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# On irreducible pseudo-prime spectrum of topological le-modules

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**Abstract.** An le-module $M$ over a ring $R$ is a complete lattice ordered additive monoid having the greatest element $e$ together with a module like action of $R$. A proper submodule element $n$ of $_RM$ is called ***pseudo-prime*** if $(n:e) = \{r \in R : re \leqslant n\}$ is a prime ideal of $R$. In this article we introduce the ***Zariski topology*** on the set $X_M$ of all pseudo-prime submodule elements of $M$ and discuss interplay between topological properties of the Zariski topology on $X_M$ and algebraic properties of $M$. If $_RM$ is pseudo-primeful, then irreducibility of $X_M$ and $\mathrm{Spec}(R/Ann(M))$ are equivalent. Also there is a one-to-one correspondence between the irreducible components of $X_M$ and the minimal pseudo-prime submodule elements in $M$. We show that if $R$ is a Laskerian ring then $X_M$ has only finitely many irreducible components.

## 1. Introduction

Inspired by the theory of multiplicative lattices [1], [17], [18], [19], [20], and lattice modules [7], [8], [9], [10], [11], [14], [21], we introduced the notion of le-modules in [2]. An le-module is a complete lattice ordered monoid endowed with a module like action of a commutative ring. Motivation behind introducing this new notion is to create a new avenue similar to what we do in module theory for studying commutative rings. In [2] and [12] we find several results on the interplay between properties of an le-module $M$ and properties of the ring $R$ acting on $M$. We considered uniqueness of primary decompositions of the primary submodule elements in a Laskerian le-module in [2].

In this article, we introduce the Zariski topology on the set $X_M$ of all pseudo-prime submodule elements of an le-module $M$ over a commutative ring $R$. Inspiration comes from the enlightening interplay between the Zariski topology on the prime spectrum $\mathrm{Spec}(R)$ of a commutative ring $R$ and the ring theoretic properties of $R$ [6], [13], [15], [16]; and interplay between the Zariski topology on the pseudo-prime spectrum of a module $A$ over $R$ and the algebraic properties of $_RA$ and $R$ [4], [5]. Besides basic characterizations of the Zariski topology on $X_M$, we find several conditions on $M$ under which $X_M$ may be an irreducible topological space.

The organization of this article is as follows. This introduction is followed by

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a section to recap definition and basic properties of le-modules. Also we recall a few notions on rings. In Section 3, we introduce the ***Zariski topology*** on $X_M$ and characterize its basic properties. We show that $X_M$ is always $T_0$ and it is $T_1$ if and only if each pseudo-prime submodule element of $_RM$ is maximal in $X_M$. Annihilator of $M$ is an ideal of $R$, which induces a natural mapping $\psi$ from $X_M$ into Spec$(R/Ann(M))$. Interplay of the properties of $X_M$ and $Spec(R/Ann(M))$ is reflected prominently in the nature of this natural map $\psi$. Here we show that if $\psi$ is surjective, then connectedness of $X_M$ implies the connectedness of Spec$(R/Ann(M))$. Section 4 characterizes irreducibility of $X_M$. If $\psi$ is surjective then irreducibility of $X_M$ and Spec$(R/Ann(M))$ are equivalent. As a consequence of the necessary and sufficient characterization of the irreducible closed subsets, presented here, we establish a bijective correspondence between the irreducible components of $X_M$ and the minimal pseudo-prime submodule elements of $_RM$. Also we prove that if a ring $R$ is Laskerian then for every le-module $_RM$, the pseudo-prime spectrum $X_M$ has only finitely many irreducible components.

## 2. Preliminaries

In this article, every ring $R$ is commutative and contains 1; and $\mathbb{N}$ denotes the set of all natural numbers. An *le-semigroup* $(M, +, \leqslant, e)$ is such that $(M, \leqslant)$ is a complete lattice with the greatest element $e$, $(M, +)$ is a commutative monoid with the zero element $0_M$ and for all $m, m_i \in M, i \in I$ it satisfies

$(S)$ $m + (\vee_{i\in I} m_i) = \vee_{i\in I}(m + m_i)$.

Let $R$ be a ring and $(M, +, \leqslant, e)$ be an le-semigroup. Then $M$ is called an *le-module* over $R$ if there is a mapping $R \times M \longrightarrow M$ which satisfies

(M1) $r(m_1 + m_2) = rm_1 + rm_2$,

(M2) $(r_1 + r_2)m \leqslant r_1m + r_2m$,

(M3) $(r_1r_2)m = r_1(r_2m)$,

(M4) $1_Rm = m; \quad 0_Rm = r0_M = 0_M$,

(M5) $r(\vee_{i\in I}(m_i)) = \vee_{i\in I}(rm_i)$,

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2, m_i \in M$, and $i \in I$.

