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Introduction to quotient ordered semirings

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Abstract. We introduce the notion of pseudoorder on an ordered semiring to define quotient ordered semirings. We also study homomorphism theorems on ordered semirings.

1. Introduction

In group theory the notion of normal subgroups play an important role in defining the quotient groups, ideals in defining quotient rings. The notion of congruence takes the place of normal subgroups in a semigroup as well as in semiring to define quotient semigroups and quotient semirings respectively. The situation is not the same for the case of ordered semigroups as given by Kehayopulu and Tsingelis in [7], where, in general given an ordered semigroup S and a congruence relation ρ on S, the quotient set S/ρ is not an ordered semigroup. To over come the situation they introduced the concept of pseudoorder on an ordered semigroup S to make S/ρ an ordered semigroup. The notion of semirings was introduced by Vandiver [8] in 1934. It has many applications in idempotent analysis, theoretical computer science, information sciences, ordered semirings etc. Recently, semirings order in semirings was studied by Han, Kim and Neggers [4]. The quotient structure of semirings one can find in [3], [5]. It is well known that each homomorphic image $(\phi(S), +, \cdot)$ of a semiring $(S, +.\cdot)$ without order is isomorphic to the congruence class semiring $(S/\kappa, +, \cdot)$ with respect to the congruence $\kappa = \phi^{-1}o\phi$ determined by ϕ, in other words, any onto homomorphism ϕ from a semiring $(S, +, \cdot)$ to another semiring $(T, +, \cdot)$ without ordered is entirely determined by the congruence $\kappa = \phi^{-1}o\phi$. But for a given ordered semiring S, and a congruence relation ρ on S, the quotient set S/ρ is not an ordered semiring, in general. So to find all homomorphisms of an ordered semiring to another ordered semiring, we need such relations which are, somehow, 'greater' than congruences. This motivates us to introduce the notion of pseudoorders.

In this paper we study ordered semirings and introduce quotient ordered semirings. In section 2, we have some preliminaries and prerequisites we need to study the underlying semirings. In section 3, we start with an example of an ordered semiring S, and a congruence ρ on S such that the quotient set S/ρ is not an ordered semiring. We introduce the notion of pseudoorder ρ on an ordered semiring S, the symmetric opening $\overline{\rho} = \rho \cap \rho^{-1}$ of ρ found to be a congruence

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relation on S. This congruence $\overline{\rho}$ relation induces the required ordering

$$\preceq_{\rho}=\{((a)_{\overline{\rho}},(b)_{\overline{\rho}}):\ \exists\, x\in(a)_{\overline{\rho}}, y\in(b)_{\overline{\rho}},(x,y)\in\rho\}$$

to make the set $S/\overline{\rho}$ a quotient ordered semiring. We also give the construction of pseudoorders induced by homomorphisms of ordered semirings. In section 4, we give the natural homomorphism of ordered semirings, prove the first and third homomorphism theorems for ordered semirings.

2. Preliminaries and prerequisites

A semiring $(S,+,\cdot)$ is a *partially ordered semiring* if there exists a partial order relation $\leqslant$ on S satisfying the following conditions:

(1) for $a,b,c\in S, a\leqslant b$ implies $a+c\leqslant b+c, c+a\leqslant c+b$,

(2) for $a,b,c, a\leqslant b$ implies $ac\leqslant bc$ and $ca\leqslant cb$.

We denote it by $(S,+,\cdot,\leqslant)$. An equivalence relation ρ on a semiring S is called a *congruence* on S if for $a,b,c,d\in S$,

$$a\rho b \text{ and } c\rho d \text{ implies } (a+c)\rho(b+d) \text{ and } ac\rho bd$$

or, equivalently,

$$a\rho b \text{ implies } (a+c)\rho(b+c),\ (c+a)\rho(c+b),\ ac\rho bc,\ ca\rho cb.$$

A mapping ϕ of a semiring $(S,+,\cdot,\leqslant_S)$ into a semiring $(T,+,\cdot,\leqslant_T)$ is called *isotone* if

$$x,y\in S,\quad x\leqslant_S y\Rightarrow\phi(x)\leqslant_T\phi(y),$$

called *reverse isotone* if

$$x,y\in S,\quad \phi(x)\leqslant_T\phi(y)\Rightarrow x\leqslant_S y,$$

and called a *homomorphism* if it is isotone and satisfy: for $x,y\in S$

$$\phi(x+y)=\phi(x)+\phi(y),\quad \phi(x\cdot y)=\phi(x)\cdot\phi(y).$$

An isotone and reverse isotone homomorphism is called an isomorphism. Two ordered semirings S and T are said to be isomorphic if there exists an isomorphism between them, and we express it by $S\cong T$.

