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Weighted means and weighted mean equations in lineated symmetric spaces

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Abstract. We develop further theory of weighted means in a lineated symmetric space. First, we illustrate fundamental examples of that space involving some classical additive/multiplicative groups of matrices and C*-algebra elements. Then we investigate properties of weighted means in which weights are arbitrary real numbers. Moreover, we investigate several (systems of) weighted mean equations in lineated symmetric spaces. In fact, every mean problem considered here is shown to has a unique solution in an explicit form.

1. Introduction

The concept of mean or midpoint shows up naturally in mathematics. One of familiar means for positive real numbers is the geometric mean $a\#b = \sqrt{ab}$, which is the solution of the algebraic equation $x^2 = ab$. This mean can be extended to positive definite matrices A and B of the same size given by

$$A \# B \;=\; A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Here, $X^{\frac{1}{2}}$ denotes the positive square root of a positive definite matrix X. This definition was exhibited in [4], and it is equivalent to that given in [19]. Significant properties of matrix geometric mean were originally established in [5]. The next observation is of significant:

Proposition 1.1 ([3]). *For positive definite matrices A and B of the same size, the geometric mean $X = A \# B$ is the unique positive definite solution of the Riccati equation $XA^{-1}X = B$.*

An alternative approach to the matrix geometric mean in terms of the Riccati equation and the congruence transformation is provided in [12]. For any $t \in [0, 1]$, the t-weighted geometric mean of positive definite matrices A and B is defined by

$$A \#_t B \;=\; A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t}A^{\frac{1}{2}}. \tag{1}$$

Mean equations for matrices have been investigated e.g. in [1, 2, 15, 18]. The formula (1) can be extended to the context of positive invertible operators on a

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Hilbert space. An abstract theory of operator means was investigated by Kubo and Ando [11]; see more information in [10, Ch. 5], [9, Sect. 3] and [7, 8]. Certain mean equations for Hilbert space operators were investigated in [6]; in particular:

Proposition 1.2. *Let A and B be positive invertible operators on the same Hilbert space, and let $t \in (0,1)$. Then the equation $A \#_t X = B$ has a unique positive solution*

$$X \;=\; A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{t}}A^{\frac{1}{2}} \;=:\; A \#_{\frac{1}{t}} B.$$

Note that, in this case, the weight $1/t$ does not belong to the interval $(0,1)$ anymore.

There are another axiomatic approaches for means in various frameworks, e.g. in symmetric cones [17] and convex bodies [20]. Let us focus on the algebraic-geometric setting of reflection quasigroups, introduced by Lawson and Lim [13]. In this setting, a fundamental idea is that each point x is associated with a "symmetry" S_x. The symmetry sending x to y ought to be the point symmetry S_m through the midpoint m of x and y. These geometric requirements as well as Proposition 1.1 motivate the following definition.

Definition 1.3 ([13, 16])**.** A *symset* is a set X together with a map $S : x \mapsto S_x$, here each $S_x : X \to X$ is called the *symmetry* or the *point reflection* through x, such that the following conditions hold for all $a, b, c \in X$:

(M1) $S_a(a) = a$;

(M2) $S_a S_a(b) = b$;

(M3) $S_a S_b(c) = S_{S_a(b)} S_a(c)$.

In addition, if X satisfies the following property

(M4) the equation $S_x(a) = b$ has a unique solution $x \in X$,

then X is called a *reflection quasigroup* or a *dyadic symset*. The solution of the equation $S_x(a) = b$ is called the (*geometric*) *mean* or the *midpoint* of a and b, denoted by $a\#b$.

For a symset X, we define the *core operation* of two elements a and b in X by $a \bullet b = S_a(b)$. Then property (M4) states precisely that for each $a, b \in X$, the equation $x \bullet a = b$ has a unique solution (cf. Proposition 1.1 with $A \bullet B = AB^{-1}A$). Hence, in particular, every reflection quasigroup is a right quasigroup with respect to the core operation. It turns out that every reflection quasigroup is equipped with weighted means in which weights are dyadic rationals. These means are defined through dyadic geodesics, see [13].

A topological version of a reflection quasigroup was also investigated, and is named a lineated symmetric space [16]. This space is equipped, through continuous symmetry homomorphism, with weighted geometric means in which weights are arbitrary reals. Certain mean equations were investigated in [14].

