Quasigroups and Related Systems **26** (2018), 149 − 154

On bi-ideals of ordered semigroups

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Abstract. The concepts of strongly quasi-prime, quasi-prime, quasi-semiprime, strongly irreducible and irreducible bi-ideals of an ordered semigroup are introduced. Moreover, we characterize regular and intra-regular ordered semigroups using bi-ideals, and investigate the ordered semigroups in which every bi-ideal is strongly quasi-prime.

1. Introduction and preliminaries

Ideal theory play an important role in characterizations of semigroups and ordered semigroups. Lajos first introduced the concept of bi-ideals in semigroups (see [7]). Li and He characterized the semigroups whose all bi-ideals are prime in [8]; the semigroups whose all bi-ideals are strongly prime were determined by Shabir in [9]. Kehayopulu did much work on characterizations of regular and intra-regular ordered semigroups by ideals, quasi-ideals and bi-ideals (see [1, 2, 3, 4, 5, 6]). The characterizations of regular and intra-regular ordered semigroups in terms of fuzzy subsets were given by Xie and Tang in [10]. In this paper, we first introduce the notions of strongly quasi-prime, quasi-prime, quasi-semiprime, strongly irreducible and irreducible bi-ideals in ordered semigroups, and then characterize regular and intra-regular ordered semigroups by bi-ideals. Finally, we characterize those ordered semigroups in which all bi-ideals are strongly quasi-prime.

We recall some basic notions in ordered semigroups. An *ordered semigroup* is a semigroup $(S, \cdot)$ endowed with an order relation $\leqslant$ such that

$$(\forall a, b, x \in S)\ a \leqslant b \Rightarrow xa \leqslant xb \text{ and } ax \leqslant bx.$$

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A non-empty subset B of S is called a *bi-ideal* of S if it satisfies the following conditions: (1) $BSB \subseteq B$; (2) $a \in B$ and $b \in S,\ b \leqslant a$ implies $b \in B$. For a nonempty subset H of S, we denote

$$(H] = \{t \in S \mid t \leqslant h \text{ for some } h \in H\}.$$

2010 Mathematics Subject Classification: 06F05, 20M12
Keywords: ordered semigroup, (strongly) quasi-prime bi-ideals, quasi-semiprime bi-ideals, (strongly) irreducible bi-ideals.
This research was supported by the National Natural Science Foundation of China (No. 11701504), the Young Innovative Talent Project of Department of Education of Guangdong Province (No. 2016KQNCX180) and the University Natural Science Project of Anhui Province (No. KJ2018A0329).

It is well known that the intersection of any number of bi-ideals of S is either empty or a bi-ideal of S. For any bi-ideals B_1, B_2 of S, $(B_1B_2]$ is a bi-ideal of S.

An ordered semigroup S is called *regular* ([2, 5]) if for every $a \in S$ there exists $x \in S$ such that $a \leqslant axa$. Equivalent definitions: (1) $A \subseteq (ASA]$ $(\forall A \subseteq S)$; (2) $a \in (aSa]$ $(\forall a \in S)$. An ordered semigroup S is called *intra-regular* ([2, 3]) if for every $a \in S$ there exist $x, y \in S$ such that $a \leqslant xa^2y$. Equivalent definitions: (1) $A \subseteq (SA^2S]$ $(\forall A \subseteq S)$; (2) $a \in (Sa^2S]$ $(\forall a \in S)$.

2. Several classes of bi-ideals

In this section, we mainly introduce and study quasi-prime, strongly quasi-prime, quasi-semiprime, irreducible and strongly irreducible bi-ideals in ordered semigroups.

Definition 2.1. Let S be an ordered semigroup and B a bi-ideal of S. B is called *quasi-prime* (*strongly quasi-prime*) if $B_1B_2 \subseteq B$ $((B_1B_2] \cap (B_2B_1] \subseteq B)$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1 and B_2 of S. B is called *quasi-semiprime* if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of S.