For undefined concepts in semigroup theory we refer to [6], for undefined concepts in semiring theory we refer to [1], [2], [3] and [5].

3. Pseudoorder on an ordered semiring

In this section we first provide an example to support that in general, for a given ordered semiring S, and a congruence relation ρ on S, which does not induce an order relation on the semiring S/ρ, in general. Then we define pseudoorder on an ordered semiring to over come the problem.

Consider the semiring $(S, +, \cdot)$ [5][Exercise 2.3, page 161], where $S = \{a, b, c, d, t\}$, and define the binary operations $+$ and $\cdot$ on S by:

$+$	a	b	c	d	t
a	a	a	a	t	t
b	a	b	b	t	t
c	a	b	b	t	t
d	t	t	t	t	t
t	t	t	t	t	t

$\cdot$	a	b	c	d	t
a	t	t	t	t	t
b	t	b	b	t	t
c	t	b	b	t	t
d	t	t	t	t	t
t	t	t	t	t	t

We define an order relation $\leqslant$ on S by:

$$\leqslant = \{(a,a),(b,b),(c,c),(d,d),(t,t),(a,d),(a,t),(b,c),(b,d),(b,t),(c,d),(c,t),(d,t)\}.$$

Then $(S, +, \cdot, \leqslant)$ is an ordered semiring. Let ρ be a congruence relation on S defined by

$$\rho = \{(a,a),(b,b),(c,c),(d,d),(t,t),(a,b),(b,a),(c,d),(d,c),(b,t),(t,b),(a,t),(t,a)\}.$$

Since ρ is a congruence relation on S, S/ρ is a semiring, where $+$ and $\cdot$ are defined by: for $a, b \in S$

$$(x)_\rho + (y)_\rho = (x+y)_\rho, \quad (x)_\rho \cdot (y)_\rho = (x \cdot y)_\rho.$$

The congruence classes are $(a)_\rho = \{a, b, t\} = (b)_\rho = (t)_\rho, (c)_\rho = \{c, d\} = (d)_\rho$. Let us define a relation $\preceq$ which could be the probable order, induced by the order relation $\leqslant$ by:

$$\preceq = \{((u)_\rho, (v)_\rho) : a \in (u)_\rho, b \in (v)_\rho, (a,b) \in \leqslant\}.$$

Here $(a)_\rho \preceq (c)_\rho$ and $(c)_\rho \preceq (a)_\rho$ but $(a)_\rho \neq (c)_\rho$, whereby the relation $\preceq$ is not anti-symmetric on S/ρ. Consequently, S/ρ is not an ordered semiring.

Now the question is that: is there any congruence relation on S so that the quotient semiring S/ρ becomes an ordered semiring? The answer is yes, and this motivates us to introduce the notion of pseudoorder on an ordered semiring.

Definition 3.1. Let $(S, +, \cdot, \leqslant)$ be an ordered semiring. A relation ρ on S is called *pseudoorder* if

1. $\leqslant \subseteq \rho$;

2. for $a, b, c \in S$, $\ a\rho b$ and $b\rho c$ imply $a\rho c$, and
3. for $a, b, c, d \in S$, $\ a\rho b \Rightarrow (a+c)\rho(b+c)$, $(c+a)\rho(c+b)$, $ac\rho bc$, $ca\rho cb$, equivalently, $a\rho b, c\rho d \Rightarrow (a+c)\rho(b+d)$, $ac\rho bd$.

Here we give the construction of a pseudoorder on a semiring induced by homomorphisms of ordered semirings.

Lemma 3.2. *Let $(S, +, \cdot, \leqslant_S)$ and $(T, +, \cdot, \leqslant_T)$ be two ordered semirings, and $f: S \longrightarrow T$ a homomorphism. Then the relation ρ_p defined by:*

$$\rho_p = \{(x, y) \in S \times S : \ f(x) \leqslant_T f(y)\}$$

is a pseudoorder on S.

