Minimal bi-ideals in regular and completely regular ordered semigroups

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Abstract. Here we characterize regular and completely regular ordered semigroups by their minimal bi-ideals. A minimal bi-ideal is expressed as a product of a minimal right ideal and a minimal left ideal. Furthermore, we show that every bi-ideal in a completely regular ordered semigroup is minimal and hence a regular ordered semigroup S is completely regular if and only if S is union its of minimal bi-ideals.

1. Introduction

As well as ring theory regularity plays a noticeable role in ordered semigroups. T. Saito [11] studied systematically ordered regular, completely regular ordered semigroups. Success attained by this school characterizing regularity on ordered semigroups are either in the semilattice and complete semilattice decompositions into different types of simple components, viz. left, $t-$, σ, λ-simple etc. or in its ideal theory.

Here our aim is to study regular and completely regular ordered semigroups by minimality of their bi-ideals. N. Kehayopulu [6] introduced the notion of bi-ideal in an ordered semigroup. Mathematicians like Lee, Kang and others studied these type of ideals in various ways. Author [3] characterized bi-ideals in Clifford and left Clifford ordered semigroup. Cao and Xu described minimal and maximal left ideals in ordered semigroup. Xu and Ma [12] studied minimality of bi-ideals in an ordered semigroup and characterized t-simplicity of ordered semigroups by minimality of their bi-ideals. In this paper we use this technique of minimality of bi-ideals to study the structure of completely regular ordered semigroups.

2. Preliminaries

In this paper $\mathbb{N}$ will provide the set of all natural numbers. An *ordered semigroup* S is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such

2010 Mathematics Subject Classification: 20M10, 06F05.
Keywords: ordered semigroup, regular, completely regular, bi-ideal, t-simple, ordered idempotents, bi-simple.

that $a \leqslant b$ implies $xa \leqslant xb$ and $ax \leqslant bx$ for all $a,\ b,\ x \in S$. It is denoted by $(S, \cdot, \leqslant)$. For an ordered semigroup S and $H \subseteq S$, denote

$$(H] := \{t \in S : t \leqslant h, \text{ for some } h \in H\}.$$

Let I be a nonempty subset of an ordered semigroup S. I is a *left* (*right*) *ideal* of S, if $SI \subseteq I$ ($IS \subseteq I$) and $(I] = I$. I is an *ideal* of S if it is both left and right ideal of S. S is left (right) *simple* if it has no non-trivial proper left (right) ideal. Similarly we define simple ordered semigroups. S is called *t-simple* ordered semigroup if it is both left and right simple. Due to Kehayopulu an ordered semigroup S is called an *regular* [7] (*completely regular* [6]) if for every $a \in S$, $a \in (aSa]$ ($a \in (a^2Sa^2]$).

A subsemigroup B of S is called a *bi-ideal* [6] if $BSB \subseteq B$ and $(B] = B$. The *principal left ideal, right ideal* [10], ideal and bi-ideal [6] generated by $a \in S$ are denoted by $L(a),\ R(a),\ I(a),\ B(a)$ respectively. It is easy to check that

$$L(a)=(a\cup Sa],\ R(a)=(a\cup aS],\ I(a)=(a\cup Sa\cup aS\cup SaS] \textit{ and } B(a)=(a\cup a^2\cup aSa],$$

and if moreover a is regular then $L(a) = (Sa],\ R(a) = (aS],\ I(a) = (Sa\cup aS\cup SaS]$ and $B(a) = (aSa]$. Kehayopulu [10] defined *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ on an ordered semigroup S as follows:

$$a\mathcal{L}b \textit{ if } L(a) = L(b),\ \ a\mathcal{R}b \textit{ if } R(a) = R(b),\ \ a\mathcal{J}b \textit{ if } I(a) = I(b),\ \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations are equivalence relations on S.

For the sake of convenience, we collect few auxiliary results.

Theorem 2.1. (cf. [2]) *An ordered semigroup S is regular if and only if for every right ideal R and left ideal L of S, $(RL] = R \cap L$.*

Theorem 2.2. (cf. [9]) *Let S be regular ordered semigroup, and B a bi-ideal of S. Then $B = (BSB]$.*

By an *ordered idempotent* [4] in an ordered semigroup S, we shall mean an element $e \in S$ such that $e \leqslant e^2$. The set of all ordered idempotents in S will denoted by $E_{\leqslant}(S)$.

