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The iterated line graphs of Cayley graphs associated to a commutative ring

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Abstract. We study the planar and outerplanar indices of Cayley graphs associated to a commutative ring, and we give a full characterization of these graphs with respect to their planar and outerplanar indices when R is a finite ring.

1. Introduction

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs considered in this framework is that of Cayley graphs. These graphs have been considered, for example in [8], [11], [12] and [5]. Let us refer the readers to the survey article [13] for extensive bibliography devoted to various applications of Cayley graphs. In particular, the Cayley graphs of semigroups are related to automata theory, as explained in [10] and the monograph [9].

Given a graph G, we denote the kth iterated line graph of G by $L^k(G)$. In particular $L^0(G) = G$ and $L^1(G) = L(G)$ is the line graph of G. The planar index of G is the smallest k such that $L^k(G)$ is non-planar. We denote the planar index of G by $\xi(G)$. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$. If H is a subgraph of G, in [7, Lemma 4], it was shown that $\xi(G) \leqslant \xi(H)$, and hence the planar index of a graph is the minimum of the planar indices of its connected components. In [7], the authors gave a full characterization of graphs with respect to their planar index.

Theorem 1.1. (cf. [7, Theorem 10]) *Let G be a connected graph. Then:*

(*a*) $\xi(G) = 0$ *if and only if G is non-planar.*

(*b*) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$.*

(*c*) $\xi(G) = 1$ *if and only if G is planar and either $\Delta(G) \geqslant 5$ or G has a vertex of degree 4 which is not a cut-vertex.*

(*d*) $\xi(G) = 2$ *if and only if $L(G)$ is planar and G contains one of the graphs H_i in Figure 1 as a subgraph.*

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(*e*) $\xi(G) = 4$ *if and only if* G *is one of the graphs* X_k *or* Y_k (*Figure* 1) *for some* $k \geqslant 2$.

(*f*) $\xi(G) = 3$ *otherwise.*

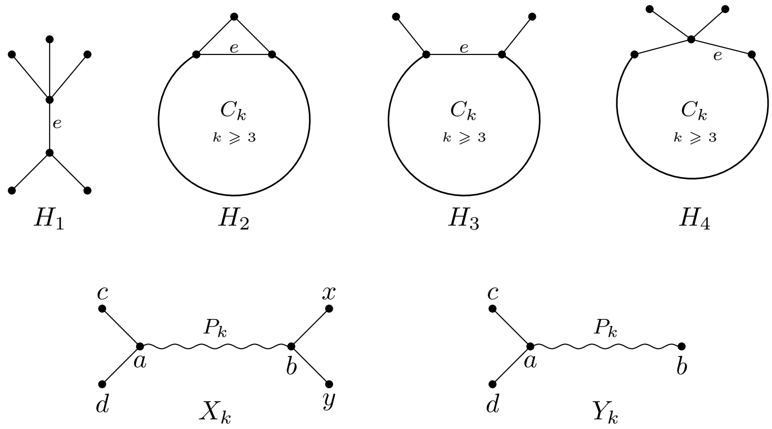

Figure 1

Recall that an undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. Clearly, every outerplanar graph is planar. Also, the outerplanar index of a graph G, which is denoted by $\zeta(G)$, is defined by the smallest integer k such that the kth iterated line graph of G is non-outerplanar. In [15], the authors gave a full characterization of all graphs with respect to their outerplanarity index.

Theorem 1.2. (cf. [15, Theorem 3.4]) *Let* G *be a connected graph. Then:*

(*a*) $\zeta(G) = 0$ *if and only if* G *is non-outerplanar.*

(*b*) $\zeta(G) = \infty$ *if and only if* G *is a path, a cycle, or* $K_{1,3}$.

(*c*) $\zeta(G) = 1$ *if and only if* G *is planar and* G *has a subgraph homeomorphic to* $K_{1,4}$ *or* $K_1 + P_3$ *in Figure 2.*

(*d*) $\zeta(G) = 2$ *if and only if* $L(G)$ *is outerplanar and* G *has a subgraph isomorphic to one of the graphs* G_2 *and* G_3 *in Figure* 2.

(*e*) $\zeta(G) = 3$ *if and only if* $G \in I(d_1, d_2, \ldots, d_t)$ *where* $d_i \geqslant 2$ *for* $i = 2, \ldots, t-1$, *and* $d_1 \geqslant 1$ (*Figure* 2).