Proof. Let $(x, y) \in \leqslant_S$. Since f is a homomorphism, one gets $f(x) \leqslant_T f(y)$, i.e., $(x, y) \in \rho_p$ so that $\leqslant_S \subseteq \rho_p$. Let $(x, y) \in \rho_p, (y, z) \in \rho_p$. Then $f(x) \leqslant_T f(y)$, $f(y) \leqslant f(z)$. Since $\leqslant_T$ is transitive on T, one has $f(x) \leqslant_T f(z)$ yielding $(x, z) \in \rho_p$. Let $x, y, z \in S$ with $x\rho_p y$. Then $f(x) \leqslant_T f(y)$ implies $f(x) + f(z) \leqslant_T f(y) + f(z), f(z) + f(x) \leqslant_T f(z) + f(y), f(x)f(z) \leqslant_T f(y)f(z)$ and $f(z)f(x) \leqslant_T f(z)f(y)$ so that $f(x+z) \leqslant_T f(y+z), f(z+x) \leqslant_T f(z+y), f(xz) \leqslant_T f(yz)$ and $f(zx) \leqslant_T f(zy)$, since f is a homomorphism. Therefore, by the definition of ρ_p, $(x+z)\rho_p(y+z), (z+x)\rho_p(z+y), xz\rho_p yz$ and $zx\rho_p zy$. Consequently, ρ_p is a pseudoorder on S. □

Example 3.3. Let $S = \{1, 2, 3, 4\}$, and define binary operations $+$ and $\cdot$ by: for $a, b \in S$, $a + b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$. Then $(S, +, \cdot)$ is a semiring. We define an order relation $\leqslant$ on S by:

$$\leqslant = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

Then ρ is a pseudoorder defined on S by:

$$\rho = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}.$$

Let ρ be a pseudoorder on an ordered semiring $(S, +, \cdot, \leqslant)$. Set

$$\overline{\rho} = \rho \cap \rho^{-1}, \text{ the symmetric opening of } \rho.$$

Lemma 3.4. *Let ρ be a pseudoorder on an ordered semiring $(S, +, \cdot, \leqslant)$. Then the relation $\overline{\rho}$ as defined above is a congruence on S.*

Proof. Let $a \in S$, then $a \leqslant a$ implies $a\rho a$ so that $(a, a) \in \overline{\rho}$. Let $(a, b) \in \overline{\rho} = \rho \cap \rho^{-1}$. Then $(a, b) \in \overline{\rho}$. Let $(a, b), (b, c) \in \overline{\rho}$. Then $(a, b) \in \rho, (b, a) \in \rho, (b, c) \in \rho, (c, b) \in \rho$ so that $(a, c) \in \rho, (c, a) \in \rho$, since ρ is a pseudoorder on S, yielding $(a, c) \in \overline{\rho}$. Thus $\overline{\rho}$ is an equivalence relation on S. Let $a, b, c \in S$ such that $a\overline{\rho}b$. Then one has $a\rho b, b\rho a$. Since ρ is a pseudoorder on S, we have $(a+c)\rho(b+c)$ and $(b+c)\rho(a+c)$, $(c+a)\rho(c+b)$, $(c+b)\rho(c+a)$, $ac\rho bc$, $bc\rho ac$, $ca\rho cb$, $cb\rho ca$. Therefore, $(a+c)\overline{\rho}(b+c)$, $(c+a)\overline{\rho}(c+b)$, $ac\overline{\rho}bc$, $ca\overline{\rho}cb$. Consequently, $\overline{\rho}$ is a congruence on S. □

Lemma 3.5. *Let $(S,+,\cdot,\leqslant)$ be an ordered semiring, ρ a pseudoorder on S. Then the set $S/\overline{\rho}$ defined by*

$$S/\overline{\rho} = \{(a)_{\overline{\rho}} :\ a \in S\}$$

with $+$ and $\cdot$ defined by: for $(a)_{\overline{\rho}}, (b)_{\overline{\rho}}$,

$$(a)_{\overline{\rho}} + (b)_{\overline{\rho}} = (a+b)_{\overline{\rho}}, \quad (a)_{\overline{\rho}} \cdot (b)_{\overline{\rho}} = (ab)_{\overline{\rho}}$$

and order relation

$$\preceq_\rho = \{((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) :\ \exists\ x \in (a)_{\overline{\rho}},\ y \in (b)_{\overline{\rho}},\ (x,y) \in \rho\}$$

is an ordered semiring.

