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On intra-regular ordered hypersemigroups

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Abstract. We present a structure theorem referring to the decomposition of ordered hypersemigroups into simple components. For an intra-regular ordered hypersemigroup H, the very simple form of its principal filters leads to a characterization of H as a semilattice of simple hypersemigroups; that is as an ordered hypersemigroup for which there exists a semilattice congruence σ such that $(x)_\sigma$ is a simple subhypersemigroup of H for every $x \in H$. This is equivalent to saying that H is a union of simple subhypersemigroups of H. In addition, an ordered hypersemigroup H is intra-regular and the hyperideals of H form a chain if and only if it is a chain of simple hypersemigroups. On this occasion, some further results related to intra-regular ordered hypersemigroups have been also given.

1. Introduction and prerequisites

The concept of the hypergroup introduced by the French Mathematician F. Marty at the 8th Congress of Scandinavian Mathematicians in 1933 is as follows: An hypergroup is a nonempty set H endowed with a multiplication xy such that (i) $xy \subseteq H$; (ii) $x(yz) = (xy)z$; (iii) $xH = Hx = H$ for every x, y, z in H. I searched, as far as I now is not possible to find the Proceedings of that Conference now, so for this definition (of the hypergroup) I will refer to Mittas [7]; and then the definitions of the hypersemigroup and the ordered hypersemigroup follows at a natural way. The first researchers who investigate hypergroups using the definition given by Marty were Mittas and Corsini [1, 6]. Since Marty introduced this concept, hundreds of papers on hyperstructures appeared using the definition given by Marty; and in the recent years, many groups in the world investigate the hyperstructures in research programs based on this definition. We will mention only few papers related to hypersemigroups in the References such as [1, 2, 6 − 9].

An interesting problem in the theory of ordered hypersemigroups is to describe the type of ordered hypersemigroups that are decomposable into left (right) simple or simple components. It is a difficult problem especially for the case of right (left) simple ordered hypersemigroups. In the present paper we managed to solve the problem just for the intra-regular ordered hypersemigroups, showing that the intra-regular ordered hypersemigroups are decomposable into simple components, and we hope that it will be a starting point for the investigation of the type of ordered hypersemigroups that are decomposable into right or left simple components. We

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prove that every intra-regular ordered hypersemigroup H is a semilattice of simple subhypersemigroups, that is for every intra-regular ordered hypersemigroup there exists a semilattice congruence σ on H such that the class $(x)_\sigma$ is a simple subhypersemigroup of H for every $x \in H$ – and so H is the union of these classes; in other words any intra-regular ordered hypersemigroup is decomposable into simple components. And, conversely, every ordered hypersemigroup that is a union of simple subhypersemigroups is intra-regular. In addition, an ordered hypersemigroup H is intra-regular and the hyperideals of H form a chain, if and only if H is a chain of simple hypersemigroups. On this occasion, some further related results concerning the intra-regular ordered hypersemigroups have been also given. We prove, for example, that for an intra-regular ordered hypersemigroup H, the set $\{(x)_\mathcal{N} \mid x \in H\}$ coincides with the set of all maximal simple subhypersemigroups of H; and an ordered hypersemigroup H is intra-regular and the hyperideals of H form a chain if and only if the hyperideals of H are prime. As an application of the results of the present paper, the corresponding results on hypersemigroup (without order) can be obtained, and this is because every hypersemigroup endowed with the equality relation $=$ is an ordered hypersemigroup.