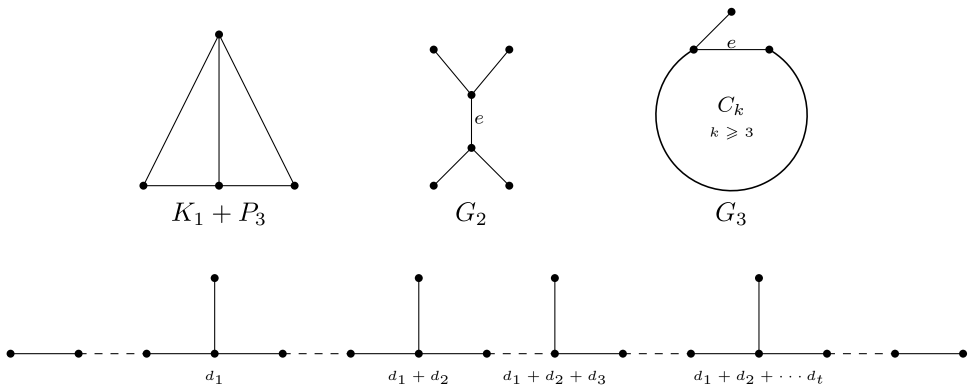

Figure 2

If H is a subgraph of G, in [15, Lemma 3.1], it was shown that $\zeta(G) \leqslant \zeta(H)$, and hence the outerplanar index of a graph is the minimum of the outerplanar indices of its connected components.

Throughout the paper, R is a finite commutative ring with non-zero identity unless otherwise stated. Also, we denote the set of all zero-divisor elements of R by $Z(R)$. For simplicity of notation, in the quotient ring $\frac{K[x]}{I}$, we denote the coset $x + I$ by X.

2. Iterated line graphs of Cay(R, U^*)

Let R be a commutative ring with unity and R^+ be the additive group of R. A non-empty proper subset U of R said to be a *multiplicative prime subset* of R if the following two conditions hold:

(a) $ab \in U$ for every $a \in H$ and $b \in R$;

(b) if $ab \in U$ for some $a, b \in R$, then either $a \in U$ or $b \in U$.

For multiplicative prime subset U of R, let $U^* = U \backslash \{0\}$. In [4], the authors definied Cay(R, U^*) as follows. The graph Cay(R, U^*) is an undirected graph with vertex set R, and two distinct vertices x and y are adjacent if and only if $xy \in U^*$. In this section, we investigate when the graph Cay(R, U^*) and its iterated line graphs are planar or outerplanar. The aim of this section is to give a full characterization of all graphs Cay(R, U^*) with respect to their planar and outerplanar indices.

Theorem 2.1. *Let R be a finite commutative ring. Then*

(a) $\xi(Cay(R, U^*)) = \infty$ *if and only if*

Ring	***multiplicative prime subset*** U
$\mathbb{Z}_4$	$\{0, 2\}$
$\mathbb{Z}_2[x]/(x^2)$	$\{(x^2), x + (x^2)\}$
$\mathbb{Z}_9$	$\{0, 3, 6\}$
$\mathbb{Z}_3[x]/(x^2)$	$\{(x^2), x + (x^2), 2x + (x^2)\}$
$\mathbb{F} \times \mathbb{Z}_2$	$\{0\} \times \mathbb{Z}_2$
$\mathbb{F} \times \mathbb{Z}_3$	$\{0\} \times \mathbb{Z}_3$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\{0\} \times \mathbb{Z}_2) \cup (\{0\} \times \mathbb{Z}_2)$

Table 1: When $\xi(\mathrm{Cay}(R, U^*)) = \infty$

(*b*) $\xi(\mathit{Cay}(R, U^*)) = 2$ *if and only if*

Ring	***Unit set*** U
$\mathbb{Z}_2[x, y]/(x, y)^2$	$\{1, 3\}$
$\mathbb{Z}_2[x]/(x^3)$	$\{1 + (x^2), 1 + x + (x^2)\}$
$\mathbb{Z}_4[x]/(2x, x^2)$	$\{1, 2, 4, 5, 7, 8\}$
$Z_4[x]/(2x, x^2 2)$	$\{1 + (x^2), 2 + (x^2), 1 + x + (x^2), 2 + x + (x^2), 1 + 2x + (x^2), 2 + 2x + (x^2)\}$
$\mathbb{Z}_8$	$U = \{0\} \times \mathbb{Z}_2$
$\mathbb{F}_4[x]/(x^2)$	$\{0\} \times \mathbb{Z}_3$
$\mathbb{Z}_4[x]/(x^2, x^2 + x + 1)$	$(\{0\} \times \mathbb{Z}_2) \cup (\{0\} \times \mathbb{Z}_2)$
$\mathbb{F} \times \mathbb{Z}_4$	$U = \{0\} \times \mathbb{Z}_4$
$\mathbb{F} \times \mathbb{Z}_2[x]/(x^2)$	$\{0\} \times \mathbb{Z}_2[x]/(x^2)$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\{0, 2\} \times \mathbb{Z}_2$
$\mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$	$U = \{0, x\} \times \mathbb{Z}_2$
$\mathbb{F} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Table 2: When $\xi(\mathrm{Cay}(R, U^*)) = 1$

(*c*) $\xi(\mathit{Cay}(R, U^*)) = 0$ *otherwise.*

Proof. We know $\xi(\mathrm{Cay}(R, U^*)) = 0$ if $\mathrm{Cay}(R, U^*)$ is non-planar. Thus we may assume that $\mathrm{Cay}(R, U^*)$ is planar. Now, by [4, Corollary 4.5], we have the following cases:

• Case 1. R is a local ring. Since U is equal to the union of some prime ideals of R, we can deduce U is a prime ideal of R. Now, by [4, Corollary 4.2], we have that $\mathrm{Cay}(R, U^*)$ is planar if and only if R is one of the following rings:

(1) $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$ and $|U| = 2$.