We denote an le-module $M$ over $R$ by $_RM$ or by $M$. From $(M5)$, we have,

(M5)$'$ $m_1 \leqslant m_2 \Rightarrow rm_1 \leqslant rm_2$, for all $r \in R$ and $m_1, m_2 \in M$.

An element $n$ of $M$ is said to be a ***submodule element*** if $n + n, rn \leqslant n$, for all $r \in R$. We call a submodule element $n$ proper if $n \neq e$. Note that $0_M = 0_Rn \leqslant n$, for every submodule element $n$ of $M$. Also $n + n = n$, i.e., every submodule element of $M$ is an idempotent. Let $\{n_i\}_{i\in I}$ be a family of submodule elements of $M$. Then their sum is defined by:

$$\sum_{i\in I} n_i = \bigvee\{(n_{i_1}+n_{i_2}+\cdots+n_{i_k}) : k\in\mathbb{N}, and\ \ i_1,i_2,\cdots,i_k\in I\}.$$

It is easy to check that $\sum_{i\in I} n_i$ is a submodule element of $M$.

For an ideal $I$ of $R$, we define

$$Ie = \bigvee\{\sum_{i=1}^{k} a_i e : k\in\mathbb{N}; a_1,a_2,\cdots,a_k\in I\}$$

Then $Ie$ is a submodule element of $M$. Also for any two ideals $I$ and $J$ of $R$, $I\subseteq J$ implies that $Ie\leqslant Je$.

Let $n$ be a submodule element of $M$. We denote

$$(n:e)=\{r\in R : re\leqslant n\}.$$

Then $(n:e)$ is an ideal of $R$. For any two submodule elements $n,l$ of $M$, $n\leqslant l$ implies that $(n:e)\subseteq(l:e)$. Also if $\{n_i\}_{i\in I}$ is an arbitrary family of submodule elements in ${}_RM$, then $(\wedge_{i\in I} n_i : e)=\cap_{i\in I}(n_i:e)$. For every submodule element $n$ of ${}_RM$ and ideal $I$ of $R$, $Ie\leqslant n$ if and only if $I\subseteq(n:e)$. This result, proved in [2], is useful here.

A proper submodule element $n$ of an le-module ${}_RM$ is called a *pseudo-prime submodule element* if $(n:e)$ is a prime ideal of $R$. The *pseudo-prime spectrum* of ${}_RM$ is the set of all pseudo-prime submodule elements of $M$ and it is denoted by $X_M$. A pseudo-prime submodule element $p$ of $M$ is said to be *maximal* if for any pseudo-prime submodule element $q$ of $M$, $p\leqslant q$ implies $p=q$. Minimal pseudo-prime submodule elements are defined dually. A submodule element $n$ of $M$ is said to be *pseudo-semiprime* if $n$ is a meet of some pseudo-prime submodule elements of $M$. A pseudo-prime submodule element $p$ of $M$ is called *extraordinary* if for any two pseudo-semiprime submodule elements $n$ and $l$ of $M$, $n\wedge l\leqslant p$ implies that either $n\leqslant p$ or $l\leqslant p$. An le-module ${}_RM$ is said to be *topological* if $X_M=\emptyset$ or every pseudo-prime submodule element of $M$ is extraordinary.

For every submodule element $n$ of $M$, we denote

$$V(n)=\{l\in X_M : n\leqslant l\}.$$

The following result have some use in this article.

**Lemma 2.1.** (cf. [12]) *Let ${}_RM$ be an le-module. Then for any ideals $I$ and $J$ of $R$, $V((IJ)e)=V(Ie)\cup V(Je)=V((I\cap J)e)$.*

Now we recall some notions from rings. We denote the set of all prime ideals of $R$ by Spec($R$). A topology, known as the *Zariski topology* is defined on Spec($R$). The closed sets in the Zariski topology on Spec($R$) are of the form

$$V^R(I)=\{P\in Spec(R) : I\subseteq P\}$$

There are many useful characterizations associating arithmetical properties of $R$ and topological properties of Spec($R$) [13], [15], [16].

## 3. Pseudo-prime spectrum of topological le-modules

Here we introduce a topology on $X_M$ analogous to the Zariski topology on the set of all pseudo-prime submodules of a module over a ring.

**Lemma 3.1.** *Let $_RM$ be an le-module. Then*

(*i*) $V(0_M) = X_M$.

(*ii*) $V(e) = \emptyset$.

(*iii*) $\cap_{i\in I} V(n_i) = V(\sum_{i\in I} n_i)$ *for any family of submodule elements* $\{n_i\}_{i\in I}$ *of* $M$.