For the sake of completeness, we give the following definitions: Let $(H, \circ, \leqslant)$ be an ordered hypersemigroup. For a subset A of H we denote by $(A]$ the subset of H defined by $(A] := \{t \in H \mid t \leqslant h \text{ for some } h \in A\}$. A nonempty subset I of H is called a *right* (resp. *left*) *hyperideal* of H if (1) $I \circ H \subseteq I$ (resp. $H \circ I \subseteq I$) and (2) if $a \in I$ and $b \in H$ such that $b \leqslant a$, then $b \in I$, that is $(I] = I$; it is called an *hyperideal* of H if it is both a right and a left hyperideal of H. A nonempty subset F of H is called an *hyperfilter* of H if (1) if $a, b \in F$, then $a \circ b \subseteq F$. (2) if $a, b \in H$ such that $a \circ b \subseteq F$, then $a \in F$ and $b \in F$. (3) For every $a, b \in H$, either $a \circ b \subseteq F$ or $(a \circ b) \cap F = \emptyset$. (4) if $a \in F$ and $H \ni b \geqslant a$, then $b \in F$; that is F is a subhypersemigroup of H satisfying the relations (2)– (4). This is equivalent to saying that for any $a, b \in H$ such that $(a \circ b) \cap F \neq \emptyset$, we have $a, b \in F$ and the property (4) given above is satisfied. An equivalence relation σ on H is called *congruence* if $(a, b) \in \sigma$ implies $(a \circ c, b \circ c) \in \sigma$ and $(c \circ a, c \circ b) \in \sigma$ for any $c \in H$; in the sense that for any $u \in a \circ c$ and any $v \in b \circ c$ we have $(u, v) \in \sigma$ and for any $u \in c \circ a$ and any $v \in c \circ b$ we have $(u, v) \in \sigma$. A congruence σ on H is called *semilattice congruence* if, for any $a, b \in H$ we have $(a, a \circ a) \in \sigma$ and $(a \circ b, b \circ a) \in \sigma$; in the sense that for every $u \in a \circ a$ we have $(a, u) \in \sigma$ and for every $u \in a \circ b$ and every $v \in b \circ a$ we have $(u, v) \in \sigma$. A semilattice congruence σ on $(H, \circ, \leqslant)$ is called *complete* if $a \leqslant b$ implies $(a, a \circ b) \in \sigma$; in the sense that if $u \in a \circ b$, then $(a, u) \in \sigma$. An ordered hypersemigroup H is called *simple* if for every hyperideal A of H we have $A = H$, that is if H is the only hyperideal of H. We denote by $\mathcal{N}$ the semilattice congruence on H defined by $a\mathcal{N}b$ if and only if $N(a) = N(b)$, where $N(a)$ is the principal hyperfilter of H. An ordered hypersemigroup H is called a *semilattice of simple hypersemigroups* if there exists a semilattice congruence σ of H such that the σ-class $(x)_\sigma$ of H containing x is a simple subhypersemigroup of H for every $x \in H$. This is equivalent to saying that there exists a semilattice $(Y, \cdot)$ and a nonempty family $\{H_\alpha \mid \alpha \in Y\}$ of simple subhypersemigroups of H such that (1) $H_\alpha \cap H_\beta = \emptyset$ for every $\alpha, \beta \in Y$, $\alpha \neq \beta$; (2) $H = \bigcup\{H_\alpha \mid \alpha \in A\}$; (3) $H_\alpha \circ H_\beta \subseteq H_{\alpha\beta}$ for every $\alpha, \beta \in Y$.

An ordered hypersemigroup H is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ such that $a \leqslant x \circ a^2 \circ y$; equivalently $a \in (H \circ a^2 \circ H]$ for every element a of H or $A \subseteq (H \circ A^2 \circ H]$ for every nonempty subset A of H. For further information we refer to [3, 4].