Remark 2.2. From Definition 2.1, we know that every strongly quasi-prime bi-ideal of an ordered semigroup S is quasi-prime, and every quasi-prime bi-ideal is quasi-semiprime. However, a quasi-prime bi-ideal is not necessarily strongly quasi-prime and a quasi-semiprime bi-ideal is not necessarily quasi-prime.

Example 2.3. (See [2]) Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the multiplication " $\cdot$ " and the order " $\leqslant$ " below:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

$$\leqslant := \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,b), (c,c), (d,d), (e,e)\}.$$

We can deduce that the bi-ideals of S are

$$\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{a,b,d\}, \{a,c,d\}, \{a,b,e\}, \{a,c,e\}, S.$$

It is easy to see that the bi-ideal $\{a,b,e\}$ is quasi-prime. But it is not strongly quasi-prime. Indeed: we have

$$(\{a,c\}\{a,d\}] = (\{a,c\}] = \{a,c\};$$

$$(\{a,d\}\{a,c\}] = (\{a,d\}] = \{a,d\};$$

$$\{a,c\} \cap \{a,d\} = \{a\} \subseteq \{a,b,e\}.$$

But neither $\{a,c\}$ nor $\{a,d\}$ is contained in $\{a,b,e\}$.

Example 2.4. (See [2]) Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the multiplication " $\cdot$ " and the order " $\leqslant$ " below:

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

$$\leqslant := \{(a,a),(a,b),(b,b),(c,a),(c,b),(c,c)(d,a),(d,b),(d,d),(e,e)\}.$$

We can obtain that the bi-ideals of S are

$$\{a,c,d\},\{a,b,c,d\},\{a,c,d,e\},S.$$

It is easy to deduce that the bi-ideal $\{a, c, d\}$ is quasi-semiprime. But it is not quasi-prime. Indeed: we have

$$\{a,b,c,d\}\{a,c,d,e\} = \{a,c,d\}.$$

However, neither $\{a, b, c, d\}$ nor $\{a, c, d, e\}$ is contained in $\{a, c, d\}$.

Definition 2.5. A bi-ideal B of an ordered semigroup S is called *irreducible* (*strongly irreducible*) if $B_1 \cap B_2 = B$ ($B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ ($B_1 \subseteq B$ or $B_2 \subseteq B$) for any bi-ideals B_1 and B_2 of S.

Remark 2.6. Clearly, every strongly irreducible bi-ideal of an ordered semigroup is irreducible. The following example shows that the converse is not true.

Example 2.7. Consider the ordered semigroup S in Example 2.3. The bi-ideal $\{a, b, d\}$ is irreducible but not strongly irreducible because

$$\{a,c\} \cap \{a,e\} = \{a\} \subseteq \{a,b,d\}.$$

But neither $\{a, c\}$ nor $\{a, e\}$ is contained in $\{a, b, d\}$.

Proposition 2.8. *The intersection of any family of quasi-prime bi-ideals of an ordered semigroup is either empty or a quasi-semiprime bi-ideal.*

Proof. Let Γ be a family of quasi-prime bi-ideals and B a bi-ideal. It is well-known that $\bigcap_{\alpha\in\Gamma} B_\alpha$ is either empty or a bi-ideal. Suppose that $\bigcap_{\alpha\in\Gamma} B_\alpha \neq \emptyset$ and $B^2 \subseteq \bigcap_{\alpha\in\Gamma} B_\alpha$. Then $B^2 \subseteq B_\alpha$ for every $\alpha \in \Gamma$. Since B_α is quasi-prime, we have $B \subseteq B_\alpha$. Thus $B \subseteq \bigcap_{\alpha\in\Gamma} B_\alpha$ and so $\bigcap_{\alpha\in\Gamma} B_\alpha$ is quasi-semiprime. □

Proposition 2.9. *Let B be a strongly irreducible quasi-semiprime bi-ideal of an ordered semigroup S. Then B is strongly quasi-prime.*

Proof. Let B_1, B_2 be two bi-ideals of S such that $(B_1B_2] \cap (B_2B_1] \subseteq B$. Since $(B_1 \cap B_2)^2 \subseteq B_1B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2B_1$, we have $(B_1 \cap B_2)^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq (B_1B_2] \cap (B_2B_1] \subseteq B$. Moreover, since B is a quasi-semiprime bi-ideal, $B_1 \cap B_2 \subseteq B$. In addition, from the strong irreducibility of B, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime bi-ideal of S. □

3. Regular and intra-regular ordered semigroups

In this section, we mainly characterize regular and intra-regular ordered semigroups by bi-ideals, and investigate the ordered semigroups in which all bi-ideals are strongly quasi-prime.