(2) $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$ and $|U| = 3$.

(3) $R \cong \mathbb{Z}_2[x, y]/(x, y)^2$, $R \cong \mathbb{Z}_2[x]/(x^3)$, $R \cong \mathbb{Z}_4[x]/(2x, x^2)$, $R \cong Z_4[x]/(2x, x^2 - 2)$, $R \cong \mathbb{Z}_8$, $R \cong \mathbb{F}_4[x]/(x^2)$ or $R \cong \mathbb{Z}_4[x]/(2x, x^2 + x + 1)$ and $|U| = 4$.

In all above cases, since U is a prime ideal of R, by Theorems 2.1 and 2.2 of [4], we can conclude that $\mathrm{Cay}(R, U^*)$ is the union of $|R/U|$ disjoint $K_{|U|}$. So, for rings of (1) and (2) we have that $\xi(\mathrm{Cay}(R, U^*)) = \infty$. Also, for rings of (3), since the line of the graph K_4 is planar and it has a subgraph isomorphic to H_2, the planar index of each connected component of the graph $\mathrm{Cay}(R, U^*)$ is 2. So, we can conclude that $\xi(\mathrm{Cay}(R, U^*)) = 2$.

• CASE 2. R is not a local ring. Since R is finite, we have that $R \cong R_1 \times R_2 \times \ldots \times R_n$ where R_i is a local ring for all $1 \leqslant i \leqslant n$. Now, consider the following cases:

• CASE 2.1. $n = 2$. By Theorem 4.4 of [4], R and U are as follows:

(1) $R \cong \mathbb{F} \times \mathbb{Z}_2$ and $U = \{0\} \times \mathbb{Z}_2$,

(2) $R \cong \mathbb{F} \times \mathbb{Z}_3$ and $U = \{0\} \times \mathbb{Z}_3$,

(3) $R \cong \mathbb{F} \times \mathbb{Z}_4$ and $U = \{0\} \times \mathbb{Z}_4$,

(4) $R \cong \mathbb{F} \times \mathbb{Z}_2[x]/(x^2)$ and $U = \{0\} \times \mathbb{Z}_2[x]/(x^2)$

(5) $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $U = \{0, 2\} \times \mathbb{Z}_2$,

(6) $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$ and $U = \{0, x\} \times \mathbb{Z}_2$,

(7) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U = (\{0\} \times \mathbb{Z}_2) \cup (\mathbb{Z}_2 \times \{0\})$,

(8) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and $U = (\{0\} \times \mathbb{Z}_3) \cup (\mathbb{Z}_2 \times \{0\})$.

For rings $(1) - (6)$, since U is a prime ideal of R, by Theorems 2.1 and 2.2 of [4], we can conclude that $\mathrm{Cay}(R, U^*)$ is the union of $|R/U|$ disjoint $K_{|U|}$. Now, for rings (1) and (2) we have that $\xi(\mathrm{Cay}(R, U^*)) = \infty$. For rings $(3) - (6)$, since each connected component of the graph $\mathrm{Cay}(R, U^*)$ is K_4, we have that $\xi(\mathrm{Cay}(R, U^*)) = 2$. For ring (7), it is easy to see that $\mathrm{Cay}(R, U^*)$ is C_4 and so $\xi(\mathrm{Cay}(R, U^*)) = \infty$. Also, for ring (8), since $|U^*| = 3$, the graph $\mathrm{Cay}(R, U^*)$ is a 3-regular graph. So, $L(\mathrm{Cay}(R, U^*))$ is planar. Futhermore, by Figure 1, one can easily see that $\mathrm{Cay}(R, U^*)$ has a subgraph isomorphic to H_2. So, by part (d) of Theorem 1.1, we have that, $\xi(\mathrm{Cay}(R, U^*)) = 2$.

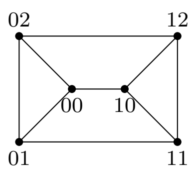

Figure 1: $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_3, ((\{0\} \times \mathbb{Z}_3) \cup (\mathbb{Z}_2 \times \{0\}))^*)$

• CASE 2.2. $n = 3$. In this case $R \cong \mathbb{F} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since U is a prime ideal of R, the graph $\mathrm{Cay}(R, U^*)$ is the union of four disjoint K_4. As we mentioned above, $L(K_4)$ is planar and it has a subgraph isomorphic to H_2. So the planar index of each connected component of the graph $\mathrm{Cay}(\mathbb{F}\times\mathbb{Z}_2\times\mathbb{Z}_2,\{0\}\times\mathbb{Z}_2\times\mathbb{Z}_2)$ is 2. So, we can conclude that $\xi(\mathrm{Cay}(\mathbb{F}\times\mathbb{Z}_2\times\mathbb{Z}_2,\{0\}\times\mathbb{Z}_2\times\mathbb{Z}_2))=2$.