2. Main results

Lemma 2.1. [4, Proposition 2.4] *For an ordered hypersemigroup H the relation $\mathcal{N}$ is a complete semilattice congruence on H.*

Lemma 2.2. *An ordered hypersemigroup $(H, \circ, \leqslant)$ is intra-regular if and only if, for every $x \in H$, we have $N(x) = \{y \in H \mid x \in (H \circ y \circ H]\}$.*

Proof. ($\Rightarrow$). Let $x \in H$ and $T := \{y \in H \mid x \in (H \circ y \circ H]\}$. The set T is an hyperfilter of H containing x. In fact: T is a nonempty subset of H containing the element x as $x \in (H \circ x^2 \circ H] \subseteq (H \circ x \circ H]$. Let $a, b \in T$. Since $x \in (H \circ a \circ H]$ and $x \in (H \circ b \circ H]$, we have

$$\begin{aligned}
x \in (H \circ x^2 \circ H] &\subseteq \Big(H \circ (H \circ b \circ H] \circ (H \circ a \circ H] \circ H\Big] \\
&= \Big((H] \circ (H \circ b \circ H] \circ (H \circ a \circ H] \circ (H]\Big] \\
&\subseteq \Big((H \circ (H \circ b \circ H) \circ (H \circ a \circ H) \circ H]\Big] \\
&= \Big(H \circ (H \circ b \circ H) \circ (H \circ a \circ H) \circ H\Big] \\
&\subseteq \Big(H \circ (b \circ H \circ a) \circ H\Big] \\
&\subseteq \bigg(H \circ \Big(H \circ (b \circ H \circ a)^2 \circ H\Big] \circ H\bigg] \\
&= \bigg((H] \circ \Big(H \circ (b \circ H \circ a)^2 \circ H\Big] \circ (H]\bigg] \\
&\subseteq \bigg(\Big(H^2 \circ (b \circ H \circ a)^2 \circ H^2\Big]\bigg] \\
&= \Big(H^2 \circ (b \circ H \circ a)^2 \circ H^2\Big] \\
&\subseteq \Big(H \circ (b \circ H \circ a)^2 \circ H\Big] \\
&= \Big(H \circ (b \circ H \circ a) \circ (b \circ H \circ a) \circ H\Big] \\
&\subseteq \Big(H \circ (a \circ b) \circ H\Big],
\end{aligned}$$

and so $a \circ b \subseteq T$. If $a, b \in H$ such that $a \circ b \subseteq T$, then $x \in (H \circ a \circ b \circ H] \subseteq (H \circ a \circ H]$, similarly $x \in (H \circ b \circ H]$ and so $a, b \in T$. Moreover, for any $a, b \in H$ we clearly have $a \circ b \subseteq T$ or $(a \circ b) \cap T = \emptyset$. If $a \in T$ and $b \in H$ such that $b \geqslant a$, then $x \in (H \circ a \circ H] \subseteq (H \circ b \circ H]$ and so $b \in T$. Let now F be an hyperfilter of H such that $x \in F$ and let $y \in T$. Since $x \in (H \circ y \circ H]$, we have $F \ni x \leqslant t \circ y \circ h$ for some $t, h \in H$. Then we have $t \circ y \circ h \subseteq F$ and then $y \in F$.

($\Leftarrow$). Let $x \in H$. Since $x \in N(x)$ and $N(x)$ is a subhypersemigroup of H, we have $x \circ x \subseteq N(x)$, then we have $x \in (H \circ x^2 \circ H]$ and so H is intra-regular. $\square$

Theorem 2.3. *Let H be an ordered hypersemigroup. If H is intra-regular, then it is a semilattice of simple hypersemigroups. Conversely, if H is a union of simple subhypersemigroups of H, then it is intra-regular.*

Proof. Let H be intra-regular. The set $(x)_{\mathcal{N}}$ is a simple subhypersemigroup of H for every $x \in H$. In fact: Let $x \in H$. Since $\mathcal{N}$ is a semilattice congruence on H, $(x)_{\mathcal{N}}$ is a subhypersemigroup of H. Let I be an hyperideal of $(x)_{\mathcal{N}}$ and $y \in (x)_{\mathcal{N}}$. Let $z \in I$ $(I \neq \emptyset)$. Since $z \in (x)_{\mathcal{N}}$, we have $(z)_{\mathcal{N}} = (x)_{\mathcal{N}}$. Since $\mathcal{N}$ is a semilattice congruence on H, we have $(z^5)_{\mathcal{N}} = (z)_{\mathcal{N}}$. On the other hand, $(z^5)_{\mathcal{N}} = (H \circ z^3 \circ H]$. Indeed:

$$\begin{aligned} t \in (z^5)_{\mathcal{N}} &\Rightarrow (t, z^5) \in \mathcal{N} \Rightarrow N(t) = N(z^5) \Rightarrow z^5 \in N(t) \\ &\Rightarrow t \in (H \circ z^5 \circ H] \subseteq (H \circ z^3 \circ H]. \end{aligned}$$

Since $y \in (x)_{\mathcal{N}}$, we have $y \in (H \circ z^3 \circ H]$. Thus we have

$$(z)_{\mathcal{N}} \ni y \leqslant a \circ z^3 \circ b = (a \circ z) \circ z \circ (z \circ b) \text{ for some } a, b \in H.$$

It is enough to prove that $a \circ z, z \circ b \subseteq (x)_{\mathcal{N}}$. Then, since $z \in I$ and I is an hyperideal of $(x)_{\mathcal{N}}$, we have $(a \circ z) \circ z \circ (z \circ b) \subseteq I$ and so $y \in I$. In fact, we have

$$\begin{aligned} a \circ z \subseteq (a \circ z)_{\mathcal{N}} :&= (a)_{\mathcal{N}} \circ (z)_{\mathcal{N}} = (a)_{\mathcal{N}} \circ (y)_{\mathcal{N}} \\ &= (a)_{\mathcal{N}} \circ (y \circ a \circ z^3 \circ b)_{\mathcal{N}} \text{ (since } y \leqslant a \circ z^3 \circ b) \\ &= (y \circ a \circ z^3 \circ b)_{\mathcal{N}} \text{ (since } \mathcal{N} \text{ is a semilattice congruence)} \\ &= (y)_{\mathcal{N}} = (x)_{\mathcal{N}} \end{aligned}$$

and

$$\begin{aligned} z \circ b \subseteq (z \circ b)_{\mathcal{N}} :&= (z)_{\mathcal{N}} \circ (b)_{\mathcal{N}} = (y)_{\mathcal{N}} \circ (b)_{\mathcal{N}} = (y \circ a \circ z^3 \circ b)_{\mathcal{N}} \circ (b)_{\mathcal{N}} \\ &= (y \circ a \circ z^3 \circ b)_{\mathcal{N}} = (y)_{\mathcal{N}} = (x)_{\mathcal{N}}. \end{aligned}$$

Therefore H is a semilattice of simple hypersemigroups; as so it is a union of simple subhypersemigroups of H.

For the converse statement, suppose $H = \bigcup\{H_\alpha \mid \alpha \in A\}$ where H_α is a simple subhypersemigroup of H for every $\alpha \in A$. Let now $x \in H$. Then $x \in H_\alpha$ for some $\alpha \in A$. On the other hand, the set $(H \circ x^2 \circ H] \cap H_\alpha$ is an hyperideal of H_α. Indeed, this is a nonempty subset of H_α as $x^4 \subseteq (Hx^2H]$ and $x^4 \subseteq H_\alpha$ and moreover we have

$$\Big((H \circ x^2 \circ H] \cap H_\alpha\Big) \circ H_\alpha \subseteq (H \circ x^2 \circ H] \circ H_\alpha \cap H_\alpha^2$$
$$\subseteq (H \circ x^2 \circ H] \circ H \cap H_\alpha \subseteq (H \circ x^2 \circ H] \cap H_\alpha$$

since $(H \circ x^2 \circ H]$ is an hyperideal of H. Also,

$$H_\alpha \circ \Big((H \circ x^2 \circ H] \cap H_\alpha\Big) \subseteq H_\alpha \circ (H \circ x^2 \circ H] \cap H_\alpha^2$$
$$\subseteq H \circ (H \circ x^2 \circ H] \cap H_\alpha \subseteq (H \circ x^2 \circ H] \cap H_\alpha.$$