*Proof.* (*i*) and (*ii*) are obvious.
(*iii*). We have $V(\sum_{i\in I} n_i) \subseteq V(n_i)$ for each $i \in I$, and hence $V(\sum_{i\in I}(n_i)) \subseteq \cap_{i\in I} V(n_i)$. Now let $p \in \cap_{i\in I} V(n_i)$. Then $n_i \leqslant p$ for all $i \in I$ implies that $\sum_{i\in I} n_i \leqslant p$, and so $p \in V(\sum_{i\in I}(n_i))$. Thus $\cap_{i\in I} V(n_i) \subseteq V(\sum_{i\in I} n_i)$. Consequently, $\cap_{i\in I} V(n_i) = V(\sum_{i\in I} n_i)$. □

Let us denote

$$\mathcal{V}_R(M) = \{V(n)\colon n \text{ is a submodule element of } M\}.$$

In general, $\mathcal{V}_R(M)$ is not closed under finite unions. If $\mathcal{V}_R(M)$ is closed under finite unions, then the le-module $_RM$ is called a *top le-module* [12]. Thus an le-module $_RM$ is a *top le-module* if and only if for every submodule elements $n, l$ of $M$ there is a submodule element $k$ of $M$ such that $V(n) \cup V(l) = V(k)$. Also we assume that every le-module $_RM$ such that $X_M = \emptyset$ is a top le-module. Following result shows that the classes of top and topological le-modules are same and establishes an useful characterization of the le-modules in this class.

**Theorem 3.2.** *The following statements are equivalent for an le-module $_RM$.*

(*i*) *$_RM$ is a top le-module.*

(*ii*) *Every pseudo-prime submodule element of $M$ is extraordinary.*

(*iii*) $V(n) \cup V(l) = V(n \wedge l)$, *for any pseudo-semiprime submodule elements $n$ and $l$ of $M$.*

*Proof.* If $X_M = \emptyset$ then the results hold trivially. Suppose $X_M \neq \emptyset$.
(*i*) $\Rightarrow$ (*ii*). Let $p$ be any pseudo-prime submodule element of $M$ and let $n$ and $l$ be two pseudo-semiprime submodule elements of $M$ such that $n \wedge l \leqslant p$. Since $_RM$ is a top le-module, there exists a submodule element $k$ of $M$ such that $V(n) \cup V(l) = V(k)$. Now $n = \wedge p_i$, for some collection of pseudo-prime submodule elements $p_i$ of $M$. Then $n \leqslant p_i$ implies that $p_i \in V(n) \subseteq V(k)$ for each $i \in I$. It follows that $k \leqslant p_i$ for each $i \in I$ and hence $k \leqslant n$. Similarly $k \leqslant l$. Thus $k \leqslant n \wedge l$ which implies that $V(n \wedge l) \subseteq V(k)$. Now $V(n) \cup V(l) \subseteq V(n \wedge l) \subseteq V(k) = V(n) \cup V(l)$. So, $V(n) \cup V(l) = V(n \wedge l)$. Also $p \in V(n \wedge l) = V(n) \cup V(l)$ shows that either

$p \in V(n)$ or $p \in V(l)$, i.e., either $n \leqslant p$ or $l \leqslant p$. Hence $p$ is extraordinary.

*(ii)* $\Rightarrow$ *(iii)*. Let $n$ and $l$ be two pseudo-semiprime submodule elements of $M$. We have $V(n) \cup V(l) \subseteq V(n \wedge l)$. Let $p \in V(n \wedge l)$. Then $p$ is a pseudo-prime submodule element and $n \wedge l \leqslant p$. Since $p$ is extraordinary, either $n \leqslant p$ or $l \leqslant p$, equivalently, either $p \in V(n)$ or $p \in V(l)$. Hence $p \in V(n) \cup V(l)$. Consequently, $V(n) \cup V(l) = V(n \wedge l)$.

*(iii)* $\Rightarrow$ *(i)*. Let $n$ and $l$ be any two submodule elements of $M$. If $V(n) = \emptyset$, then $V(n) \cup V(l) = V(l)$ and the result holds. Assume that both $V(n)$ and $V(l)$ are nonempty. Then $V(n) \cup V(l) = V(\wedge_{p \in V(n)} p) \cup V(\wedge_{p \in V(l)} p) = V((\wedge_{p \in V(n)} p) \wedge (\wedge_{p \in V(l)} p))$, by *(iii)*. Thus ${}_R M$ is a top le-module. □

From the equivalence of *(i)* and *(ii)* in the above result, we have:

**Corollary 3.3.** *An le-module ${}_R M$ is a top le-module if and only if it is a topological le-module.*

Thus in view of Lemma 3.1, it follows that $\mathcal{V}_R(M)$ satisfies the axioms of a topological space for the closed subsets if and only if ${}_R M$ is topological. If ${}_R M$ is a topological le-module, then this topology is said to be the *Zariski topology* on $X_M$.

Henceforth, in this article, we assume that every le-module ${}_R M$ is a topological le-module.