For example consider the ordered semigroup $(\mathbb{R}^+, \cdot, \leqslant)$. Then $(\mathbb{R}^+, \cdot)$ is not regular as a semigroup but it is ordered regular, as for example $2 \leqslant 2 \cdot 2 \cdot 2$. Again 1 is the only idempotent in the semigroup $(\mathbb{R}^+, \cdot)$ where as each natural number n is an ordered idempotent.

In an ordered semigroup S, every left (right) ideal a quasi-ideal and every quasi-ideal is a bi-ideal. Keeping in mind that every t-simple ordered semigroup is a t-simple ordered semigroup, we restate the result of Kehayopulu [9].

Theorem 2.3. (cf. [9]) *An ordered semigroup S is t-simple ordered semigroup if and only if it has no proper bi-ideal.*

Theorem 2.4. (cf. [1]) *An ordered semigroup S is completely regular if and only if S is union of t-simple ordered semigroups.*

Lemma 2.5. *Let S be a completely regular ordered semigroup. Then following statements hold in S:*

(1) *For every $a \in S$ there is $h \in S$ such that $a \leqslant aha$, $a \leqslant a^2h$, and $a \leqslant ha^2$.*

(2) *For every $a \in S$ there is $h \in S$ such that $a\mathcal{H}ah$ and $a\mathcal{H}ha$.*

(3) *For every $a \in S$ there is $e, f \in E_{\leqslant}(S)$ such that $e\mathcal{H}f$.*

Proof. (1). Let $a \in S$. Then there is $t \in S$ such that $a \leqslant a^2ta^2$. Now $a \leqslant a^3ta^2ta^2 \leqslant a^3ta^2ta^2ta^3 \leqslant aha$, where $h = a^2ta^2ta^2ta^2$. Also $a \leqslant ha^2$ and $a \leqslant a^2h$ are obvious.

(2). Let $a \in S$. Then $a \leqslant aha$, where $h = a^2ta^2ta^2ta^2$, follows from the proof of (1). Now $ah = a(a^2ta^2ta^2ta^2) = ua$ for some $u = a(a^2ta^2ta^2ta) \in S$. Then from (1) it follows that $a\mathcal{H}ah$. Similarly $a\mathcal{H}ha$.

(3). This statement is fairly straightforward. $e = ah$, $f = ha \in E_{\leqslant}(S)$ as in (1) serves our purpose. □

For a semigroup S (without order), the set $P(S)$ of all finite subsets of S is a semilattice ordered semigroup with the operation $\cdot$ and $\leqslant$ defined as follows:

For $A, B \in P(F)$, $AB = \{ab \mid a \in A, b \in B\}$ and $A \leqslant B$ if and only if $A \subseteq B$(cf.[1]).

Lemma 2.6. *Let S be a regular semigroup. Then $P(S)$ is regular.*

Lemma 2.7. *Let S be a regular ordered semigroup. Then the following statements hold in S:*

(1) *For every $a \in S$, $B(a) = (R(a)L(a)]$.*

(2) *$(SA] \cap (AS] = (SA \cap AS]$ for any non empty subset A of S.*

Proof. (1). Let $x \in B(a)$. Since S is ordered regular there is $s \in S$ such that $x \leqslant asa$. Note that $a \in R(a)$ and this yields that $as \in R(a)$. Also $a \in L(a)$. Thus $asa \in R(a)L(a)$, so $x \in R(a)L(a)$. Therefore $B(a) \subseteq (R(a)L(a)]$.

Again for some $y \in (R(a)L(a)]$ there is $s, t \in S$ such that $y \leqslant asta$. Then $y \in (aSa] = B(a)$. Hence $B(a) = (R(a)L(a)]$.