Let now $y \in (H \circ x^2 \circ H] \cap H_\alpha$ and $z \in H_\alpha$ such that $z \leqslant y$. Since $H \ni z \leqslant y \in (H \circ x^2 \circ H]$ and $(H \circ x^2 \circ H]$ is an hyperideal of H, we have $z \in (H \circ x^2 \circ H]$ and thus $z \in (H \circ x^2 \circ H] \cap H_\alpha$. Since H_α is a simple subhypersemigroup of H, we have $(H \circ x^2 \circ H] \cap H_\alpha = H_\alpha$. Since $x \in H_\alpha$, we get $x \in (H \circ x^2 \circ H]$ and so H is intra-regular. □

For an ordered hypersemigroup H and a semilattice congruence σ on H, we denote by "$\preceq$" the order on the hypersemigroup H/σ defined by

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (x \circ y)_\sigma.$$

An ordered hypersemigroup H is called a *chain of simple hypersemigroups* if there exists a semilattice congruence σ on H such that the class $(x)_\sigma$ is a simple subhypersemigroup of H for every $x \in H$ and $(H/\sigma, \preceq)$ is a chain.

In the following, when we say that the hyperideals of H form a chain we suppose them endowed with the inclusion relation "$\subseteq$".

A nonempty subset T of an hypersemigroup H is said to be *prime* if for any nonempty subsets A, B of H such that $A \circ B \subseteq T$, we have $A \subseteq T$ or $B \subseteq T$; equivalently, for any elements a, b of T such that $a \circ b \subseteq T$, we have $a \in T$ or $b \in T$ [5].

Lemma 2.4. [5, Corollary 24] *If H is an ordered hypersemigroup, then the hyperideals of H are prime if and only if they form a chain and H is intra-regular.*

Lemma 2.5. *Let H be an ordered hypersemigroup. If H is intra-regular and the hyperideals of H form a chain, then for every $x, y \in H$, we have*

$$x \in (H \circ x \circ y \circ H] \quad \text{or} \quad y \in (H \circ x \circ y \circ H] \qquad (*)$$

The converse statement also holds.

Proof. Let H be intra-regular and the hyperideals of H form a chain. Then, by Lemma 2.4, the hyperideals of H are prime. Since $(H \circ x \circ y \circ H]$ is an hyperideal of H and $x^2 \circ y^2 \subseteq (H \circ x \circ y \circ H]$, we have $x^2 \subseteq (H \circ x \circ y \circ H]$ or $y^2 \subseteq (H \circ x \circ y \circ H]$ and then we have $x \in (H \circ x \circ y \circ H]$ or $y \in (H \circ x \circ y \circ H]$.

For the converse statement, suppose the relation $(*)$ is satisfied. To prove that H is intra-regular, by Lemma 2.4, it is enough to prove that the hyperideals of H are prime. For this purpose, let I be an hyperideal of H and $a, b \in H$ such that $a \circ b \subseteq I$. Since $a, b \in H$, by hypothesis, we have $a \in (H \circ a \circ b \circ H]$ or $b \in (H \circ a \circ b \circ H]$. If $a \in (H \circ a \circ b \circ H]$ then, since $a \circ b \subseteq I$, we have $a \in (H \circ I \circ H] \subseteq (I] = I$ and so $a \in I$. If $b \in (H \circ I \circ H]$, then $b \in (H \circ I \circ H \subseteq (I] = I$ and so $b \in I$ and the proof is complete. □

Theorem 2.6. *An ordered hypersemigroup H is intra-regular and the hyperideals of H form a chain if and only if H is a chain of simple hypersemigroups*