Recall that a topological space $X$ is $T_1$ if and only if every singleton subset of $X$ is a closed subset. For each subset $Y$ of $X_M$, we denote the closure of $Y$ in $X_M$ by $\overline{Y}$, and meet of the elements of $Y$ by $\Im(Y)$, i.e., $\Im(Y) = \wedge_{p \in Y} p$. If $Y = \emptyset$, then we take $\Im(Y) = e$.

A subset $Y$ of a topological space $X$ is called *dense* in $X$ if $Y$ has non-empty intersection with every non-empty open subset of $X$. Equivalently, $Y$ is dense in $X$ if and only if $\overline{Y} = X$.

**Proposition 3.4.** *Let ${}_R M$ be an le-module and $Y \subseteq X_M$.*

*(i) Then $\overline{Y} = V(\Im(Y))$. Hence $Y$ is closed if and only if $Y = V(\Im(Y))$. In particular, $\overline{\{l\}} = V(l)$, for every $l \in X_M$.*

*(ii) If $0_M \in Y$, then $Y$ is dense in $X_M$.*

*(iii) $X_M$ is a $T_0$-space.*

*(iv) $X_M$ is a $T_1$-space if and only if each pseudo-prime submodule element of $M$ is a maximal element in $X_M$.*

*Proof.* *(i)*. Clearly $Y \subseteq V(\Im(Y))$. Let $V(n)$ be any closed subset of $X_M$ containing $Y$. Since $\Im(V(n)) \leqslant \Im(Y)$, we have $V(\Im(Y)) \subseteq V(\Im(V(n))) = V(n)$. Thus $V(\Im(Y))$ is the smallest closed subset of $X_M$ containing $Y$. Hence, $\overline{Y} = V(\Im(Y))$.

*(ii)*. This is clear by *(i)*.

*(iii)*. Let $n$ and $l$ be two distinct elements of $X_M$. Then by *(i)*,

$$\overline{\{n\}} = V(n) \neq V(l) = \overline{\{l\}}.$$

Now by the fact that a topological space is a $T_0$-space if and only if the closures of distinct elements are distinct, we conclude that $X_M$ is a $T_0$-space.

*(iv)*. Let $X_M$ be a $T_1$-space and let $p$ be a pseudo-prime submodule element of $M$. Then $\{p\}$ is closed, hence

$$\{p\} = \overline{\{p\}} = V(p), \text{ by } (i).$$

Thus $p$ is a maximal element in $X_M$.

Conversely, suppose $p$ is a maximal element in $X_M$, then by (i), we have

$$\overline{\{p\}} = V(p) = \{p\}.$$

Thus $\{p\}$ is closed and hence $X_M$ is a $T_1$-space. □

Let ${}_RM$ be an le-module. Then the ideal $(0_M : e)$ of $R$ is called the *annihilator* of $M$. It is denoted by $Ann(M)$. Thus

$$Ann(M) = \{r \in R : re \leqslant 0_M\} = \{r \in R : re = 0_M\}.$$

Consider the canonical epimorphism $\phi : R \to R/Ann(M)$. The image of every element $r$ and every ideal $I$ of $R$ such that $Ann(M) \subseteq I$ under $\phi : R \to R/Ann(M)$ will be denoted by $\overline{r}$ and $\overline{I}$ respectively. It is well known in quotient rings that for every prime ideal $P$ of $R$ such that $Ann(M) \subseteq P$, the ideal $\overline{P} = P/Ann(M)$ is prime in $\overline{R} = R/Ann(M)$. Hence the mapping $\psi : X_M \to Spec(\overline{R})$ defined by

$$\psi(p) = \overline{(p : e)} \text{ for every } p \in X_M$$

is well defined. We call $\psi$ the *natural map* on $X_M$. An le-module ${}_RM$ is called *pseudo-primeful* if either $M = 0_M$ or $M \neq 0_M$ and the natural map $\psi$ is surjective. Also ${}_RM$ is called *pseudo-injective* if the natural map $\psi$ is injective.

Recall that if $I$ is an ideal of a ring $R$, then the *radical* of $I$ is defined by

$$\text{Rad}(I) = \{a \in R : a^n \in I, \text{ for some positive integer } n\}$$

Since $R$ is commutative $\text{Rad}(I)$ is also an ideal of $R$ and $I \subseteq Rad(I)$. Also $\text{Rad}(I)$ is the intersection of all prime ideals $P$ such that $I \subseteq P$. An ideal $I$ of $R$ is called a *radical ideal* if $I = Rad(I)$.