Theorem 3.1. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(i) *S is both regular and intra-regular;*

(ii) *$(B^2] = B$ for every bi-ideal B of S;*

(iii) *$B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1]$ for all bi-ideals B_1 and B_2 of S;*

(iv) *Every bi-ideal of S is quasi-semiprime.*

Proof. $(i) \Rightarrow (ii)$. Let B be a bi-ideal of S. Then $BSB \subseteq B$. Since S is regular and intra-regular, $B \subseteq (BSB]$ and $B \subseteq (SB^2S]$. Thus $B \subseteq (BSB] \subseteq ((BSB](SB]] = (BSBSB] \subseteq ((BS](SB^2S](SB]] \subseteq (BSSB^2SSB] \subseteq (BSBBSB] \subseteq (B^2]$. Also, $(B^2] \subseteq ((BSB](BSB]] = (BSBBSB] \subseteq (BSB] \subseteq (B] = B$.

$(ii) \Rightarrow (i)$. Let $a \in S$. Then $B(a) = (a \cup aSa]$. Since $B = (B^2]$ for every bi-ideal B of S, we have $a \in B(a) = (B^2(a)] = ((B^2(a)](B(a)]] = (B^3(a)] = ((a \cup aSa](a \cup aSa](a \cup aSa]] \subseteq ((a \cup aSa)(a \cup aSa)(a \cup aSa)] \subseteq (aSa]$. Hence S is regular.

Similarly, we have $a \in B(a) = (B^2(a)] = ((B^2(a)](B^2(a)]] = (B^4(a)] = ((a \cup aSa](a \cup aSa](a \cup aSa](a \cup aSa]] \subseteq ((a \cup aSa)(a \cup aSa)(a \cup aSa)(a \cup aSa)] \subseteq (Sa^2S]$. Thus S is intra-regular.

$(ii) \Rightarrow (iii)$. Let B_1 and B_2 be two bi-ideals of S. Then $B_1 \cap B_2$ is either empty or a bi-ideal of S.

Case 1). Suppose that $B_1 \cap B_2 = \emptyset$. Next we prove that $(B_1B_2] \cap (B_2B_1] = \emptyset$. Otherwise, $(B_1B_2] \cap (B_2B_1]$ is a bi-ideal (Since $(B_1B_2]$ and $(B_2B_1]$ are bi-ideals). Thus $(B_1B_2] \cap (B_2B_1] = (((B_1B_2] \cap (B_2B_1])((B_1B_2] \cap (B_2B_1])] \subseteq ((B_1B_2](B_2B_1]] \subseteq ((B_1B_2B_2B_1]] = (B_1B_2B_2B_1] \subseteq (B_1SB_1] \subseteq (B_1] = B_1$. Similarly, $(B_1B_2] \cap (B_2B_1] \subseteq B_2$. Hence $(B_1B_2] \cap (B_2B_1] \subseteq B_1 \cap B_2 = \emptyset$, which is impossible.