Proof. $(\Rightarrow)$. Since H is intra-regular, the relation $\mathcal{N}$ is a semilattice congruence on H (see the proof of Theorem 2.3), and the class $(x)_{\mathcal{N}}$ is a simple subhypersemigroup of H for every $x \in H$. Let now $(x)_{\mathcal{N}}, (y)_{\mathcal{N}} \in S/\mathcal{N}$. By hypothesis and Lemma 2.5, we have $x \in (H \circ x \circ y \circ H]$ or $y \in (H \circ x \circ h \circ H]$. Let $x \in (H \circ x \circ y \circ H]$. Since $N(x) \ni x \leqslant t \circ x \circ y \circ h$ for some $t, h \in H$, we have $x \circ y \subseteq N(x)$, thus $N(x \circ y) \subseteq N(x)$. Let $y \in (H \circ x \circ h \circ H]$. Since $N(y) \ni y \leqslant z \circ x \circ y \circ k$ for some $z, k \in H$, we have $x \circ y \subseteq N(y)$ and then $N(x \circ y) \subseteq N(y)$. On the other hand, $x \circ y \subseteq N(x \circ y)$ implies $x, y \in N(x \circ y)$, then $N(x) \subseteq N(x \circ y)$ and $N(y) \subseteq N(x \circ y)$. Hence we obtain $N(x \circ y) = N(x)$ or $N(x \circ y) = N(y)$, Thus $(x)_{\mathcal{N}} = (x \circ y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} = (x \circ y)_{\mathcal{N}} = (y \circ x)_{\mathcal{N}}$, that is $(x)_{\mathcal{N}} \preceq (y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$.

$(\Leftarrow)$. Let σ be a semilattice congruence on H such that $(x)_\sigma$ is a simple subhypersemigroup of H for every $x \in H$ and let $(S/\sigma, \preceq)$ be a chain. We have to prove that H is intra-regular and the hyperideals of H form a chain. By Lemma 2.4, it is enough to prove that the hyperideals of H are prime. Let I be an hyperideal of H and $a, b \in H$ such that $a \circ b \subseteq I$. The set $(a \circ b)_\sigma \cap I$ is an hyperideal of $(a \circ b)_\sigma$. Indeed: it is a nonempty subset of $(a \circ b)_\sigma$ as $a \circ b \subseteq (a \circ b)_\sigma$ and $a \circ b \subseteq I$; and we also have

$$\Big((a \circ b)_\sigma \cap I\Big) \circ (a \circ b)_\sigma \subseteq (a \circ b)_\sigma^2 \cap I \circ (a \circ b)_\sigma \subseteq (a \circ b)_\sigma \cap I \circ H \subseteq (ab)_\sigma \cap I$$

and

$$(a \circ b)_\sigma \circ \Big((a \circ b)_\sigma \cap I\Big) \subseteq (a \circ b)_\sigma^2 \cap (a \circ b)_\sigma \circ I \subseteq (a \circ b)_\sigma \cap S \circ I \subseteq (a \circ b)_\sigma \cap I.$$

Let now $x \in (a \circ b)_\sigma \cap I$ and $y \in (a \circ b)_\sigma$ such that $y \leqslant x$. Since $H \ni y \leqslant x \in I$ and I is an hyperideal of H, we have $y \in I$, and so $y \in (a \circ b)_\sigma \cap I$. By hypothesis, $(a \circ b)_\sigma$ is a simple subhypersemigroup of H, so we have $(a \circ b)_\sigma \cap I = (a \circ b)_\sigma$. Again by hypothesis, we have $(a)_\sigma \preceq (b)_\sigma$ or $(b)_\sigma \preceq (a)_\sigma$. If $(a)_\sigma \preceq (b)_\sigma$, then $(a)_\sigma = (a \circ b)_\sigma$, then $(a)_\sigma \cap I = (a)_\sigma$ and so $a \in I$. If $(b)_\sigma \preceq (a)_\sigma$ then, since σ is a semilattice congruence on H, we have $(b)_\sigma = (b \circ a)_\sigma = (a \circ b)_\sigma$, then $(b)_\sigma \cap I = (b)_\sigma$ and so $b \in I$. □