**Proposition 3.5.** *Let ${}_RM$ be a nonzero pseudo-primeful le-module and $I$ be a radical ideal of $R$. Then $(Ie : e) = I$ if and only if $Ann(M) \subseteq I$. In particular, $Pe$ is pseudo-prime submodule element of $M$ for every prime ideal $P$ of $R$ containing $Ann(M)$.*

*Proof.* Assume that $Ann(M) \subseteq I$. Since $I$ is a radical ideal, $Ann(M) \subseteq I = \cap_{I \subseteq P_i} P_i$, where $P_i$ are prime ideals of $R$. Since ${}_RM$ is a pseudo-primeful le-module and $Ann(M) \subseteq P_i$, there exists a pseudo-prime submodule element $p_i$ of $M$ such that $(p_i : e) = P_i$. Therefore $I \subseteq (Ie : e) = ((\cap_{I \subseteq P_i} P_i)e : e) \subseteq \cap_{I \subseteq P_i} (P_i e : e) = \cap_{I \subseteq P_i} P_i = I$. Hence $(Ie : e) = I$. □

It is well known that the prime spectrum Spec$(R)$ of a ring $R$ is connected if and only if $R$ contains no idempotents other than 0 and 1 [3]. Now we have the following:

**Theorem 3.6.** *Let ${}_RM$ be a pseudo-primeful le-module and the pseudo-prime spectrum $X_M$ be connected. Then Spec$(\overline{R})$ is connected and hence the ring $\overline{R}$ contains no idempotents other than $\overline{0}$ and $\overline{1}$.*

*Proof.* First we show that the natural map $\psi : X_M \to Spec(\overline{R})$ is continuous. Let $I$ be an ideal of $R$ such that $Ann(M) \subseteq I$ and $p \in \psi^{-1}(V^{\overline{R}}(\overline{I})$. Then there exists $\overline{J} \in V^{\overline{R}}(\overline{I})$ such that $\psi(p) = \overline{J}$, i.e, $\overline{(p : e)} = \overline{J}$. This implies that $(p : e) = J \supseteq I$ and so $Ie \leqslant (p : e)e \leqslant p$. Hence $p \in V(Ie)$. Therefore $\psi^{-1}(V^{\overline{R}}(\overline{I})) \subseteq V(Ie)$. Now let $q \in V(Ie)$. Then $I \subseteq (Ie : e) \subseteq (q : e)$ implies that $\overline{I} \subseteq \overline{(q : e)}$. Hence $q \in \psi^{-1}(V^{\overline{R}}(\overline{I}))$. Thus $V(Ie) \subseteq \psi^{-1}(V^{\overline{R}}(\overline{I}))$. Therefore $\psi^{-1}(V^{\overline{R}}(\overline{I})) = V(Ie)$. Hence $\psi$ is continuous. Thus the theorem follows from the fact that the map $\psi$ is surjective and the continuous image of a connected set is connected. □

## 4. Irreducible pseudo-prime spectrum

A topological space $X$ is *irreducible* if and only if for every pair of closed subsets $Y_1, Y_2$ of $X$, $X = Y_1 \cup Y_2$ implies $X = Y_1$ or $X = Y_2$. A nonempty subset $Y$ of a topological space $X$ is called an *irreducible subset* if the subspace $Y$ of $X$ is irreducible. An *irreducible component* of a topological space $X$ is a maximal irreducible subset of $X$. A subset $Y$ of $X$ is irreducible if and only if its closure $\overline{Y}$ is irreducible. Thus irreducible components of $X$ are closed. Since every singleton subset of $X_M$ is irreducible, its closure is also irreducible.

The following result is a direct consequence of Proposition 3.4$(i)$ and hence we omit the proof.

**Lemma 4.1.** *$V(l)$ is an irreducible closed subset of $X_M$ for every pseudo-prime submodule element $l$ of an le-module ${}_RM$.*

**Theorem 4.2.** *Let ${}_RM$ be a nonzero pseudo-primeful le-module. Then the following statements are equivalent:*

- $(i)$ *$X_M$ is an irreducible space;*
- $(ii)$ *Spec$(\overline{R})$ is an irreducible space;*
- $(iii)$ *$V^R(Ann(M))$ is an irreducible space;*
- $(iv)$ *Rad$(Ann(M))$ is a prime ideal of $R$;*
- $(v)$ *$X_M = V(Ie)$ for some $I \in V^R(Ann(M))$.*

*Proof.* $(i) \Rightarrow (ii)$. In the proof of Theorem 3.6, we have seen that the mapping $\psi : X_M \to Spec(\overline{R})$ is continuous. Thus $(ii)$ follows from the fact that $\psi$ is surjective and continuous image of an irreducible space is irreducible.

$(ii) \Rightarrow (iii)$. Note that the mapping $\phi : Spec(\overline{R}) \to Spec(R)$ defined by $\overline{P} \mapsto P$ is a homeomorphism. Hence $V^R(Ann(M))$ is an irreducible space.

$(iii) \Rightarrow (iv)$. Obvious.