• CASE 2.3. $n\geqslant 4$. It is easy to see that every prime ideal of R has at least 6 elements. Now, by [4, Lemma 4.1], we have that $\mathrm{Cay}(R,U^*)$ is not planar which implies that $\xi(\mathrm{Cay}(R,U^*))=0$. □

In [4], the outerplanarity of $\mathrm{Cay}(R,U^*)$ is investigated. In the following theorem, we determine the outerplanar index of $\mathrm{Cay}(R,U^*)$.

Theorem 2.2. *Let R be a finite commutative ring. Then*

(*a*) $\zeta(Cay(R,U^*))=\infty$ *if and only if* $R\cong\mathbb{Z}_4$, $R\cong\mathbb{Z}_2[x]/(x^2)$, $R\cong\mathbb{Z}_9$, $R\cong\mathbb{Z}_3[x]/(x^2)$, $R\cong\mathbb{F}\times\mathbb{Z}_2$ *and* $U=\{0\}\times\mathbb{Z}_2$ *or* $R\cong\mathbb{F}\times\mathbb{Z}_3$ *and* $U=\{0\}\times\mathbb{Z}_3$ *or* $R\cong\mathbb{Z}_2\times\mathbb{Z}_2$ *and* $U=(\{0\}\times\mathbb{Z}_2)\cup(\{0\}\times\mathbb{Z}_2)$.

(*b*) $\zeta(Cay(R,U^*))=0$ *otherwise.*

Proof. We know $\zeta(\mathrm{Cay}(R,U^*))=0$ if $\mathrm{Cay}(R,U^*)$ is not outerplanar. Thus we may assume that $\mathrm{Cay}(R,U^*)$ is outerplanar. Also, it is well-known that every outerplanar graph is planar. So, we must check the following cases:

• CASE 1. R is a local ring. By [4, Corollary 4.2], $\mathrm{Cay}(R,U^*)$ is planar if and only if R is one of the following rings:

(1) $R\cong\mathbb{Z}_4$ or $R\cong\mathbb{Z}_2[x]/(x^2)$ and $|U|=2$.

(2) $R\cong\mathbb{Z}_9$ or $R\cong\mathbb{Z}_3[x]/(x^2)$ and $|U|=3$.

(3) $R\cong\mathbb{Z}_2[x,y]/(x,y)^2$, $R\cong\mathbb{Z}_2[x]/(x^3)$, $R\cong\mathbb{Z}_4[x]/(2x,x^2)$,
$R\cong Z_4[x]/(2x,x^22)$, $R\cong\mathbb{Z}_8$, $R\cong\mathbb{F}_4[x]/(x^2)$ or
$R\cong\mathbb{Z}_4[x]/(2x,x^2+x+1)$ and $|U|=4$.

For all above rings, U is a prime ideal of R, so the graph $\mathrm{Cay}(R,U^*)$ is the union of $|R/U|$ disjoint $K_{|U|}$. Therefore, for rings of (1) and (2), $\mathrm{Cay}(R,U^*)$ is the union of K_2 and K_3, respectively. Hence, $\zeta(\mathrm{Cay}(R,U^*))=\infty$. Also, for rings of (3), note that each connected component of the graph $\mathrm{Cay}(R,U^*)$ is K_4. So, for these rings, $\mathrm{Cay}(R,U^*)$ is not outerplanar which implies that $\zeta(\mathrm{Cay}(R,U^*))=0$.

• CASE 2. R is not a local ring. Let $R\cong R_1\times R_2\times\ldots\times R_n$ where R_i is a local ring for all $1\leqslant i\leqslant n$. Since $\mathrm{Cay}(R,U^*)$ is planar, we may consider the following cases:

• CASE 2.1. $n=2$. So, by Theorem 4.4 of [4], R and U are as follows:

(1) $R\cong\mathbb{F}\times\mathbb{Z}_2$ and $U=\{0\}\times\mathbb{Z}_2$ or $R\cong\mathbb{F}\times\mathbb{Z}_3$ and $U=\{0\}\times\mathbb{Z}_3$. Since U is a maxiaml ideal of R, the Cayley graph $\mathrm{Cay}(R,U^*)$ is the union of $|\mathbb{F}|$ disjoint K_2 and K_3, respectively. Thus, $\zeta(\mathrm{Cay}(R,U^*))=\infty$.

(2) $R \cong \mathbb{F} \times \mathbb{Z}_4$ and $U = \{0\} \times \mathbb{Z}_4$, $R \cong \mathbb{F} \times \mathbb{Z}_2[x]/(x^2)$ and $U = \{0\} \times \mathbb{Z}_2[x]/(x^2)$, $R \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $U = \{0, 2\} \times \mathbb{Z}_2$, or $R \cong \mathbb{Z}_2[x]/(x^2) \times \mathbb{Z}_2$ and $U = \{0, x\} \times \mathbb{Z}_2$. It is easy to see that in all cases, U is a prime ideal and $|U| = 4$. So $\mathrm{Cay}(R, U^*)$ is the union of disjoint K_4. Thus $\mathrm{Cay}(R, U^*)$ is not outerplanar and we have that $\zeta(\mathrm{Cay}(R, U^*)) = 0$.