Proposition 2.7. *If H is an intra-regular ordered hypersemigroup then, for any $x, y \in H$, we have $(H \circ x \circ y \circ H] = (H \circ x \circ y \circ H]$.*

Proof. Since H is intra-regular, we have

$$\begin{aligned} x \circ y &\subseteq \Big(H \circ (x \circ y)^2 \circ H\Big] = (H \circ x \circ y \circ x \circ y \circ H] \\ &\subseteq (H^2 \circ y \circ x \circ H^2] \subseteq (H \circ y \circ x \circ H]. \end{aligned}$$

Then we have

$$\begin{aligned} H \circ x \circ y \circ H &\subseteq H \circ (H \circ y \circ x \circ H] \circ H \\ &= (H] \circ (H \circ y \circ x \circ H] \circ (H] \subseteq \Big((H^2 \circ y \circ x \circ H^2]\Big] \\ &= (H^2 \circ y \circ x \circ H^2] \subseteq (H \circ y \circ x \circ H], \end{aligned}$$

from which $(H \circ x \circ y \circ H] \subseteq \Big((H \circ y \circ x \circ H]\Big] = (H \circ y \circ x \circ H]$. By symmetry, in a similar way we prove that $(H \circ y \circ x \circ H] \subseteq (H \circ x \circ y \circ H]$, and equality holds. □

Proposition 2.8. *For an intra-regular ordered hypersemigroup H, the set of all $(x)_{\mathcal{N}}$ such that $x \in H$, coincides with the set of all maximal simple subhypersemigroups of H.*

Proof. Let $x \in H$. Then $(x)_{\mathcal{N}}$ is a maximal simple subhypersemigroup of H. In fact: First of all, since H is intra-regular, the set $(x)_{\mathcal{N}}$ is a simple subhypersemigroup of H (see the proof of Theorem 2.3). Let now T be a simple subhypersemigroup of H such that $T \supseteq (x)_{\mathcal{N}}$. Then $T = (x)_{\mathcal{N}}$. Indeed: Let $y \in T$. Since $x \in T$, the set $(H \circ x \circ H] \cap T$ is an hyperideal of T. Since T is simple, we have $(H \circ x \circ H] \cap T = T$. Since $y \in T$, we have $y \in (H \circ x \circ H]$. Since H is intra-regular and $y \in H \circ x \circ H$, by Lemma 2.2, we have $x \in N(y)$ and so $N(x) \subseteq N(y)$. Similarly, since $y \in T$, the set $(H \circ y \circ H] \cap T$ is an hyperideal of T. Since T is simple, we have $(H \circ y \circ H] \cap T = T$. Since $x \in T$, we obtain $x \in (H \circ y \circ H]$ and then, by Lemma 2.2, we get $y \in N(x)$ and so $N(y) \subseteq N(x)$. Thus we have $N(x) = N(y)$, that is $y \in (x)_{\mathcal{N}}$. Therefore we have $T \subseteq (x)_{\mathcal{N}}$ and so $T = (x)_{\mathcal{N}}$.

For the converse statement, let T be a maximal simple subhypersemigroup of H. Then there exists $x \in H$ such that $T = (x)_{\mathcal{N}}$. Indeed: Take an element $x \in T$ $(T \neq \emptyset)$. In a similar way as in the case above, we prove that $T \subseteq (x)_{\mathcal{N}}$. Since H is intra-regular, the class $(x)_{\mathcal{N}}$ is a simple subhypersemigroup of H. Since T is a maximal simple subhypersemigroup of H, we have $T = (x)_{\mathcal{N}}$ and the proof is complete. □

We wrote this paper at the usual way, and we will come back to this paper in a forthcoming paper.

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