In the present paper, we develop further theory of weighted means on lineated symmetric spaces. First, we provide fundamental examples for lineated symmetric spaces; see Section 3. Indeed, weighted means on some classical additive/multiplicative groups are given by weighted arithmetic means and weighted geometric means, respectively. We then move to establish properties of weighted means in which weights are arbitrary real numbers; see Section 4. Some of them generalize those of dyadic rational numbers in [14]. Moreover, we investigate weighted mean equations in lineated symmetric spaces in both single equations and systems of equations; see Sections 5 and 6. All equations considered here are consistent and, in fact, have unique solutions in explicit forms. Proposition 1.2 and the mean equations discussed in [14] are special cases of our particular results.

2. Preliminaries

In this section, we recall some terminologies and results for reflection quasigroups and lineated symmetric spaces. We start with a morphism in the category of reflection quasigroups as follows.

Definition 2.1 ([13]). A function $f : (X, \bullet_X) \to (Y, \bullet_Y)$ between reflection quasigroups is called a *symmetry homomorphism* or a *$\bullet$-homomorphism* if

$$f(a \bullet_X b) \;=\; f(a) \bullet_Y f(b)$$

for all $a, b \in X$.

Let $\mathbb{D}$ be the set of dyadic rationals in $\mathbb{R}$. Then $\mathbb{D}$ is a reflection quasigroup with respect to the symmetry $S_a(b) = 2a - b$ for each $a, b \in \mathbb{D}$.

Theorem 2.2 ([13]). *For any reflection quasigroup X and two elements $x, y \in X$, there exists a unique symmetry homomorphism $\gamma : \mathbb{D} \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

A symmetry homomorphism from $\mathbb{D}$ into a reflection quasigroup X is called a *dyadic geodesic* in X. The function γ in Theorem 2.2 is always mean-preserving, i.e. $\gamma(a \,\#\, b) = \gamma(a) \,\#\, \gamma(b)$ for all $a, b \in \mathbb{D}$. Theorem 2.2 allows us to define the t-weighted mean $x \,\#_t\, y = \gamma(t)$ for each $t \in \mathbb{D}$, where γ is the unique dyadic geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$. It turns out that $\#_{-1} = \bullet$, see [14].

Theorem 2.3 ([16]). *In a reflection quasigroup X, the mean $\#_t$ satisfies the following properties for all $u, v, w, z \in X$, and all $r, s, t \in \mathbb{D}$:*

(1) (*idempotency*) $u \,\#_t\, u = u$,

(2) (*commutativity*) $u \,\#_t\, v = v \,\#_{1-t}\, u$,

(3) (*exponential law*) $u \#_r (u \#_s v) = u \#_{rs} v$,

(4) (*affine change of parameter*) $(u \#_r v) \#_t (u \#_s v) = u \#_{(1-t)r+ts} v$,

(5) (*limited mediality*) *if* $u \# w = m = v \# z$, *then* $(u \#_t v) \# (w \#_t z) = m$,

(6) (*cancellability*) $u \#_t v = u \#_t w$ *for some* $t \neq 0$ *implies* $v = w$,

(7) $u \bullet (v \#_t w) = (u \bullet v) \#_t (u \bullet w)$.

A topological version of a reflection quasigroup is of interest.

Definition 2.4 ([16]). A *lineated symmetric space* is a symset X together with a Hausdorff topology such that

(1) The map $(x, y) \mapsto S_x(y) : X \times X \to X$ is continuous.

(2) For each $x, y \in X$, there is a unique continuous symmetry homomorphism $\alpha_{x,y} : \mathbb{R} \to X$ such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. Here, $\mathbb{R}$ is equipped with the natural symmetry homomorphism $S_a(b) = 2a - b$.

(3) The map $(t, x, y) \mapsto x \#_t y : \mathbb{R} \times X \times X \to X$ is continuous.

The image $\alpha_{x,y}(t)$ is also denoted $x \#_t y$, and is called the *t-weighted mean* of x and y.

Every lineated symmetric space satisfies (M4) and, thus, is a reflection quasigroup ([16, Proposition 3.3]). By [16, Remark 3.4], we have that for all $x, y \in X$ and $t \in \mathbb{R}$,

$$x \#_t y \;=\; y \#_{1-t} x. \tag{2}$$

A *pointed reflection quasigroup* $(X, \bullet, \varepsilon)$ is a reflection quasigroup $(X, \bullet)$ together with a fixed element $\varepsilon \in X$, called a *base point*. In this case, we define $x^t = \varepsilon \#_t x$ for each $x \in X$ and $t \in \mathbb{D}$. The concept of *pointed lineated symmetric space* is defined similarly. In this space, $x^t = \varepsilon \#_t x$ is defined for any $t \in \mathbb{R}$.