(4) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U = (\{0\} \times \mathbb{Z}_2) \cup (\{0\} \times \mathbb{Z}_2)$. Since $\mathrm{Cay}(R, U^*)$ is a cycle on 4 vertices, we have that $\zeta(\mathrm{Cay}(R, U^*)) = \infty$.

(5) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and $U = (\{0\} \times \mathbb{Z}_3) \cup (\mathbb{Z}_2 \times \{0\})$. By Figure 1, it is easy to see that $\mathrm{Cay}(R, U^*)$ has a subdivision of $K_{2,3}$. Hence it is not outerplanar and we have that $\zeta(\mathrm{Cay}(R, U^*)) = 0$.

• CASE 2.2. $n = 3$. In this case $R \cong \mathbb{F} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $U = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The graph $\mathrm{Cay}(R, U^*)$ is the union of K_4. So, $\mathrm{Cay}(R, U^*)$ is not outerplanar and $\zeta(TCay(R, U^*)) = 0$. □

3. Iterated line graphs of $\mathrm{Cay}(I(R), I^*)$

In this section, we investigate the planar and outerplanar index of the graph $\mathrm{Cay}(I(R), I^*)$. This graph is defined and studied in [3]. Let R be a commutative ring and $I(R)$ be the set of all ideals of R and $I^* = I(R) \setminus \{0\}$. The Cayley sum graph $\mathrm{Cay}(I(R); I^*)$ is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices I and J are adjacent whenever $I + K = J$ or $J + K = I$, for some ideal K in I^*. In the next theorem we classify all rings with respect to planar index of their Cayley sum graphs.

Theorem 3.1. *Let R be a finite commutative ring. Then*

(a) $\xi(Cay(I(R), I^*)) = \infty$ *if and only if* $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 = 0$ *and* $I(R) = \{0, (x), R\}$, *where* $x \in m$.

(b) $\xi(Cay(I(R), I^*)) = 1$ *if and only if* $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 2$ *and* $I(R) = \{0, (x), (y), (x, y), R\}$, *where* $x, y \in m$.

(c) $\xi(Cay(I(R), I^*)) = 2$ *if and only if* $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 \neq 0$ *and* $I(R) = \{0, (x^2), (x), R\}$, *where* $x \in m$. $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, *where the* $\mathbb{F}_1, \mathbb{F}_2$ *are fields,*

(d) $\xi(Cay(I(R), I^*)) = 0$ *otherwise.*

Proof. Since $\xi(\mathrm{Cay}(I(R), I^*)) = 0$ for every non-planar graphs, we only consider the rings whose the Cayley sum graph $\mathrm{Cay}(I(R), I^*)$ is planar. By Theorem 3.5 of [3], we have the following cases:

• CASE 1. $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where the $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. The Cayley sum $\mathrm{Cay}(I(R), I^*)$ is pictured in Figure 2. Since $\Delta(\mathrm{Cay}(I(R), I^*)) = 3$, we have that $L(\mathrm{Cay}(I(R), I^*))$ is planar. Also, $\mathrm{Cay}(I(R), I^*)$ has a subgraph homomorphic to H_2. So, $\xi(\mathrm{Cay}(I(R), I^*)) = 2$.

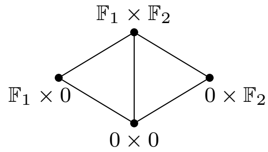

Figure 2: $\mathrm{Cay}(I(\mathbb{F}_1 \times \mathbb{F}_2), I^*)$

• CASE 2. (R, m) is a local ring and it satisfies in one of the following conditions:

• CASE 2.1. $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 2$ and $I(R) = \{0, (x), (y), (x, y), R\}$, where $x, y \in m$. In this situation, the graph $\mathrm{Cay}(I(R), I^*)$ is $K_5 \setminus \{e\}$. So it has a vertex of degree 4 which is not a cut vertex. Then $L(\mathrm{Cay}(I(R), I^*))$ is not planar which implies that $\xi(\mathrm{Cay}(I(R), I^*)) = 1$.

• CASE 2.2. $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 \neq 0$ and $I(R) = \{0, (x^2), (x), R\}$, where $x \in m$. It is easy to see that the graph $\mathrm{Cay}(I(R), I^*)$ is K_4. Thus $\xi(\mathrm{Cay}(I(R), I^*)) = 2$.