Proof. Since $\overline{\rho}$ is a congruence relation on S, the binary operations are well defined. Before we go to the main proof we prove the following equivalent definitions of the relation $\preceq_\rho$ which we will frequently use:

(1) There are $x \in (a)_{\overline{\rho}}, y \in (b)_{\overline{\rho}}$ with $(x,y) \in \rho$.

(2) For all $u \in (a)_{\overline{\rho}}$, and for all $v \in (b)_{\overline{\rho}}$ one has $(u,v) \in \rho$.

(3) $(a,b) \in \rho$.

Proof: (1) $\Rightarrow$ (2). Let $x \in (a)_{\overline{\rho}}, y \in (b)_{\overline{\rho}}$ with $(x,y) \in \rho$, and $u \in (a)_{\overline{\rho}}, v \in (b)_{\overline{\rho}}$. Then one gets $u\overline{\rho}x, v\overline{\rho}y$ so that $u\rho x, x\rho u, v\rho y$ and $y\rho v$ yielding $u\rho x\rho y\rho v$, whence $u\rho v$, since ρ is transitive. Thus (2) holds.

(2) $\Rightarrow$ (3). Since $a \in (a)_{\overline{\rho}}, b \in (b)_{\overline{\rho}}$, by (2), we have $(a,b) \in \rho$.

(3) $\Rightarrow$ (1). Let $a, b \in S$ with $(a,b) \in \rho$. Then $a \in (a)_{\overline{\rho}}, b \in (b)_{\overline{\rho}}$ with $(a,b) \in \rho$. We show that $\preceq_\rho$ is a partial order on $S/\overline{\rho}$. Let $(a)_{\overline{\rho}} \in S/\overline{\rho}$. Then $a \in S$ and $(a,a) \in \leqslant \subseteq \rho$ so that $(a)_{\overline{\rho}} \preceq_\rho (a)_{\overline{\rho}}$. Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}} \in S/\overline{\rho}$ with $(a)_{\overline{\rho}} \preceq_\rho (b)_{\overline{\rho}}$ and $(b)_{\overline{\rho}} \preceq_\rho (a)_{\overline{\rho}}$. Then $(a,b) \in \rho, (b,a) \in \rho$ imply $(a,b) \in \rho \cap \rho^{-1} = \overline{\rho}$ yielding $(a)_{\overline{\rho}} = (b)_{\overline{\rho}}$. Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}}, (c)_{\overline{\rho}} \in S/\overline{\rho}$ such that $(a)_{\overline{\rho}} \preceq_\rho (b)_{\overline{\rho}}$ and $(b)_{\overline{\rho}} \preceq_\rho (c)_{\overline{\rho}}$. Then $(a,b) \in \rho, (b,c) \in \rho$, whence $(a,c) \in \rho$, by transitivity of ρ. Therefore $(a)_{\overline{\rho}} \preceq_\rho (c)_{\overline{\rho}}$. Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}}, (c)_{\overline{\rho}} \in S/\overline{\rho}$ such that $(a)_{\overline{\rho}} \preceq_\rho (b)_{\overline{\rho}}$. Then $(a,b) \in \rho$. Since ρ is a pseudoorder on S, $(a+c)\rho(b+c), (c+a)\rho(c+b), ac\rho bc, ca\rho cb$ so that $(a+c)_{\overline{\rho}} \preceq_\rho (b+c)_{\overline{\rho}}, (c+a)_{\overline{\rho}} \preceq_\rho (c+b)_{\overline{\rho}}, (ac)_{\overline{\rho}} \preceq_\rho (bc)_{\overline{\rho}}, (ca)_{\overline{\rho}} \preceq_\rho (cb)_{\overline{\rho}}$. Consequently, $\preceq_\rho$ is a partial order on $S/\overline{\rho}$. Hence $S/\overline{\rho}$ is an ordered semiring. $\square$

Example 3.6. We take the ordered semiring given in Example 3.3, and the pseudoorder ρ as defined on S there. Now

$$\rho^{-1} = \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),(4,4)\},$$

and

$$\overline{\rho} = \rho \cap \rho^{-1} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\}.$$