(2). First consider a nonempty subset A of S. Let $x \in (SA] \cap (AS]$. Then there are $s, t \in S$ and $a, b \in A$ such that $x \leqslant sa$, $x \leqslant bt$. Since S is ordered regular $x \leqslant xzx$ for some $z \in S$ so that $x \leqslant btzsa$. Now $b(tzsa) \in AS$, $(btzs)a \in SA$. This yields that $btzsa \in SA \cap AS$. Therefore $x \in (SA \cap AS]$ and hence $(SA] \cap (AS] \subseteq (SA \cap AS]$. Also it is obvious that $(SA \cap AS] \subseteq (SA] \cap (AS]$. So finally $(SA] \cap (AS] = (SA \cap AS]$. □

Theorem 2.8. *In a regular ordered semigroup a nonempty subset A of S is a bi-ideal of S if and only if $A = (RL]$ for a right ideal R and a left ideal L of S.*

Following Xu and Ma [12] we define minimality of bi-ideals in an ordered semigroup as follows.

Definition 2.9. (cf. [12]) A bi-ideal M of an ordered semigroup S is called a *minimal bi-ideal* if there is no non trivial bi-ideal B such that $M \subset B$.

Theorem 2.10. (cf. [12]) *A bi-ideal B of an ordered semigroup S is minimal if and only if B is t-simple.*

3. Regular ordered semigroups and minimal bi-ideals

In this section we characterize regular ordered by their minimal bi-ideals. We prove that a bi-ideal in a regular ordered semigroup is minimal if and only if it is a $\mathcal{H}$-class. Also a regular ordered semigroup is completely regular if and only if it is union of minimal bi-ideals.

The following result makes a natural analogy between a bi-ideal in a semigroup and a bi-ideal in an ordered semigroup.

Theorem 3.1. *Let S be an ordered semigroup S. Then for any bi-ideal A of S, $P(A)$ is a bi-ideal of $P(S)$.*

Proof. Let $X \in P(A)P(S)P(A)$. Then there are $X_1, X_2 \in P(A)$ and $Y \in P(S)$ such that $X = X_1YX_2$. Since $X_1, X_2 \in P(A)$ we have that $X_1, X_2 \subseteq A$ and so $X \subseteq ASA$. Since A is a bi-ideal of S we have $X \in A$. Therefore $X \in P(A)$. Hence $P(A)P(S)P(A) \subseteq P(A)$.

Next let $Y \in (P(A)]$. Then there is $Z \in P(A)$ such that $Y \subseteq Z$. Then $Y \subseteq Z \subseteq A$, since $Z \in P(A)$. Therefore $Y \in P(A)$ and so $(P(A)] = P(A)$. Hence $P(A)$ is a bi-ideal of $P(S)$. □

Theorem 3.2. *Let S be a regular ordered semigroup. Then a non empty subset B of S is a minimal bi-ideal of S if and only if $B = (RL]$ for some minimal right-ideal R and minimal left ideal L of S.*

Proof. First suppose that B is a minimal bi-ideal of S. Then for every $a \in B$, $B(a) = B$ and hence $B = (R(a)L(a)]$, by Lemma 2.7(1). Let R be a right ideal of S such that $R \subseteq R(a)$. Since S is regular, $(R(a)L(a)] = R(a) \cap L(a)$, by Theorem 2.1. Now $(RL(a)] = R \cap L(a) \subseteq R(a) \cap L(a) = B$. Also by Theorem 2.8, $(RL(a)]$ itself a bi-ideal of S contained in B. By the minimality of B it follows that $B = (RL(a)]$. That is $R \cap L(a) = R(a) \cap L(a)$. Note that $a \in R(a) \cap L(a) = R \cap L(a)$. Thus $a \in R$. Then for every $x \in R(a)$, $x \in R$. This implies that $R = R(a)$. Therefore $R(a)$ is a minimal right ideal of S. Similarly $L(a)$ is a minimal left ideal of S. Thus the condition is necessary.

Conversely, let B be a nonempty subset of S such that $B = (RL]$ for a minimal left ideal L and a minimal right ideal R of S. Then B is a bi-ideal of S, by Theorem 2.8. To prove the minimality of B let us choose a bi-ideal B' of S such that $B' \subseteq B$. Then $(SB'] \subseteq (SB] \subseteq (S(RL]] \subseteq (SL] \subseteq L$, since L is a left ideal of S.