• CASE 2.3. $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 = 0$ and $I(R) = \{0, (x), R\}$, where $x \in m$. In this case the graph $\mathrm{Cay}(I(R), I^*)$ is K_3 and so $\xi(\mathrm{Cay}(I(R), I^*)) = \infty$. □

In the sequel, we give a characterization of graphs $\mathrm{Cay}(I(R), I^*)$ with respect to the outerplanar index. In order to achieve this goal, we will use the characterization of outerplanar $\mathrm{Cay}(I(R), I^*)$ which was presented in [3].

Theorem 3.2. *Let R be a finite commutative ring. Then*

(a) *$\zeta(Cay(I(R), I^*)) = \infty$ if and only if $dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 = 0$ and $I(R) = \{0, (x), R\}$, where $x \in m$.*

(b) *$\zeta(Cay(I(R), I^*)) = 1$ if and only if $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1, \mathbb{F}_2$ are fields.*

(c) *$\zeta(Cay(I(R), I^*)) = 0$ otherwise.*

Proof. Since for every non-outerplanar graph, we have that $\xi(\mathrm{Cay}(I(R), I^*)) = 0$, we may assume that the graph $\mathrm{Cay}(I(R), I^*)$ is outerplanar. By Proposition 3.6 of [3], we have the following cases:

• CASE 1. $R \cong \mathbb{F}_1 \times \mathbb{F}_2$, where the $\mathbb{F}_1$ and $\mathbb{F}_2$ are fields. By Figure 2, Since $\mathrm{Cay}(I(R), I^*) \cong K_1 + P_3$, we have that $L(\mathrm{Cay}(I(R), I^*))$ is not outerplanar. So, $\zeta(\mathrm{Cay}(I(R), I^*)) = 1$.

• CASE 2. (R, m) is a local ring and $\dim_{\frac{R}{m}}(\frac{m}{m^2}) = 1$, $m^2 = 0$ and $I(R) = \{0, (x), R\}$, where $x \in m$. In this case the graph $\mathrm{Cay}(I(R), I^*)$ is K_3 and so $\zeta(\mathrm{Cay}(I(R), I^*)) = \infty$. □

4. Iterated line graphs of $\Gamma(R,S)$

Let R be a commutative ring with nonzero identity and G be a multiplicative subgroup of $U(R)$, where $U(R)$ is the multiplicative group of unit elements of R. Also suppose that S is a nonempty subset of G such that $S^{-1} = \{s^{-1} \mid s \in S\} \subseteq S$. A generalization of the unit and unitary Cayley graphs of R, which is denoted by $\Gamma(R,G,S)$, is defined ans studied in [14]. $\Gamma(R,G,S)$ is a graph with vertex set R and two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$.

In this section, we investigate the planar and outerplanar index of the graph $\Gamma(R,G,S)$ in the case that $G = U(R)$. For simplicity of notation, we denote $\Gamma(R,U(R),S)$ by $\Gamma(R,S)$. Also, if we have no restriction on S, we denote $\Gamma(R,S)$ by $\Gamma(R)$.

Theorem 4.1. (cf. [14, Theorem 3.7]) *Let R be an Artinian ring. Then $\Gamma(R,S)$ is planar if and only if one of the following conditions hold.*

- (a) $R \cong \mathbb{Z}_2^l \times T$, *where* $l \geqslant 0$ *and* T *is isomorphic to one of the following rings:* $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ *or* $\mathbb{Z}_2[x]/(x^2)$.
- (b) $R \cong \mathbb{F}_4$
- (c) $R \cong \mathbb{Z}_2^l \times \mathbb{F}_4$, *where* $l > 0$ *with* $S = \{1\}$.
- (d) $R \cong \mathbb{Z}_5$ *with* $S = \{1\}$.
- (e) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ *with* $S = \{(1,1)\}$, $S = \{(1,-1)\}$ *or* $S = \{(-1,1)\}$.

In the next theorem we classify all Artinian rings with respect to planar index of their $\Gamma(R,S)$.

Theorem 4.2. *Let R be an Artinian ring. Then*

- (a) $\xi(\Gamma(R,S)) = \infty$ *if and only if* $R \cong (\mathbb{Z}_2)^l \times T, l \geqslant 0, T \cong \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)$, *or* $R \cong \mathbb{Z}_3, (\mathbb{Z}_2)^l \times \mathbb{Z}_3$ *with* $l > 0$, $|S| = 1$.
- (b) $\xi(\Gamma(R,S)) = 1$ *if and only if* $R \cong \mathbb{Z}_5$ *with* $S = \{1\}$, *or* $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ *with* $S = \{(1,1)\}$, $S = \{(1,-1)\}$ *or* $S = \{(-1,1)\}$.
- (c) $\xi(\Gamma(R,S)) = 2$ *if and only if* $R \cong (\mathbb{Z}_2)^l \times \mathbb{Z}_3$ *with* $l > 0$, $|S| > 1$, *or* $R \cong \mathbb{F}_4$, *or* $R \cong (\mathbb{Z}_2)^l \times \mathbb{F}_4$ *with* $l > 0$, $|S| = 1$.
- (d) $\xi(\Gamma(R,S)) = 0$ *otherwise.*