Then by Lemma 3.4, $\overline{\rho}$ is a congruence on S. Now $(1)_{\overline{\rho}} = \{1,2\} = (2)_{\overline{\rho}}$, $(3)_{\overline{\rho}} = \{3\}$, $(4)_{\overline{\rho}} = \{4\}$, and $S/\overline{\rho} = \{\{1,2\},\{3\},\{4\}\}$. Define addition and multiplication on $S/\overline{\rho}$ as given in the composition table below:

+	{1, 2}	{3}	{4}
{1, 2}	{1, 2}	{3}	{4}
{3}	{3}	{3}	{4}
{4}	{4}	{4}	{4}

·	{1, 2}	{3}	{4}
{1, 2}	{1, 2}	{1, 2}	{1, 2}
{3}	{1, 2}	{3}	{3}
{4}	{1, 2}	{3}	{4}

The order relation

$$\preceq_{\rho} = \{((1)_{\overline{\rho}},(1)_{\overline{\rho}}),((1)_{\overline{\rho}},(3)_{\overline{\rho}}),((1)_{\overline{\rho}},(4)_{\overline{\rho}}),((3)_{\overline{\rho}},(3)_{\overline{\rho}}),((3)_{\overline{\rho}},(4)_{\overline{\rho}}),((4)_{\overline{\rho}},(4)_{\overline{\rho}})\}$$

on the semiring $S/\overline{\rho}$ makes it a quotient ordered semiring.

4. Homomorphism theorems in ordered semirings

As there always exists a natural homomorphism from a given group (resp. ring) G (resp. R) onto the quotient group (resp. ring) G/H (resp. R/I), H being a normal subgroup of G(I being an ideal of R), similarly, the same thing happens in the case of ordered semiring, where the congruence relation induced by a pseudoorder plays the role of a normal subgroup or an ideal.

Lemma 4.1. (Natural homomorphism) *Let $(S,+,\cdot,\leqslant)$ be an ordered semiring, ρ – a pseudoorder on S. Define*

$$\psi : S \longrightarrow S/\overline{\rho}$$

by for $a \in S$,

$$\psi(a) = (a)_{\overline{\rho}}.$$

Then ψ is an onto homomorphism.

Proof. The proof is straightforward. □

In the ring theory, we know that the kernel $ker\phi$ of a homomorphism ϕ from a ring R into another ring S is an ideal of R, and $R/ker\phi$ becomes a quotient semiring. Analogous to this concept we define the kernel of an ordered semiring homomorphism, and find that the kernel becomes a congruence, moreover, it will be a pseudoorder. Let $(S,+,\cdot,\leqslant_S)$ and $(T,+,\cdot,\leqslant_T)$ be two ordered semirings and $\phi : S \longrightarrow T$ a homomorphism. Define kernel of ϕ by:

$$ker\phi = \{(a,b) \in S \times\ S :\ \phi(a) = \phi(b)\}.$$

It is easy see that $ker\phi$ is an equivalence relation on S. Here we show that $ker\phi$ is a congruence on S, and $\ker\phi = \overline{\rho_p}$, a pseudo order on S.

Lemma 4.2. *Let $(S,+,\cdot,\leqslant_S)$ and $(T,+,\cdot,\leqslant_T)$ be two ordered semirings and let $\phi : S \longrightarrow T$ a homomorphism. Then $ker\phi$ is a congruence on S, and $ker\phi = \overline{\rho_p}$.*