Likewise $(B'S] \subseteq R$. Also $(SB']$ and $(B'S]$ are left and right ideals of S respectively. Then the minimality of L and R yields that $(SB'] = L$ and $(B'S] = R$. Therefore $B = (RL] = ((B'S](SB']] \subseteq (B'SB'] = B'$, by Theorem 2.2. Thus $B' = B$ and hence B is a minimal bi-ideal of S. □

Then the following corollary follows from Theorem 2.10 and Theorem 2.1.

Corollary 3.3. *Let S be an ordered semigroup. If R is a minimal right ideal and L is a minimal left ideal of S then $(RL]$ is a t-simple ordered subsemigroup of S.*

By the Theorem 2.1, Theorem 2.8 and 2.1 we immediately have the following corollary.

Corollary 3.4. *Let B be a bi-ideal of a regular ordered semigroup S. Then B is a minimal bi-ideal of S if and only if B is an intersection of a minimal left ideal and a minimal right ideal.*

We have the following lemma that characterizes the minimality of bi-ideal in respect of producing same principal bi-ideals.

Lemma 3.5. *Let S be an ordered semigroup. A bi-ideal B of S is minimal if and only if $B(a) = B(b)$ for all $a, b \in B$.*

Proof. First assume that B is a minimal bi-ideal of S. Let $a, b \in B$. Then $B(a) = B = B(b)$.

Conversely, suppose that the given condition holds in S. Let K be a bi-ideal of S such that $K \subseteq B$. Let $z \in K$. Then for every $x \in B$, $B(x) = B(z)$ implies $x \in B(x) = B(z) \subseteq K$ and so $K = B$. Hence B is a minimal bi-ideal of S. □

Now we introduce an equivalence relation which is determined in respect of producing same principal bi-ideals. Let S be an ordered semigroup. Define a relation β on S by:

$$a\beta b \Leftrightarrow B(a) = B(b).$$

It requires only routine verification to see that β is an equivalence relation.

Lemma 3.6. *The following conditions hold in an ordered semigroup S:*

(1) $\beta \subseteq \mathcal{H}$.

(2) *If S is regular then $\beta = \mathcal{H}$.*

Proof. (1). This is obvious.

(2). First suppose that S is regular. Let $a, b \in S$ be such that $a\mathcal{H}b$. Then $a\mathcal{L}b$ and $a\mathcal{R}b$ implies that $L(a) = L(b)$ and $R(a) = R(b)$. Since S is regular $B(a) = L(a) \cap R(a)$, by Lemma 2.7 and Theorem 2.1. Thus $B(a) = L(b) \cap R(b) = B(b)$, and so $a\beta b$. Hence by (1) $\beta = \mathcal{H}$. □

Theorem 3.7. *Let S be an ordered semigroup. Then every bi-ideal is a union of β-class.*

Proof. Let B be a bi-ideal of S and $b \in B$. Let $a \in S$ be such that $a\beta b$. Then $B(a) = B(b) \subseteq B$ implies that $a \in B$. Thus the results follows. □

Theorem 3.8. *Let S be an ordered semigroup. A bi-ideal B of S is minimal if and only if it is a β-class.*

Proof. First suppose that B is a minimal bi-ideal of S. Let $a, b \in B$. Then by the minimality of B it follows from Lemma 3.5 that $B(a) = B(b)$, and this implies that $a\beta b$. Therefore B is a β-class.

Conversely, assume that a bi-ideal B of S is a β-class. Choose a bi-ideal K of S such that $K \subseteq B$. Let $x \in B$ be arbitrary. Consider $y \in K$. Then $B(x) = B(y)$, since $x, y \in B$. Therefore $x \in B(y) \subseteq K$ which implies $K = B$. Hence B is contained in a $\beta-$class and hence by Theorem 3.7, B is a $\beta-$class. □

Immediately we have the following corollary. It requires only routine verification and so its proof is omitted.

Corollary 3.9. *Let B be a bi-ideal of a regular ordered semigroup S. Then B is minimal bi-ideal of S if and only if B is an $\mathcal{H}$-class of S.*

Let us consider the following example of [8].