Proof. Since $\xi(\mathrm{Cay}(I(R), I^*)) = 0$ for every non-planar graphs, we only consider the rings whose $\Gamma(R,S)$ is planar, according to Theorem 4.1. If $R \cong \mathbb{Z}_2$, then $\Gamma(R)$ is isomorphic to K_2. If $R \cong \mathbb{Z}_3$, then it is easy to check that $\Gamma(R,S)$ is isomorphic to K_3, or it is a path with three vertices. If R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$, then $\Gamma(R)$ is isomorphic to C_4, a cycle with four vertices. Therefore, by Theorem 1.1, in the above situationes we have $\xi(\Gamma(R,S)) = \infty$. If $R \cong \mathbb{F}_4$, then $S = \{1\}$ or $|S| \geq 2$. By [14, Theorem 2.7], $\Gamma(R,S)$ is isomorphic to K_4, and so $\xi(\Gamma(R,S)) = 2$. If $R \cong \mathbb{Z}_2^l \times T$, where $l > 0$ and T is isomorphic to one of the ring $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)$ or $\mathbb{F}_4$, then as it is mentioned in the proof of Theorem 3.7 of [14], we have $\Gamma(R) \cong 2^{l-1}\Gamma(\mathbb{Z}_2 \times T)$. So it is sufficient to check the planar index of $\Gamma(\mathbb{Z}_2 \times T)$. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is isomorphic to $2K_2$ and so $\xi(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) = \infty$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. If $|S| = 1$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, S)$ is isomorphic to the cycle C_6, ans so $\xi(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, S)) = \infty$. If $S = U(\mathbb{Z}_2 \times \mathbb{Z}_3)$, then $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, S)$ is pictured in Figure 3, and so $\xi(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, S)) = 1$.

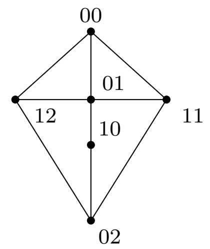

Figure 3: $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3, U(\mathbb{Z}_2 \times \mathbb{Z}_3))$

It is routine to check that $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_4, S) \cong \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2), S) \cong 2C_4$, and so in this situation the planar index is ∞. $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4, S = \{1\})$ is pictured in Figure 4, where $a, a^2 \in \mathbb{F}_4 \setminus \{0,1\}$, and hence its planar index is 2.

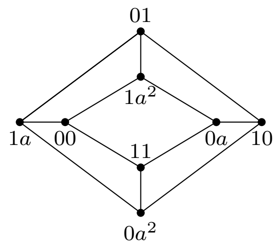

Figure 4: $\Gamma(\mathbb{Z}_2 \times \mathbb{F}_4, S = \{1\})$

If $R \cong \mathbb{Z}_5$ with $S = \{1\}$, then $\Gamma(\mathbb{Z}_5, \{1\})$ is picthured in Figure 5, and so $\xi(\Gamma(\mathbb{Z}_5, \{1\})) = 1$.

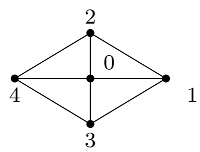

Figure 5: $\Gamma(\mathbb{Z}_5, \{1\})$

Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(-1,1)\}) \cong \Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,-1)\})$ which is pictured in Figure 6. Also $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1)\})$ is pictured in Figure 7. By Theorem 1.1, we see that in each of the situations, the planar index is equal to 1.

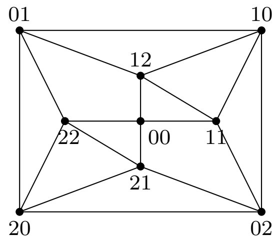

Figure 6: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,-1)\})$

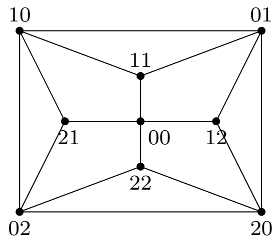

Figure 7: $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_3, \{(1,1)\})$

Now, by the above discusion the results hold. □

Since every outerplanar graph is planar, by using the proof of Theorem 4.2, and also in view of Theorem 1.2, we have the following corollary.

Corollary 4.3. *Let R be an Artinian ring. Then*

(*i*) $\Gamma(R,S)$ *is outerplanar if and only if* $R \cong (\mathbb{Z}_2)^l \times T, l \geqslant 0, T \cong \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2)$, *or* $R \cong \mathbb{Z}_3, (\mathbb{Z}_2)^l \times \mathbb{Z}_3$ *with* $l > 0$, $|S| = 1$.

(*ii*) $\zeta(\Gamma(R,S)) = \infty$ *if and only if* $\Gamma(R,S)$ *is outerplanar. Otherwise* $\zeta(\Gamma(R,S)) = 0$.