$(iv) \Rightarrow (v)$. Assume that $\mathrm{Rad}(Ann(M))$ is a prime ideal of $R$. Then by Proposition 3.5, $(Rad(Ann(M)))e$ is a pseudo-prime submodule element of $M$. Let $p \in X_M$. Then $Rad(Ann(M)) \subseteq (p : e)$ which implies that $(Rad(Ann(M)))e \leqslant (p : e)e \leqslant p$. Thus $p \in V((Rad(Ann(M)))e)$ and hence $X_M = V(Ie)$, where $I = Rad(Ann(M)) \in V^R(Ann(M))$.

$(v) \Rightarrow (i)$. This is a direct consequence of the Proposition 3.5 and Lemma 4.1. □

For a submodule element $n$ of $M$, the ***pseudo-prime radical*** of $n$, denoted by $\mathbb{P}rad(n)$, is the meet of all pseudo-prime submodule elements of $M$ containing $n$, that is,

$$\mathbb{P}rad(n) = \wedge_{p \in V(n)} p.$$

If $V(n) = \emptyset$, then we set $\mathbb{P}rad(n) = e$. Note that $n \leqslant \mathbb{P}rad(n)$ and that $\mathbb{P}rad(n) = e$ or $\mathbb{P}rad(n)$ is a pseudo-semiprime submodule element of $M$. Also $V(n) = V(\mathbb{P}rad(n))$. A submodule element $n$ of $M$ is said to be a ***pseudo-prime radical submodule element*** if $n = \mathbb{P}rad(n)$.

It is well-known that in a ring $R$, a subset $Y$ of $\mathrm{Spec}(R)$ is irreducible if and only if $\Im(Y)$ is a prime ideal of R [3]. The next theorem is a analogue of this fact for topological le-modules.

**Theorem 4.3.** *Let $_RM$ be an le-module and $Y \subseteq X_M$. Then $\Im(Y)$ is a pseudo-prime submodule element of $M$ if and only if $Y$ is irreducible in $X_M$.*

*Proof.* Let $Y$ be irreducible, $I$ and $J$ be two ideals of $R$ such that $IJ \subseteq (\Im(Y) : e)$. Then $(IJ)e \leqslant \Im(Y)$. Now, we have

$$Y \subseteq V(\Im(Y)) \subseteq V((IJ)e) = V(Ie) \cup V(Je), \text{ by Lemma 2.1.}$$

Since $Y$ is irreducible, so either $Y \subseteq V(Ie)$ or $Y \subseteq V(Je)$. Hence, either $Ie \leqslant (\mathbb{P}rad(Ie)) = \Im(V(Ie)) \leqslant \Im(Y)$ or $Je \leqslant (\mathbb{P}rad(Je)) = \Im(V(Je)) \leqslant \Im(Y)$. This implies that $I \subseteq (\Im(Y) : e)$ or $J \subseteq (\Im(Y) : e)$. Thus $\Im(Y)$ is a pseudo-prime submodule element of $M$.

Conversely let $\Im(Y)$ be a pseudo-prime submodule element of $M$ and let $Y \subseteq Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ are two closed subset of $X_M$. Then there exist submodule elements $n$ and $l$ of $M$ such that $Y_1 = V(n)$ and $Y_2 = V(l)$. Hence

$$\mathbb{P}rad(n) \wedge \mathbb{P}rad(l) = \Im(V(n)) \wedge \Im(V(l)) = \Im(V(n) \cup V(l)) = \Im(Y_1 \cup Y_2) \leqslant \Im(Y).$$

Since ${}_RM$ is a topological le-module, $\Im(Y)$ is an extraordinary submodule element. Hence, We have $\mathbb{I}rad(n) \leqslant \Im(Y)$ or $\mathbb{P}rad(l) \leqslant \Im(Y)$. Thus $Y \subseteq V(\Im(Y)) \subseteq V(\mathbb{P}rad(n)) = V(n) = Y_1$ or $Y \subseteq Y_2$. Therefore $Y$ is irreducible. □

For every $I \in Spec(R)$, we denote

$$X_{M,I} = \{p \in X_M : (p : e) = I\}.$$

**Corollary 4.4.** *Let ${}_RM$ be an le-module, $n$ be a submodule element of $M$ and $I \in Spec(R)$. Then*

- $(i)$ *$V(n)$ is irreducible in $X_M$ if and only if $\mathbb{P}rad(n)$ is a pseudo-prime submodule element of $M$.*
- $(ii)$ *$X_M$ is an irreducible topological space if and only if $\mathbb{P}rad(0_M)$ is a pseudo-prime submodule element of $M$.*
- $(iii)$ *If $X_{M,I} \neq \emptyset$ then $X_{M,I}$ is an irreducible space.*