Theorem 2.5 ([16]). *Let $(X, \bullet, \varepsilon)$ be a pointed reflection quasigroup endowed with a Hausdorff topology such that*

(*i*) *the map $(x, y) \mapsto x \bullet y : X \times X \to X$ is continuous;*

(*ii*) *the map $(q, x) \mapsto x^q : \mathbb{D} \times X \to X$ can be extended to a continuous map $(t, x) \mapsto x^t : \mathbb{R} \times X \to X$.*

Then X is a pointed lineated symmetric space.

3. Examples of lineated symmetric spaces

In this section, we provide fundamental examples of lineated symmetric spaces.

Example 3.1. Recall that $(\mathbb{R}, +)$ is an additive group. The natural core operation on $\mathbb{R}$ is defined for each $x, y \in \mathbb{R}$ by $x \bullet y = 2x - y$. Then $(\mathbb{R}, \bullet)$ is a lineated symmetric space.

Proof. It is easy to see that $(\mathbb{R}, \bullet)$ is a symset. We equip $\mathbb{R}$ with the usual topology, which is Hausdorff. The map $(x, y) \mapsto x \bullet y$ is clearly continuous on $\mathbb{R} \times \mathbb{R}$. Let $x, y \in \mathbb{R}$ and define $\alpha_{x,y} : (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet)$ by

$$\alpha_{x,y}(t) \;=\; (1-t)x + ty.$$

Then $\alpha_{x,y}$ is a continuous $\bullet$-homomorphism such that $\alpha_{x,y}(0) = x$ and $\alpha_{x,y}(1) = y$. For uniqueness, let $\beta : (\mathbb{R}, \bullet) \to (\mathbb{R}, \bullet)$ be another continuous $\bullet$-homomorphism such that $\beta(0) = x$ and $\beta(1) = y$. Note that $\beta(t) \in \mathbb{D}$ for any $t \in \mathbb{D}$. By [14, Remark 3.3], the restriction $\beta|_{\mathbb{D}}$ must be of the form $\beta(t) = at + b$ for some $a, b \in \mathbb{D}$. Since both $\alpha_{x,y}$ and β are continuous on $\mathbb{R}$, $\alpha_{x,y} = \beta$ on $\mathbb{D}$, and $\mathbb{D}$ is dense in $\mathbb{R}$, we conclude $\alpha_{x,y} = \beta$ on $\mathbb{R}$. The map $(t, x, y) \mapsto x \,\#_t\, y = \alpha_{x,y}(t) = (1-t)x + ty$ is clearly continuous. Therefore, $(\mathbb{R}, \bullet)$ is a lineated symmetric space. $\square$

Example 3.2. Recall that the set $M_n(\mathbb{R})$ of n-by-n real matrices is a group under addition. Its natural core operation is defined by $A \odot B = 2A - B$. Then $(M_n(\mathbb{R}), \odot)$ is a lineated symmetric space.

Proof. It is easy to see that $(M_n(\mathbb{R}), \odot)$ is a symset. We equip $M_n(\mathbb{R})$ with the topology induced by a matrix norm (e.g. the Frobenius norm). Clearly, the map $(A, B) \mapsto 2A - B$ is continuous. Let $A, B \in M_n(\mathbb{R})$ and define

$$\alpha_{A,B} : (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot), \quad \alpha_{A,B}(t) \;=\; (1-t)A + tB.$$

Then, $\alpha_{A,B}(0) = A$ and $\alpha_{A,B}(1) = B$. The map $\alpha_{A,B}$ is a symmetry homomorphism since for any $s, t \in \mathbb{R}$, we have

$$\begin{aligned}
\alpha_{A,B}(s) \odot \alpha_{A,B}(t) &= [(1-s)A + sB] \odot [(1-t)A + tB] \\
&= [1 - (s \bullet t)]A + (s \bullet t)B \\
&= \alpha_{A,B}(s \bullet t).
\end{aligned}$$

To show the uniqueness, let $\beta : (\mathbb{R}, \bullet) \to (M_n(\mathbb{R}), \odot)$ be another continuous symmetry homomorphism such that $\beta(0) = A$ and $\beta(1) = B$. For each $X \in M_n(\mathbb{R})$, we define $\pi_{ij}(X) = X_{ij}$, the (i, j)-th entry of X. For any $t, s \in \mathbb{R}$, we have

$$\begin{aligned}
(\pi_{ij} \circ \beta)(t \bullet s) \;&=\; \pi_{ij}(\beta(t) \odot \beta(s)) \;=\; \pi_{ij}(2\beta(t) - \beta(s)) \\
&=\; (2\beta(t) - \beta(s))_{ij} \;=\; 2\beta(t)_{ij} - \beta(s)_{ij} \\
&=\; \beta(t)_{ij} \bullet \beta(s)_{ij} \quad\;=\; (\pi_{ij} \circ \beta)(t) \bullet (\pi_{ij} \circ \beta)(s).
\end{aligned}$$