5. Iterated line graphs of Cay(R)

Let R be a commutative ring with nonzero identity and R^+ and $Z^*(R)$ be the additive group and the set of nonzero zero-divisors of R, respectively. We denote by Cay(R), the Cayley graph Cay$(R^+, Z^*(R))$. For simplicity of notation, we denote Cay$(R^+, Z^*(R))$ by Cay(R), a graph whose vertices are elements of R and two distinct vertices x and y are adjacent if and only if $x - y \in Z(R)$. Clearly if

R is an integral domain, then $\mathrm{Cay}(R)$ has no edge. In [2], several properties of $\mathrm{Cay}(R)$ are investigated and studied.

In the next theorem, we investigate the planar index of $\mathrm{Cay}(R)$.

Theorem 5.1. *Let R be a ring which is not an integral domain. Then*

(a) *$\xi(Cay(R) = \infty$ if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$.*

(b) *$\xi(Cay(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_8$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.*

(c) *$\xi(Cay(R) = 0$ otherwise.*

Proof. Since $\xi(\mathrm{Cay}(R) = 0$ for every non-planar graphs, we only consider the rings whose $\mathrm{Cay}(R)$ is planar. By Theorem 13 of [2], we have that $\mathrm{Cay}(R)$ is planar if and only if R is one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_9,\ \mathbb{Z}_3[x]/(x^2),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_2[x]/(x^3)$$
$$\mathbb{Z}_4[x]/(2x,x^2),\ \mathbb{Z}_4[x]/(2x,x^2-2),\ \mathbb{Z}_8,\ \mathbb{F}_4[x]/(x^2)\ \text{ and }\ \mathbb{Z}_4[x]/(x^2+x+1).$$

Now, we follow the below cases:

• Case 1. Let R be one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_3$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathrm{Cay}(R)$ is isomorphic to C_4, a cycle with four vertices, and so, by Theorem 1.1, $\xi(\mathrm{Cay}(R)) = \infty$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, then, by Figure 8, we have that $\xi(\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_3)) = 2$.

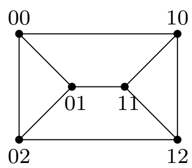

Figure 8: $\mathrm{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_3)$

• Case 2. Let R be one of the rings

$$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_9,\ \mathbb{Z}_3[x]/(x^2),\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \mathbb{Z}_2[x]/(x^3),$$
$$\mathbb{Z}_4[x]/(2x,x^2),\ \mathbb{Z}_4[x]/(2x,x^2-2),\ \mathbb{Z}_8,\ \mathbb{F}_4[x]/(x^2),\ \mathbb{Z}_4[x]/(x^2+x+1).$$

By [1, Lemma 1(ii)], it is shown that if $(R,\mathfrak{m})$ is an Artinian local ring, then $\mathrm{Cay}(R)$ is a disjoint union of $|\frac{R}{\mathfrak{m}}|$ copies of the complete graph $K_{|\mathfrak{m}|}$. Now, since the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$ are local rings with $|\mathfrak{m}| = 2$, the graph $\mathrm{Cay}(R)$ is isomorphic to $2K_2$, and so

$$\xi(\mathrm{Cay}(\mathbb{Z}_4)) = \xi(\mathrm{Cay}(\mathbb{Z}_2[x]/(x^2))) = \infty.$$

The rings $\mathbb{Z}_9$ and $\mathbb{Z}_3[x]/(x^2)$ are local rings with $|\mathfrak{m}| = 3$, and so their $\mathrm{Cay}(R)$ is isomorphic to $3K_3$. Thus

$$\xi(\mathrm{Cay}(\mathbb{Z}_9)) = \xi(\mathrm{Cay}(\mathbb{Z}_3[x]/(x^2))) = \infty.$$

Also, by [2, Lemma 9], the rings $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_8$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$ are local rings with $|\mathfrak{m}| = 4$. So, for these rings, the graph $\mathrm{Cay}(R)$ is a disjoint union of $|\frac{R}{\mathfrak{m}}|$ copies of the complete graph K_4. Therefore

$$\begin{aligned}
\xi(\mathrm{Cay}(\mathbb{Z}_2[x,y]/(x,y)^2)) &= \xi(\mathrm{Cay}(\mathbb{Z}_2[x]/(x^3))) \\
&= \xi(\mathrm{Cay}(\mathbb{Z}_4[x]/(2x,x^2))) \\
&= \xi(\mathrm{Cay}(\mathbb{Z}_4[x]/(2x,x^2-2))) \\
&= \xi(\mathrm{Cay}(\mathbb{Z}_8)) \\
&= \xi(\mathrm{Cay}(\mathbb{F}_4[x]/(x^2))) \\
&= \xi(\mathrm{Cay}(\mathbb{Z}_4[x]/(x^2+x+1))) \\
&= 2. \qquad \square
\end{aligned}$$

Since every outerplanar graph is planar, by using the proof of Theorem 5.1, and also in view of Theorem 1.2, we have the following corollary.

Corollary 5.2. *Let R be a ring which is not an integral domain.*

(i) Cay(R) is outerplanar if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$.

(ii) ζ(Cay(R)) $= \infty$ if and only if Cay(R) is outerplanar. Otherwise, ζ(Cay(R)) $=$ 0.

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