*Proof.* $(i)$. Since $\mathbb{P}rad(n) = \Im(V(n))$, the result follows from Theorem 4.3.
$(ii)$. This is obvious.
$(iii)$. We have $(\Im(X_{M,I}) : e) = (\wedge_{p \in X_{M,I}} p : e) = \cap_{p \in X_{M,I}} (p : e) = I \in \mathrm{Spec}(R)$ and hence the result follows from Theorem 4.3. □

**Corollary 4.5.** *Let ${}_RM$ be an le- module such that $0_M \in X_M$. Then $X_M$ is an irreducible space.*

Let $Y$ be closed subset of a topological space $X$. An element $y \in Y$ is called a *generic point* of $Y$ if $Y = \overline{\{y\}}$. In Proposition 3.4, we have seen that every element $l$ of $X_M$ is a generic point of the irreducible closed subset $V(l)$. The next theorem shows that the irreducible closed subset of $X_M$ are determined completely by the pseudo-prime submodule elements of $M$. Also there is a one-to-one correspondence between the set of minimal pseudo-prime submodule elements of $M$ and the set of irreducible components of $X_M$.

**Theorem 4.6.** *Let ${}_RM$ be an le-module and $Y \subseteq X_M$.*

- $(i)$ *Then $Y$ is an irreducible closed subset of $X_M$ if and only if $Y = V(p)$ for some $p \in X_M$. Thus every irreducible closed subset of $X_M$ has a generic point.*
- $(ii)$ *The correspondence $V(p) \mapsto p$ is a bijection of the set of all irreducible components of $X_M$ onto the set of all minimal pseudo-prime submodule elements of $M$.*

*Proof.* $(i)$. Let $Y$ be an irreducible closed subset of $X_M$. Then there exists a submodule element $n$ of $M$ such that $Y = V(n)$. By Theorem 4.3,

$$\Im(Y) = \Im(V(n)) = \mathbb{P}rad(n) \in X_M.$$

Hence $Y = V(n) = V(\mathbb{P}rad(n))$. Converse part follows from the Lemma 4.1.

$(ii)$. Let $Y$ be an irreducible component of $X_M$. Then $Y$ is an irreducible closed subset of $X_M$ and so by $(i)$, we have $Y = V(p)$ for some $p \in X_M$. Since each irreducible component is a maximal irreducible closed subset, $V(p)$ is a maximal irreducible closed subset of $X_M$. Let $q$ be a pseudo-prime submodule element of $M$ such that $q \leqslant p$. Then $V(q)$ is an irreducible closed subset and $V(p) \subseteq V(q)$ implies that $V(p) = V(q)$. Thus $p = q$. Hence $p$ is a minimal element of $X_M$.

Now let $p$ be a minimal element of $X_M$. Then by Corollary 4.1, $V(p)$ is an irreducible closed subset of $X_M$. Let $V(p) \subseteq V(q)$ for some $q \in X_M$. Then

$$q = \mathbb{P}rad(q) = \Im(V(q)) \leqslant \Im(V(p)) = \mathbb{P}rad(p) = p,$$

and hence $p = q$. Therefore $V(p) = V(q)$. Thus $V(p)$ is an irreducible component of $X_M$. □

**Theorem 4.7.** *Let ${}_RM$ be a pseudo-primeful le-module. Then the mapping $\phi : V(p) \mapsto \overline{(p:e)}$ is a bijection from the set of all irreducible components of $X_M$ onto the set of all minimal prime ideals of $\overline{R}$.*

*Proof.* Let $V(p)$ be an irreducible component of $X_M$. Then by Theorem 4.6$(ii)$, $p$ is a minimal pseudo-prime submodule element of $M$ and so $(p:e)/Ann(M)$ is a prime ideal of $\overline{R}$. We show that $(p:e)/Ann(M)$ is a minimal prime ideal of $\overline{R}$. Let $J/Ann(M) \in Spec(R/Ann(M))$ be such that $J/Ann(M) \subseteq (p:e)/Ann(M)$. Then $Je \leqslant (p:e)e \leqslant p$. Since ${}_RM$ is pseudo-primeful and $Je$ is a proper submodule element of $M$, $Je$ is a pseudo-prime submodule element of $M$ with $(Je:e) = J$, by Proposition 3.5. By the minimality of $p$, $Je = p$ and hence $(p:e)/Ann(M) = J/Ann(M)$. Thus $(p:e)/Ann(M)$ is a minimal prime ideal of $\overline{R}$. Thus $\phi$ is well-defined.