By Example 3.1, $\pi_{ij}\circ\beta$ must be of the form $(\pi_{ij}\circ\beta)(t)=(1-t)A_{ij}+tB_{ij}$. Hence,

$$\begin{aligned}\beta(t) &= [(\pi_{ij}\circ\beta)(t)]_{ij} \;=\; [(1-t)A_{ij}+tB_{ij}]_{ij}\\ &= (1-t)A+tB \;=\; \alpha_{A,B}(t).\end{aligned}$$

Finally, note that the map $(t,A,B)\mapsto(1-t)A+tB$ is continuous. $\square$

To provide the next example of lineated symmetric spaces, recall that a subset T of a group is termed a *twisted subgroup* if it contains the identity and is closed under the core operation $x\bullet y=xy^{-1}x$. In addition, if for each $x\in T$ there is a unique element $a\in T$ such that $a^2=x$, then T is said to be a 2-*powered twisted subgroup*. In this case, we call a the square root of x, denoted by $x^{\frac{1}{2}}$.

Lemma 3.3. *Let X be a 2-powered twisted subgroup of a group. Then, for each $a,b\in X$, the equation $xa^{-1}x=b$ has a unique solution given by*

$$x \;=\; a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}. \tag{3}$$

Proof. The proof is similar to that for matrices; see e.g. [12]. It is easy to see that if $x=a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{\frac{1}{2}}a^{\frac{1}{2}}$, then $xa^{-1}x=b$. For uniqueness, let $x,y\in X$ be such that $xa^{-1}x=b=ya^{-1}y$. Then

$$(a^{-\frac{1}{2}}xa^{-\frac{1}{2}})^2 \;=\; a^{-\frac{1}{2}}xa^{-1}xa^{-\frac{1}{2}} \;=\; a^{-\frac{1}{2}}ya^{-1}ya^{-\frac{1}{2}} \;=\; (a^{-\frac{1}{2}}ya^{-\frac{1}{2}})^2.$$

By the uniqueness of the square root, we have $a^{-\frac{1}{2}}xa^{-\frac{1}{2}}=a^{-\frac{1}{2}}ya^{-\frac{1}{2}}$ and hence $x=y$. $\square$

Example 3.4. Let $\mathcal{A}$ be a C*-algebra and let $\mathcal{A}^+$ be the set of positive invertible elements in $\mathcal{A}$. Then $\mathcal{A}^+$ is a 2-powered twisted subgroup of the group of invertible elements in $\mathcal{A}$. For each $a,b\in\mathcal{A}^+$ we define

$$a\boxtimes b \;=\; ab^{-1}a.$$

Then $(\mathcal{A}^+,\boxtimes)$ is a lineated symmetric space in which the weighted mean $\#_t$ is given by

$$a\,\#_t\,b \;=\; a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}},\quad a,b\in\mathcal{A}^+,\ t\in\mathbb{R}. \tag{4}$$

Proof. It is straightforward to show that $(\mathcal{A}^+,\boxtimes)$ is a symset. Lemma 3.3 tells us that the equation $x\boxtimes a=b$ has a unique solution. Thus, $(X,\boxtimes)$ is a reflection quasigroup. We equip $\mathcal{A}^+$ with the subspace topology inherited from the norm topology on $\mathcal{A}$. The map $(a,b)\mapsto ab^{-1}a$ is continuous on $\mathcal{A}^+\times\mathcal{A}^+$ by the continuities of the multiplication and the inversion. Consider the map $(q,a)\mapsto a^q:\mathbb{D}\times\mathcal{A}^+\to\mathcal{A}^+$ and its extension $\Phi:\mathbb{R}\times\mathcal{A}^+\to\mathcal{A}^+$, $(t,a)\mapsto a^t=e^{t\log a}$. Here, e^a and $\log a$ are defined via the continuous functional calculus on the spectrum of a. Since the exponential map and the logarithm map on C*-algebras are continuous, Φ is continuous. Therefore, $\mathcal{A}^+$ is a lineated symmetric space by Theorem 2.5.