Proof. For the first part we are only to show that $ker\phi$ is compatible with respect to addition and multiplication. Let $a, b, c \in S$ with $(a, b) \in ker\phi$. Then $\phi(a) = \phi(b)$. Now $\phi(a+c) = \phi(a)+\phi(c) = \phi(b)+\phi(c) = \phi(b+c), \phi(ac) = \phi(a)\phi(c) = \phi(b)\phi(c) = \phi(bc)$. Similarly one gets $\phi(c + a) = \phi(c) + \phi(a) = \phi(c) + \phi(b) = \phi(c + b), \phi(ca) = \phi(c)\phi(a) = \phi(c)\phi(b) = \phi(cb)$. Therefore, $(a + c, b + c), (c + a, c + b), (ac, bc), (ca, cb)$ are all in $\ker\phi$ showing that $ker\phi$ is a congruence on S. Let $(x, y) \in ker\phi$. Then $\phi(x) = \phi(y)$ so that $\phi(x) \leqslant_T \phi(y)$ and $\phi(y) \leqslant_T \phi(x)$. Therefore $x\rho_p y$ and $y\rho_p x$, whence $(x, y) \in \overline{\rho_p}$. Consequently, $ker\phi \subseteq \overline{\rho_p}$. For the reverse inclusion, let $(x, y) \in \overline{\rho_p}$. Then $x\rho_p y$ and $y\rho_p x$ imply $\phi(x) \leqslant_T \phi(y)$ and $\phi(y) \leqslant_T \phi(x)$ so that $\phi(x) = \phi(y)$ yielding $(x, y) \in ker\phi$. Thus $ker\phi = \overline{\rho_p}$. □

Now we are in a position to present the fundamental theorem of homomorphism for ordered semirings.

Theorem 4.3. *Let $(S,+,\cdot,\leqslant_S)$ and $(T,+,\cdot,\leqslant_T)$ be two ordered semirings and $\phi : S \longrightarrow T$ a homomorphism. If ρ is a pseudoorder on S with $\rho \subseteq ker\phi$, then the mapping $f : S/\overline{\rho} \longrightarrow T$ defined by: for all $(a)_{\overline{\rho}} \in S/\overline{\rho}$,*

$$f((a)_{\overline{\rho}}) = \phi(a)$$

is the unique homomorphism of $S/\overline{\rho}$ into T such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "\psi"] & T \\ S/\overline{\rho} \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, i.e., $fo\psi = \phi$, where ψ is the natural homomorphism, and $f(S/\overline{\rho}) = \phi(S)$. Conversely, if ρ is a pseudoorder on S for which there exists a homomorphism $f : S/\overline{\rho} \longrightarrow T$ such that the above diagram commutes, then $\rho \subseteq ker\phi$.

Proof. Let ρ be a pseudoorder on S, $\rho \subseteq ker\phi$, and $f : S/\overline{\rho} \longrightarrow T$ be a mapping defined by: for all $(a)_{\overline{\rho}} \in S/\overline{\rho}$,

$$f((a)_{\overline{\rho}}) = \phi(a).$$

f is well defined: Let $a, b \in S$ such that $(a)_{\overline{\rho}} = (b)_{\overline{\rho}}$. Then by definition of $\overline{\rho}$ one has $(a, b) \in \rho$ so that $(a, b) \in ker\phi$, by hypothesis. Therefore $\phi(a) = \phi(b)$.

f is a homomorphism: Let $a, b \in S$. Then $f[(a)_{\overline{\rho}} + (b)_{\overline{\rho}}] = f[(a + b)_{\overline{\rho}}] = \phi(a + b) = \phi(a) + \phi(b) = f[(a)_{\overline{\rho}}] + f[(b)_{\overline{\rho}}]$, and $f[(a)_{\overline{\rho}}(b)_{\overline{\rho}}] = f[(ab)_{\overline{\rho}}] = \phi(ab) = \phi(a)\phi(b) = f[(a)_{\overline{\rho}}]f[(b)_{\overline{\rho}}]$. Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}} \in S/\overline{\rho}$ such that $(a)_{\overline{\rho}} \preceq_\rho (b)_{\overline{\rho}}$. Then $(a, b) \in \rho \subseteq ker\phi = \overline{\rho_p}$ yielding $\phi(a) \leqslant_T \phi(b)$.

$\underline{fo\psi = \phi}$: For $a \in S$, we have $(fo\psi)(a) = f(\psi(a)) = f[(a)_{\overline{\rho}}] = \phi(a)$.

$\underline{f \text{ is unique}}$: Let $g : S/\overline{\rho} \longrightarrow T$ be a homomorphism such that $go\psi = \phi$. Then for $\underline{a \in S, f[(a)_{\overline{\rho}}]} = \phi(a) = (go\psi)(a) = g(\psi(a)) = g[(a)_{\overline{\rho}}]$, whence $f = g$.