Case 2). Suppose that $B_1 \cap B_2 \neq \emptyset$. By hypothesis, $B_1 \cap B_2 = ((B_1 \cap B_2)^2] = ((B_1 \cap B_2)(B_1 \cap B_2)] \subseteq (B_1B_2]$. In the same way, we have $B_1 \cap B_2 \subseteq (B_2B_1]$. Thus,

$$B_1 \cap B_2 \quad \subseteq \quad (B_1B_2] \cap (B_2B_1]. \tag{1}$$

Hence $(B_1B_2] \cap (B_2B_1] \neq \emptyset$ and so $(B_1B_2] \cap (B_2B_1]$ is a bi-ideal. Similar to the proof of Case 1), we have

$$(B_1B_2] \cap (B_2B_1] \subseteq B_1 \cap B_2. \tag{2}$$

By (1) and (2), we obtain that

$$B_1 \cap B_2 = (B_1B_2] \cap (B_2B_1].$$

$(iii) \Rightarrow (iv)$. Let B_1 and B be two bi-ideals of S such that $B_1^2 \subseteq B$. By hypothesis, $B_1 = B_1 \cap B_1 = (B_1^2] \cap (B_1^2] = (B_1^2]$. Thus, we have $B_1 = (B_1^2] \subseteq (B] = B$. Hence every bi-ideal of S is quasi-semiprime.

$(iv) \Rightarrow (ii)$. Let B be a bi-ideal of S. Then $(B^2]$ is a bi-ideal. By hypothesis, $(B^2]$ is quasi-semiprime. Since $B^2 \subseteq (B^2]$, we have $B \subseteq (B^2]$. Furthermore, $(B^2] \subseteq ((B^2](B]] = (B^3] \subseteq (BSB] \subseteq (B] = B$. Hence $B = (B^2]$. □

The following result can be directly obtained from Theorem 3.1.

Proposition 3.2. *Let S be a regular and intra-regular ordered semigroup and B a bi-ideal of S. Then the following statements are equivalent:*

(i) *B is strongly irreducible;*

(ii) *B is strongly quasi-prime.*

Next we characterize those ordered semigroups in which every bi-ideal is strongly quasi-prime and also those ordered semigroups in which every bi-ideal is strongly irreducible.

Lemma 3.3. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(i) *The set of bi-ideals of S is totally ordered under inclusion;*

(ii) *Every bi-ideal of S is strongly irreducible and $B_1 \cap B_2 \neq \emptyset$ for any bi-ideals B_1 and B_2 of S;*

(iii) *Every bi-ideal of S is irreducible and $B_1 \cap B_2 \neq \emptyset$ for any bi-ideals B_1 and B_2 of S.*

Proof. $(i) \Rightarrow (ii)$. By condition (i), it is obvious that $B_1 \cap B_2 \neq \emptyset$ for any bi-ideals B_1 and B_2 of S. Let B be a bi-ideal of S and B_1, B_2 two bi-ideals such that $B_1 \cap B_2 \subseteq B$. Since the set of bi-ideals of S is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is strongly irreducible.

$(ii) \Rightarrow (iii)$. The conclusion is obvious.

$(iii) \Rightarrow (i)$. Let B_1 and B_2 be two bi-ideals of S. Since $B_1 \cap B_2 \neq \emptyset$, $B_1 \cap B_2$ is a bi-ideal. By hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of S is totally ordered. □

Theorem 3.4. *Let S be an ordered semigroup. Then every bi-ideal of S is strongly quasi-prime and $B_1 \cap B_2 \neq \emptyset$ for any bi-ideals B_1 and B_2 of S if and only if S is regular, intra-regular and the set of bi-ideals of S is totally ordered.*

Proof. (⇒). Let every bi-ideal of S be strongly quasi-prime. Then every bi-ideal of S is quasi-semiprime. From Theorem 3.1, we have S is regular and intra-regular. Furthermore, we know that every bi-ideal of S is strongly irreducible from Proposition 3.2. Thus by Lemma 3.3, the set of bi-ideals of S is totally ordered under inclusion.

(⇐). Since the set of bi-ideals of S is totally ordered under inclusion, we have $B_1 \cap B_2 \neq \emptyset$ for any bi-ideals B_1 and B_2 of S and every bi-ideal of S is strongly irreducible from Lemma 3.3. Since S is regular and strongly regular, from Proposition 3.2, we obtain that every bi-ideal of S is strongly quasi-prime. □

Acknowledgments

The author are extremely grateful to the referees for their valuable comments and helpful suggestions which help to improve the presentation of this paper.

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Received August 8, 2017

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