In the rest, we shall show that the weighted mean on $\mathcal{A}^+$ is given by (4). By uniqueness, it suffices to prove that, for each $a, b \in \mathcal{A}^+$, the map $\alpha_{a,b} : \mathbb{R} \to \mathcal{A}^+$, $\alpha_{a,b}(t) = a \#_t b$ is a continuous symmetry homomorphism such that $\alpha_{a,b}(0) = a$ and $\alpha_{a,b}(1) = b$. Indeed, for each $s, t \in \mathbb{R}$, we have

$$\begin{aligned}
\alpha_{a,b}(s) \boxtimes \alpha_{a,b}(t) &= [a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^s a^{\frac{1}{2}}][a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}}]^{-1}[a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^s a^{\frac{1}{2}}] \\
&= a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^s a^{\frac{1}{2}} a^{-\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{-t} a^{-\frac{1}{2}} a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^s a^{\frac{1}{2}} \\
&= a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^{2s-t} a^{\frac{1}{2}} \\
&= \alpha_{a,b}(s \bullet t).
\end{aligned}$$

Clearly, the map $\alpha_{a,b}$ is continuous, $\alpha_{a,b}(0) = a$ and $\alpha_{a,b}(1) = b$. □

4. Properties of weighted means

In this section, we establish further properties of weighted means in lineated symmetric spaces in which weights are arbitrary real numbers.

Theorem 4.1. *Let X be a lineated symmetric space. Then the following properties hold for all $x, y, z, w \in X$ and $r, s, t \in \mathbb{R}$:*

(1) $(x \#_r y) \#_t (x \#_s y) = x \#_{(1-t)r+ts} y$,

(2) *If* $x \# w = m = y \# z$, *then* $(x \#_t y) \# (w \#_t z) = m$,

(3) $x \#_r (x \#_s y) = x \#_{rs} y$,

(4) $x \bullet (y \#_t z) = (x \bullet y) \#_t (x \bullet z)$,

(5) $(x \#_t y) \#_s x = x \#_t (y \#_s x)$,

(6) $(y \#_t z)^{-1} = y^{-1} \#_t z^{-1}$, *here we fix a base point.*

Proof. To prove (1), let $t, r, s \in \mathbb{R}$. Since $\mathbb{D}$ is dense in $\mathbb{R}$, there are sequences (t_n), (r_n) and (s_n) in $\mathbb{D}$ such that $t_n \to t$, $r_n \to r$ and $s_n \to s$. By (sequential) continuity, we have $x \#_{r_n} y \to x \#_r y$ and $x \#_{s_n} y \to x \#_s y$. It follows that

$$(x \#_{r_n} y) \#_{t_n} (x \#_{s_n} y) \to (x \#_r y) \#_t (x \#_s y)$$

and hence $x \#_{(1-t_n)r_n+t_n s_n} y \to x \#_{(1-t)r+ts} y$. By Theorem 2.3, we have

$$(x \#_{r_n} y) \#_{t_n} (x \#_{s_n} y) = x \#_{(1-t_n)r_n+t_n s_n} y.$$

Since X is Hausdorff, we get $(x \#_r y) \#_t (x \#_s y) = x \#_{(1-t)r+ts} y$. The assertions (2)-(4) can be proved in a similar manner to (1). To prove (5), we have by using (3) that

$$\begin{aligned}
(x \#_t y) \#_s x &= x \#_{1-s} (x \#_t y) = x \#_{(1-s)t} y \\
&= x \#_t (x \#_{1-s} y) = x \#_t (y \#_s x).
\end{aligned}$$

The assertion (6) follows immediately from (4) by setting the base point $\varepsilon = x$. □

The property (5) in Theorem 4.1 had been not noticed before in the literature.

Theorem 4.2. *The following hold in a lineated symmetric space X:*

(1) *For each $w \in X$ and $t \in \mathbb{R} - \{1\}$, the map $x \mapsto x \#_t w$ is bijective.*

(2) *For each $w \in X$ and $t \in \mathbb{R} - \{0\}$, the map $x \mapsto w \#_t x$ is bijective.*

(3) *(right cancellability) $y \#_t x = z \#_t x$ for some $t \neq 1$ implies $y = z$.*

(4) *(left cancellability) $x \#_t y = x \#_t z$ for some $t \neq 0$ implies $y = z$.*

(5) *For each $a, b \in X$ such that $a \neq b$, the map $t \mapsto a \#_t b$ is injective.*