$\underline{f(S/\overline{\rho}) = \phi(S)}$: One has $f(S/\overline{\rho}) = \{f[(a)_{\overline{\rho}}] : a \in S\} = \{\phi(a) : a \in S\} = \phi(S)$. Conversely, suppose that ρ is a pseudoorder on S for which there is a homomorphism $f : S/\overline{\rho} \longrightarrow T$ such that the given diagram commutes. Let $(a, b) \in \rho$. Then $(a)_{\overline{\rho}} \preceq_\rho (b)_{\overline{\rho}}$ so that $f[(a)_{\overline{\rho}}] \leqslant_T f[(b)_{\overline{\rho}}]$. This implies $f(\psi(a)) \leqslant_T f(\psi(b))$, i.e., $(fo\psi)(a) \leqslant_T (fo\psi(b))$ yielding $\phi(a) \leqslant_T \phi(b)$. Therefore $(a, b) \in ker\phi = \rho_p$ so that $\preceq_\rho \subseteq ker\phi = \rho_p$. □

Remark. Let $(S, +, \cdot, \leqslant_S)$ and $(T, +, \cdot, \leqslant_T)$ be two ordered semirings and let $\phi : S \longrightarrow T$ a homomorphism. If ϕ is reverse isotone, then $S \cong f(S)$.

Corollary 4.4. *Let $(S, +, \cdot, \leqslant_S)$ and $(T, +, \cdot, \leqslant_T)$ be two ordered semirings and $\phi : S \longrightarrow T$ a homomorphism. Then $S/ker\phi \cong \phi(S)$.*

Proof. By Lemma 3.2, the relation ρ_p (induced by ϕ in this theorem) is a pseudoorder on S. Now applying the first part of the Theorem 4.3 for $\rho = \rho_p$, $ker\phi = \overline{\rho_p}$, and $f : S/\overline{\rho_p} \longrightarrow T$ defined by: for all $(a)_{\overline{\rho_p}} \in S/\overline{\rho_p}$, $f[(a)_{\overline{\rho_p}}] = \phi(a)$ is a homomorphism. For f to be reverse isotone, let $a, b \in S$ with $\phi(a) \leqslant_T \phi(b)$. Then $(a, b) \in \rho_p$, a pseudoorder on S. Therefore $(a)_{\overline{\rho_p}} \preceq_{\overline{\rho_p}} (b)_{\overline{\rho_p}}$. So by the above Remark, one gets $S/\overline{\rho_p} \cong f(S/\overline{\rho_p})$. By Theorem 4.3, we have $f(S/\overline{\rho_p}) = \phi(S)$ yielding $S/\overline{\rho_p} \cong \phi(S)$. Since $ker\phi = \overline{\rho_p}$, $S/ker\phi \cong \phi(S)$. □

Let $(S, +, \cdot, \leqslant)$ be an ordered semiring, ρ, σ be pseudoorders on S with $\rho \subseteq \sigma$. We define a relation σ/ρ on $S/\overline{\rho}$ by

$$\sigma/\rho = \{((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in S/\overline{\rho} \times S/\overline{\rho} : \exists x \in (a)_{\overline{\rho}},\ y \in (b)_{\overline{\rho}},\ (x, y) \in \sigma\}.$$

The proof of the following lemma is similar to that of the equivalent definitions in Lemma 3.5.

Lemma 4.5. *Let $(S, +, \cdot, \leqslant)$ be an ordered semiring, ρ, σ pseudoorders on S with $\rho \subseteq \sigma$. Then the following are equivalent:*

1. *For $a, b \in S$, $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$.*

2. *For all $x \in (a)_{\overline{\rho}}$, and for all $y \in (b)_{\overline{\rho}}$ one has $(x, y) \in \sigma$.*

3. *$(a, b) \in \sigma$.*

Finally we present the third homomorphism theorem for ordered semirings analogous to that of the semirings or ordered semigroups.

Theorem 4.6. *Let $(S, +, \cdot, \leqslant)$ be an ordered semiring, ρ, σ be pseudoorders on S with $\rho \subseteq \sigma$. Then σ/ρ is a pseudoorder on $S/\overline{\rho}$ and $S/\overline{\rho}/\overline{\sigma/\rho} \cong S/\overline{\sigma}$.*

Proof. We know that $\preceq_\rho$ is an ordered relation on $S/\overline{\rho}$.