Example 3.10. Let $S = \{a, b, c, d, e\}$ be the ordered semigroup defined by the multiplication and the order below:

.	a	b	c	d	e
a	a	a	c	a	c
b	a	a	c	a	c
c	a	a	c	a	c
d	d	d	e	d	e
e	d	d	e	d	e

Define a relation $\leqslant$ on S as follows:

$$\leqslant := \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

In this example $(S, \cdot, \leqslant)$ is an ordered semigroup. And in S, $\{a, b\}$ and $\{a, b, c\}$ are bi-ideals of S. This shows that $\{a, b, c\}$ is not a minimal bi-ideal of S. It is interesting to note that S is not regular and so not a complete regular. We now see that every bi-ideal in a complete regular ordered semigroup is bi-ideal.

Theorem 3.11. *Let S be a completely regular ordered semigroup. Then a bi-ideal B of S is minimal if and only if $B(a) = B(e)$ for all $a \in B$ and $e \in E_{\leqslant}(B)$.*

Proof. First suppose that B is minimal. Let $a \in B$ and $e \in E_{\leqslant}(B)$. Then $a, e \in B$. Since S is regular we have $B(a) = B(e)$, by Lemma 3.5.

Conversely, assume that the given conditions hold in S. Let K be a bi-ideal of S such that $K \subseteq B$. Let $z \in K$ and $b \in B$. Since S is completely regular there is $h \in S$ be such that $b\mathcal{H}bh$, by Lemma 2.5(2). So $B = B(b) = B(bh)$. Also from the proof of Lemma 2.5(3) we have $bh \in E_{\leqslant}(S)$, infact $bh \in E_{\leqslant}(B)$. So by given condition $B(z) = B(bh)$. Therefore $z \in B(z) = B$, and so $K = B$. Hence B is minimal bi-ideal of S. □

Corollary 3.12. *Let S be a completely regular ordered semigroup. Then a bi-ideal B of S is minimal if and only if $B(e) = B(f)$ for all $e, f \in E_{\leqslant}(B)$.*

Proof. This is obvious. □

Next we discuss about the bi-simplicity of an ordered semigroup.

Definition 3.13. An ordered semigroup S is called *bi-simple* if S has no proper bi-ideal.

In the following theorem bi-simplicity of regular ordered semigroup has been described by its any two ordered idempotents.

Theorem 3.14. *Let S be a regular ordered semigroup. Then S is bi-simple if and only if for every $e, f \in E_{\leqslant}(S), B(e) = B(f)$.*

Proof. Suppose that S is bi-simple. Consider $e, f \in E_{\leqslant}(S)$. Clearly $B(e) = S = B(f)$.

Conversely, assume that the given condition hold in S and choose a bi-ideal B of S. Let $a \in S$ and $b \in B$. Since S is regular there are $x, y \in S$ such that $a \leqslant axa$ and $b \leqslant byb$. Clearly ax, xa, by and $yb \in E_{\leqslant}(S)$. Now by given condition we have $B(ax) = B(by)$ and $B(xa) = B(yb)$. This yields that $ax\mathcal{H}by$ and $xa\mathcal{H}yb$, in otherwords $ax\mathcal{R}by$ and $xa\mathcal{L}yb$. Now $ax\mathcal{R}by$ gives $ax \leqslant byz$ and $by \leqslant axw$ for some $z, w \in S$. So from $a \leqslant axa$ and $b \leqslant byb$ we have $a \leqslant byza$ and $b \leqslant axwb$, which gives that $a\mathcal{R}b$. In like manner $a\mathcal{L}b$ follows from $yb\mathcal{L}xa$. Thus $a\mathcal{H}b$ and so $B(a) = B(b)$, by Lemma 3.6. So $a \in B(b) = B$. Hence $S = B$. This completes the proof. □

Every minimal bi-ideal is a bi-simple ordered semigroup. It is interesting to note down that there are bi-ideals which are neither left ideal nor a right ideal, but an ordered semigroup S is bi-simple if and only if it is both left and right simple. Thus we have the following theorem.