Proof. We shall provide proofs of (1) and (5); the assertion (2) can be similarly proved. The assertions (3)-(4) follow from (1)-(2). To prove (1), let $z \in X$. Then, by Theorem 4.1, we have

$$(w \#_{\frac{1}{1-t}} z) \#_t w \;=\; w \#_{1-t} (w \#_{\frac{1}{1-t}} z) \;=\; w \#_1 z \;=\; z.$$

Hence, the map $x \mapsto x \#_t w$ is surjective. To show the injectivity, let $w, y, z \in X$ be such that $y \#_t w = z \#_t w$. By Theorem 4.1, we get

$$y \;=\; w \#_1 y \;=\; w \#_{\frac{1}{1-t}} (w \#_{1-t} y) \;=\; w \#_{\frac{1}{1-t}} (w \#_{1-t} z) \;=\; z.$$

Thus, $x \mapsto w \#_t x$ is bijective. To prove (5), let $a, b \in X$ with $a \neq b$. Then the associated symmetry homomorphism $\alpha_{a,b}$ is nonconstant since $\alpha_{a,b}(0) \neq \alpha_{a,b}(1)$. By [16, Proposition 3.3], the map $\alpha_{a,b}$ is injective. It follows that if $a \#_t b = a \#_s b$, then $\alpha_{a,b}(t) = \alpha_{a,b}(s)$ and hence $t = s$. □

5. Weighted mean equations

In this section, we investigate certain weighted mean equations in lineated symmetric spaces. Every equation considered here is shown to have a unique solution in an explicit form. Our results generalizes the mean equations in [14]. Moreover, we discuss new mean equations.

From now on, let X be a lineated symmetric space and $a, b \in X$. To solve weighted mean equations in X, the property (2) will be frequently used without recalling.

Theorem 5.1. *Let $t \in \mathbb{R} - \{0\}$. Then the equation $a \#_t x = b$ has a unique solution $x = a \#_{\frac{1}{t}} b$.*

Proof. By Theorem 4.1, we have $a \#_t (a \#_{\frac{1}{t}} b) = a \#_1 b = b$. For uniqueness, let $y \in X$ be such that $a \#_t y = b$. Then

$$a \#_{\frac{1}{t}} b \;=\; a \#_{\frac{1}{t}} (a \#_t y) \;=\; a \#_1 y \;=\; y.$$

□

Theorem 5.1 includes Proposition 1.2 as a special case.

Corollary 5.2. *Let $s \in \mathbb{R} - \{0\}$ and $t \in \mathbb{R} - \{1\}$. Then, the equation*

$$(a \#_s x) \#_t a = b$$

has a unique solution $x = a \#_{\frac{1}{s(1-t)}} b$.

Proof. By Theorem 4.1, we have

$$(a \#_s x) \#_t a = a \#_{1-t} (a \#_s x) = a \#_{s(1-t)} x.$$

Thus, the equation $(a \#_s x) \#_t a = b$ is equivalent to $a \#_{s(1-t)} x = b$. Theorem 5.1 implies that $x = a \#_{\frac{1}{s(1-t)}} b$ is a unique solution of this equation. □

Corollary 5.3. *Let $s, t \in \mathbb{R}$ be such that $s(1-t) \neq 1$. Then the equation*

$$(x \#_s a) \#_t x = b$$

has a unique solution $x = a \#_{\frac{1}{1-s(1-t)}} b$.

Proof. Using Theorem 4.1, we get

$$\begin{aligned}(x \#_s a) \#_t x &= x \#_{1-t} (x \#_s a) \\ &= x \#_{s(1-t)} a \\ &= a \#_{1-s(1-t)} x.\end{aligned}$$

Hence, the equation $(x \#_s a) \#_t x = b$ is equivalent to $a \#_{1-s(1-t)} x = b$. By Theorem 5.1, we conclude that $x = a \#_{\frac{1}{1-s(1-t)}} b$ is the unique solution of this equation. □

Theorem 5.4. *Let $s, t \in \mathbb{R}$ be such that $st - s \neq -1$. Then the equation*

$$(a \#_s x) \#_t b = x \tag{5}$$

has a unique solution $x = a \#_{\frac{t}{st-s+1}} b$.