Now suppose that $P/Ann(M)$ is a minimal prime ideal of $R/Ann(M)$. Then by Proposition 3.5, $(Pe:e) = P$ and $Pe$ is a pseudo-prime submodule element of $M$. To show $Pe$ is a minimal pseudo-prime submodule element of $M$ let $q \leqslant Pe$ for some pseudo-prime submodule element $q$ of $M$. Then $(q:e)/Ann(M) \subseteq (Pe:e)/Ann(M) = P/Ann(M)$. By the minimality of $P/Ann(M)$ we have $(q:e)/Ann(M) = P/Ann(M)$ and so $(q:e) = P$. Thus $Pe = (q:e)e \leqslant q \leqslant Pe$ which implies that $q = Pe$. Hence $Pe$ is a minimal pseudo-prime submodule element of $M$. Therefore $V(Pe)$ is a irreducible component of $X_M$ by Theorem 4.6(ii). Thus $\phi$ is a surjection. Now let $V(p)$ and $V(q)$ be two irreducible components of $X_M$ such that $\overline{(p:e)} = \overline{(q:e)}$. Then by Theorem 4.6$(ii)$, both $p$ and $q$ are minimal pseudo-prime submodule elements of $M$. It follows from $\overline{(p:e)} = \overline{(q:e)}$ that $(p:e) = (q:e)$ which implies that $(p:e)e \leqslant (q:e)e \leqslant q$. Now by Proposition 3.5, $(p:e)e$ is a pseudo-prime submodule element, and hence, by the minimality of $q$, $(p:e)e = q$. Then $q \leqslant p$ and so $q = p$. Therefore, $V(p) = V(q)$. Hence $\phi$ is an injection. □

A ring $R$ is called *Laskerian* if every proper ideal of $R$ has a primary decomposition. In the following result we show that if $R$ is a Laskerian ring then the irreducible components of $X_M$ are precisely determined by the primary decomposition of the ideal $\text{Ann}(M)$ of $R$ and they are finite in numbers.

**Theorem 4.8.** *Let ${}_RM$ be a nonzero pseudo-primeful le-module. Then the following statements hold:*

($i$) *The set of all irreducible components of $X_M$ is of the form*

$$T = \{V(Ie) : I \text{ is a minimal element of } V^R(Ann(M)).$$

($ii$) *If $R$ is a Laskerian ring then $X_M$ has only finitely many irreducible components.*

*Proof.* ($i$). Let $Y$ be an irreducible component of $X_M$. Then by Theorem 4.6($i$), $Y = V(n)$ for some $n \in X_M$. Now $(n : e)$ is a prime ideal of $R$ containing $\text{Ann}(M)$ so by Proposition 3.5, $(n : e)e$ is a pseudo-prime submodule element of $M$. Also $(n : e)e \leqslant n$ implies that $Y = V(n) \subseteq V((n : e)e)$. Since $Y$ is irreducible component of $X_M$, $V(n) = V((n : e)e)$. Thus $(n : e)e = n$. We show that $(n : e)$ is a minimal element of $V^R(Ann(M))$. Let $J \in V^R(Ann(M))$ be such that $J \subseteq (n : e)$. Then $J/Ann(M) \in Spec(R/Ann(M))$. Since ${}_RM$ is a pseudo-primeful le-module, there exists $l \in X_M$ such that $(l : e) = J$. Also $(l : e)e$ is a pseudo-prime submodule element of $M$, by Proposition 3.5. Then $Y = V(n) \subseteq V((l : e)e)$ and so $V(n) = V((l : e)e)$, since $Y$ is irreducible component. Thus $n = (l : e)e \leqslant l$ which implies that $(n : e) \subseteq (l : e) = J \subseteq (n : e)$. Hence $(n : e) = J$.

Now let $Y \in T$. Then there exists a minimal element $J$ of $V^R(Ann(M))$ such that $Y = V(Je)$. Since ${}_RM$ is a pseudo-primeful le-module, $Je$ is a pseudo-prime submodule element of $M$ and $(Je : e) = J$, by Proposition 3.5. Thus $V(Je)$ is an irreducible space, by Lemma 4.1. Let $Y = V(Je) \subseteq V(l)$ for some $l \in X_M$. Then $Je \in V(l)$ implies that $l \leqslant Je$ which implies that $(l : e) \subseteq (Je : e) = J$. By the minimality of $J$ we have $(l : e) = J$. Thus $Je = (l : e)e \leqslant l$ and so $V(l) \subseteq V(Je)$. Hence $Y = V(Je) = V(l)$ and so $Y$ is an irreducible component of $X_M$.

($ii$). Let $R$ be a Laskerian ring then every proper ideal of $R$ has a primary decomposition. Let $I$ be a minimal element of $V^R(Ann(M))$ and $Ann(M) = \cap_{i=1}^{n} Q_i$ is a minimal primary decomposition. Then there exists $1 \leqslant i \leqslant n$ such that $Q_i \subseteq I$ and hence by minimality of $I$ we have $I = Rad(Q_i)$. Thus irreducible components of $X_M$ are $V(Rad(Q_i)e)$, by ($i$). □

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