$\preceq_\rho \subseteq \sigma/\rho$: Let $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \preceq_\rho$. Then $(a,b) \in \rho \subseteq \sigma$ implies $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$, by Lemma 4.5.

Transitivity: Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}}, (c)_{\overline{\rho}}$ be in $S/\overline{\rho}$ with $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$, $((b)_{\overline{\rho}}, (c)_{\overline{\rho}}) \in \sigma/\rho$. Then $(a,b) \in \sigma$ and $(b,c) \in \sigma$ so that $(a,c) \in \sigma$, whence $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$.

Compatible with addition and multiplication: Let $(a)_{\overline{\rho}}, (b)_{\overline{\rho}}, (c)_{\overline{\rho}}$ be in $S/\overline{\rho}$ with $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$. Then $(a,b) \in \sigma$. Since σ is a pseudoorder on S, for $c \in S$, one has $(a+c)\sigma(b+c), (c+a)\sigma(c+b)$, $ac\sigma bc$, $ca\sigma cb$. Therefore, $((a+c)_{\overline{\rho}}, (b+c)_{\overline{\rho}}) \in \sigma/\rho, ((c+a)_{\overline{\rho}}, (c+b)_{\overline{\rho}}) \in \sigma/\rho, ((ac)_{\overline{\rho}}, (bc)_{\overline{\rho}}) \in \sigma/\rho, ((ca)_{\overline{\rho}}, (cb)_{\overline{\rho}}) \in \sigma/\rho$. Consequently, σ/ρ is a pseudoorder on $S/\overline{\rho}$.

Define a mapping $f : S/\overline{\rho} \longrightarrow S/\overline{\sigma}$ by: for all $(a)_{\overline{\rho}} \in S/\overline{\rho}$

$$f((a)_{\overline{\rho}}) = (a)_{\overline{\sigma}}.$$

f is well defined: For $a,b \in S, (a)_{\overline{\rho}} = (b)_{\overline{\rho}} \Rightarrow (a,b) \in \overline{\rho} = \rho \cap \rho^{-1}$ so that $(a,b) \in \sigma, (b,a) \in \rho \subseteq \sigma$. Therefore, $(a,b) \in \sigma \cap \sigma^{-1}$, whence $(a)_{\overline{\sigma}} = (b)_{\overline{\sigma}}$.

f is a homomorphism: Let $a,b \in S$, $f((a)_{\overline{\rho}} + (b)_{\overline{\rho}}) = f[(a+b)_{\overline{\rho}}] = (a+b)_{\overline{\sigma}}$ $= (a)_{\overline{\sigma}} + (b)_{\overline{\sigma}} = f[(a)_{\overline{\rho}}] + f[(b)_{\overline{\rho}}]$, and $f((a)_{\overline{\rho}}(b)_{\overline{\rho}}) = f[(ab)_{\overline{\rho}}] = (ab)_{\overline{\sigma}} = (a)_{\overline{\sigma}}(b)_{\overline{\sigma}} = f[(a)_{\overline{\rho}}]f[(b)_{\overline{\rho}}]$. Now by Corollary 4.4, we have $(S/\overline{\rho})/kerf \cong f(S)$. Also $f_p = \{((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in S/\overline{\rho} \times S/\overline{\rho} : \ f((a)_{\overline{\rho}}) \preceq_\sigma f((b)_{\overline{\rho}})\}$. Then $((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in f_p \Leftrightarrow f((a)_{\overline{\rho}}) \preceq_\sigma f((b)_{\overline{\rho}}) \Leftrightarrow (a)_{\overline{\sigma}} \preceq_\sigma (b)_{\overline{\sigma}} \Leftrightarrow (a,b) \in \sigma \Leftrightarrow ((a)_{\overline{\rho}}, (b)_{\overline{\rho}}) \in \sigma/\rho$, by Lemma 4.5, whence $f_p = \sigma/\rho$. Thus $kerf = \overline{f_p} = \overline{\sigma/\rho}$. Now $f(S) = \{f((a)_{\overline{\rho}}) : \ a \in S\} = \{(a)_{\overline{\sigma}} : \ a \in S\} = S/\overline{\sigma}$. Consequently, one gets $S/\overline{\rho}/\overline{\sigma/\rho} \cong S/\overline{\sigma}$. $\square$

Open problem. *What would be the second homomorphism theorem, if it exists?*

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