Proof. The cases $s = 0$ and $t = 0$ are trivial. Now, consider the case $s, t \neq 0$. Set $y = a \#_s x$. From the equation (5) and Theorem 5.1, we have $y \#_t b = x = a \#_{\frac{1}{s}} y$. It follows from Theorem 4.1 that

$$\begin{aligned}(a \#_s x) \#_{\frac{1}{t}} x &= y \#_{\frac{1}{t}} (a \#_{\frac{1}{s}} y) = y \#_{\frac{1}{t}} (y \#_{1-\frac{1}{s}} a) \\ &= y \#_{\frac{s-1}{st}} a \qquad = a \#_{\frac{st-s+1}{st}} y.\end{aligned}$$

Hence, the equation (5) is equivalent to $a \#_{\frac{st-s+1}{st}} y = b$. By Theorem 5.1, this equation has a unique solution

$$y = a \#_{\frac{st}{st-s+1}} b.$$

Therefore, the solution of (5) is given by

$$x \;=\; a \,\#_{\frac{1}{s}}\, y \;=\; a \,\#_{\frac{1}{s}}\, (a \,\#_{\frac{st}{st-s+1}}\, b) \;=\; a \,\#_{\frac{t}{st-s+1}}\, b.$$

□

Corollary 5.5. *For each $s, t \in \mathbb{R}$ such that $s \neq t$, the equation*

$$a \,\#_s\, x \;=\; b \,\#_t\, x$$

has a unique solution $x = a \,\#_{\frac{1-t}{s-t}}\, b$.

Proof. If $s = 0$, then by Theorem 5.1 the equation $a = b \,\#_t\, x$ has a unique solution $x = a \,\#_{\frac{t-1}{t}}\, b$. Now, assume that $s \neq 0$. By Theorem 5.1, one can transform the equation $a \,\#_s\, x = b \,\#_t\, x$ to the following equation

$$(b \,\#_t\, x) \,\#_{\frac{s-1}{s}}\, a \;=\; x.$$

Theorem 5.4 implies that this equation has a unique solution

$$x \;=\; b \,\#_{\frac{s-1}{s-t}}\, a \;=\; a \,\#_{\frac{1-t}{s-t}}\, b.$$

□

Theorem 5.6. *Let $r, s, t \in \mathbb{R}$ be such that $s - sr + rt \neq 0$. Then the equation*

$$(x \,\#_s\, a) \,\#_r\, (x \,\#_t\, b) \;=\; x \tag{6}$$

has a unique solution

$$x \;=\; a \,\#_{\frac{rt}{rt-rs+s}}\, b. \tag{7}$$

Proof. If $r = 0$, then the condition $s - sr + rt \neq 0$ implies $s \neq 0$. In this case, the equation (6) is reduced to $x \,\#_s\, a = x$, which has a unique solution $x = a$. Now, assume $r \neq 0$. According to Theorem 5.1, the equation (6) is equivalent to

$$x \,\#_t\, b \;=\; (x \,\#_s\, a) \,\#_{\frac{1}{r}}\, x.$$

Using Theorem 4.1, we transform this equation to

$$a \,\#_{\frac{r+s-rs}{r}}\, x \;=\; b \,\#_{1-t}\, x.$$

By Corollary 5.5, the above equation has a unique solution given by (7). □

As a special case of Theorem 5.6 when $s = t$, we have:

Corollary 5.7. *Let $r, t \in \mathbb{R}$ with $t \neq 0$. Then the mean $x = a \,\#_r\, b$ is a unique solution of the equation $(x \,\#_t\, a) \,\#_r\, (x \,\#_t\, b) = x$.*

6. Systems of weighted mean equations

In this final section, we solve certain systems of weighted mean equations in lineated symmetric spaces. Our results include systems of mean equations in [14] as special cases.

Theorem 6.1. *For each $s, t \in \mathbb{R}$ such that $s \neq t$, the system of equations*

$$\begin{aligned} x \#_s y &= a, \\ x \#_t y &= b \end{aligned} \tag{8}$$

has a unique solution $x = a \#_{\frac{s}{s-t}} b$ and $y = a \#_{\frac{s-1}{s-t}} b$.

Proof. If $s \neq 0$ and $t = 0$, then the system (8) has a unique solution $x = b$ and $y = a \#_{\frac{s-1}{s}} b$. If $s = 0$ and $t \neq 0$, then (8) has a unique solution $x = a$ and $y = a \#_{\frac{1}{t}} b$. Now, consider the case $s, t \neq 0$ with $s \neq t$. For uniqueness, let $x, y \in X$ be such that (8) holds. Note that the equation $x \#_s y = a$ is equivalent to $y = x \#_{\frac{1}{s}} a$ by Theorem 5.1. Similarly, the equation $x \#_t y = b$ is equivalent to $y = x \#_{\frac{1}{t}} b$. We have

$$a = x \#_s (x \#_{\frac{1}{t}} b) = x \#_{\frac{s}{t}} b = b \#_{1-\frac{s}{t}} x.$$

