^0 + a) + b = (a^0 + a) + (b^0 + b) = \phi(a) + \phi(b)$ and $\phi(a)\phi(b) = (a^0 + a)(b^0 + b) = a^0b^0 + a^0b + ab^0 + ab = (ab)^0 + ab = \phi(ab)$. Hence ϕ is a homomorphism. Moreover for any $x \in K$, $\phi(x) = x^0 + x = x$. Consequently, S is a retractive nil-extension of b-lattice of skew-rings K.

(ii) $\Rightarrow$ (i): This part is obvious. □

References

[1] **S. Bogdanović**, *Semigroups with a system of subsemigroups*, Novi Sad (1985).

[2] **S. Bogdanović and M. Ćirić**, *Retractive nil-extensions of regular semigroups I*, Proc. Japan Acad. **68** (5), Ser. A (1992), 115 – 117.

[3] **S. Bogdanović and M. Ćirić**, *Retractive nil-extensions of regular semigroups II*, Proc. Japan Acad. **68** (6), Ser. A (1992), 126 – 130.

[4] **S. Bogdanović and M. Cirić**, *Retractive nil-extensions of bands of groups*, Facta Univ. Niš, Ser. Math. Inform. **8** (1993), 11 – 20.

[5] **S. Bogdanović and S. Milić**, *A nil-extension of a completely simple semigroup*, Publ. Inst. Math. **36(50)** (1984), 45 – 50.

[6] **R. Debnath, S.K. Maity and A.K. Bhuniya**, *Subdirect products of an idempotent semiring and a b-lattice of skew-rings*, submitted for publication.

[7] **J.S. Golan**, *The theory of semirings with applications in mathematics and theoretical computer science*, Pitman Monographs and Surveys in Pure and Appl. Math. 54, Longman Scientific (1992).

[8] **S.K. Maity and R. Ghosh**, *Nil-extensions of completely simple semirings*, Discussiones Math. General Algebra Appl. **33** (2013), 201 – 209.

[9] **S.K. Maity and R. Ghosh**, *On quasi completely regular semirings*, Semigroup Forum **89** (2014), 422 – 430.

[10] **F. Pastijn and Y.Q. Guo**, *The lattice of idempotent distributive semiring varieties*, Science in China (Series A) **42** (8) (1999), 785 – 804.

[11] **M. Petrich and N.R. Reilly**, *Completely regular semigroups*, Wiley, New York (1999).

[12] **M.K. Sen, S.K. Maity and K.P. Shum**, *Clifford semirings and generalized Clifford semirings*, Taiwanese J. Math. **9** (2005), 433 – 444.

[13] **M.K. Sen, S.K. Maity and K.P. Shum**, *On completely regular semirings*, Bull. Cal. Math. Soc. **98** (2006), 319 – 328.

[14] **M.K. Sen, S.K. Maity and H.J. Weinert**, *Completely simple semirings*, Bull. Cal. Math. Soc. **97** (2005), 163 – 172.

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S.K. Maity
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India.
E-mail: skmpm@caluniv.ac.in

R. Chatterjee
Department of Mathematics, The University of Burdwan, Golapbag, Burdwan-713104, India.
E-mail: rumpachatterjee13@gmail.com

R. Ghosh
Department of Mathematics, Heritage Institute of Technology, Anandapur, P.O East Kolkata Township, Kolkata - 700 107, India.
E-mail: rituparna.ghosh@heritageit.edu