By Theorem 5.1, $x = b \#_{\frac{t}{t-s}} a = a \#_{\frac{s}{s-t}} b$. It follows that

$$\begin{aligned} y &= x \#_{\frac{1}{t}} b \quad = b \#_{1-\frac{1}{t}} (b \#_{\frac{t}{t-s}} a) \\ &= b \#_{\frac{t-1}{t-s}} a = a \#_{\frac{s-1}{s-t}} b. \end{aligned}$$

For existence, it is straightforward to verify that the above formulas of x and y satisfy the system (8). $\square$

Corollary 6.2. *For each $t \in \mathbb{R} - \{-1\}$, the system of equations*

$$\begin{aligned} x \bullet y &= a, \\ x \#_t y &= b \end{aligned}$$

has a unique solution $x = a \#_{\frac{1}{1+t}} b$ and $y = a \#_{\frac{2}{1+t}} b$.

Proof. From Theorem 6.1, put $s = -1$ and use the fact that $\#_{-1} = \bullet$. $\square$

Theorem 6.3. *Let $s, t, p, q \in \mathbb{R}$ be such that $q \neq 1$ and $s(1-q)+(q-t)(1-p) \neq 0$. Then the system*

$$\begin{aligned} y \#_p a &= b \#_s x, \\ y \#_q x &= a \#_t x \end{aligned} \tag{9}$$

has a unique solution given by $x = a \#_k b$ and $y = a \#_l b$ where

$$k = \frac{(s-1)(1-q)}{s(1-q)+(q-t)(1-p)}, \quad l = \frac{(s-1)(t-q)}{s(1-q)+(q-t)(1-p)}.$$

Proof. Suppose that x and y satisfy the system (9). By Theorems 4.1 and 5.1, we can deduce from the equation $y \,\#_p\, a = b \,\#_s\, x$ that

$$y \;=\; a \,\#_{\frac{1}{1-p}}\, (b \,\#_s\, x) \;=\; (b \,\#_s\, x) \,\#_{\frac{p}{p-1}}\, a. \tag{10}$$

Similarly, the equation $y \,\#_q\, x = a \,\#_t\, x$ implies

$$y \;=\; x \,\#_{\frac{1-t}{1-q}}\, a. \tag{11}$$

From (10) and (11), we have by using Corollary 5.5 that $a = (b \,\#_s\, x) \,\#_r\, x$, where

$$r \;:=\; \frac{pt - t - pq + q}{pt - t - pq + 1}.$$

Rewrite it as $a = (x \,\#_{1-s}\, b) \,\#_r\, x$. Corollary 5.3 now implies that

$$x \;=\; b \,\#_{\frac{1}{1-(1-s)(1-r)}}\, a \;=\; a \,\#_k\, b.$$

It follows from (11) that

$$y \;=\; a \,\#_{\frac{(t-q)k}{1-q}}\, b \;=\; a \,\#_l b.$$

For existence, it is straightforward to prove that $x = a \,\#_k\, b$ and $y = a \,\#_l\, b$ satisfy the system (9). □

Corollary 6.4. *Let $s, t \in \mathbb{R}$ be such that $s \neq t + 1$. Then the system*

$$\begin{aligned} y \bullet a &= b \,\#_s\, x, \\ y \bullet x &= a \,\#_t\, x \end{aligned}$$

has a unique solution given by $x = a \,\#_{\frac{s-1}{s-t-1}}\, b$ and $y = a \,\#_{\frac{(t+1)(s-1)}{2(s-t-1)}}\, b$.

Proof. Put $p = q = -1$ in Theorem 6.3 and use the fact that $\#_{-1} = \bullet$. □

Corollary 6.5. *Let $p, q \in \mathbb{R}$ be such that $2q - p - pq \neq 0$. Then the system*

$$\begin{aligned} y \,\#_p\, a &= b \bullet x, \\ y \,\#_q\, x &= a \bullet x \end{aligned} \tag{12}$$

has a unique solution given by $x = a \,\#_{\frac{2(q-1)}{2q-p-pq}}\, b$ and $y = a \,\#_{\frac{2(q+1)}{2q-p-pq}}\, b$.

Proof. If $q = 1$ and $2q - p - pq \neq 0$, then $p \neq 1$ and the system (12) has a unique solution given by $x = a$ and $y = a \,\#_{\frac{2}{1-p}}\, b$. To treat the case $q \neq 1$ and $2q - p - pq \neq 0$, we apply Theorem 6.3 when $s = t = -1$, and use the fact that $\#_{-1} = \bullet$. □

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