Theorem 3.15. *The following conditions are equivalent on an ordered semigroup S:*

(1) *S is bi-simple.*

(2) *S is t-simple ordered semigroup.*

(3) *For every $a \in S$, $S = B(a)$.*

(4) *For every $a \in S$, $S = L(a)$ and $S = R(a)$.*

Proof. (1) $\Rightarrow$ (2). This is obvious.

(2) $\Rightarrow$ (3). This implication follows from Theorem 2.3.

(3) $\Rightarrow$ (4). This follows from from the fact that every left ideal and every right ideal are bi-ideals.

(4) $\Rightarrow$ (1). Let the given conditions hold in S. Consider a bi-ideal B of S. Let $a \in B$. Then $B(a) = B$. Now $S = L(a^2)$ and $S = R(a^2)$, by condition (4). Let $x \in S$. Then $x \in L(a) \cap R(a)$. This implies that $x \leqslant sa^2$ for some $s \in S^1$. Since $sa \in S$ there is $t \in S^1$ such that $sa \leqslant a^2t$. Thus $x \leqslant sa^2$ implies that $x \leqslant a^2ta = a(at)a$. Since $a(at)a \in BSB$ and B is a bi-ideal of S we have that $a^2ta \in B$. Thus $x \in B$, and hence $S = B$. This shows that S is bi-simple. □

Theorem 3.16. *Let S be an ordered semigroup. Then a bi-ideal B is minimal if and only if it is bi-simple.*

Proof. First suppose that B is a minimal bi-ideal of S. Consider a bi-ideal T of B. Let $x \in T$. Then $(xBx] \subseteq (TBT] \subseteq T$. This implies that $(xBx] \subseteq T \subseteq B$. Also $(xBx]$ is a bi-ideal of S, by Lemma 3 of [12] . Then the minimality of B yields that $(xBx] = B$, and so $T = B$. Therefore B is bi-simple.

Conversely assume that B is bi-simple. Consider a bi-ideal Y of S such that $Y \subseteq B$. Choose $y \in Y$ arbitrarily. Then by Lemma 3 of [12], $(yBy]$ is a bi-ideal of S. Since B is bi-simple, $(yBy] = B$. Then $B \subseteq (YBY] \subseteq (YSY] \subseteq Y$. Therefore $Y = B$ and hence B is a minimal bi-ideal of S. □

In the next theorem we characterize completely regular ordered semigroups in terms of their bi-ideals.

Theorem 3.17. *An ordered semigroup S is a completely regular ordered semigroup if and only if the following conditions hold in S:*

(1) *For every bi-ideal B of S there is some $e \in E_{\leqslant}(S)$ such that $B = B(e)$.*

(2) *For every $x \in B$, $B(x^2) = B(e)$.*

Proof. Let S be a completely regular ordered semigroup. Consider a bi-ideal B of S. Choose $a \in B$. Then by the Theorem 2.5, there is $h \in S$ such that $a \leqslant aha$, $a \leqslant a^2h$ and $a \leqslant ha^2$. Then $B(a) = (aSa] \subseteq (ahaSa^2h] \subseteq (ahSah] = B(e)$, where $e = ah \in E_{\leqslant}(S)$. Also $B(e) = (eSe] = (ahSah]$. Now $h = a^2ta^2ta^2ta^2$, by the proof of Lemma 2.5(1), and so $B(e) = (ahSah] \subseteq (aSa] = B(a)$. Therefore $B(a) = B(e)$. Thus condition (1) follows.

For condition (2) let $x \in B$. Clearly $B(x^2) = B = B(a)$. So by condition (1) we have $B(x^2) = B(e)$.

Conversely, assume that the given conditions hold in S. Let $a \in S$. Consider the bi-ideal $B(a)$ of S. Then there is $e \in E_{\leqslant}(S)$ be such that $B(a) = B(e) = B(a^2)$. Then $B(a) = B(a^2) = (a^2 \cup a^4 \cup a^2Sa^2]$. Thus $a \leqslant a^2$ or $a \leqslant a^4$ or $a \in (a^2Sa^2]$. Hence in either case S is completely regular ordered semigroup. □

Corollary 3.18. *An ordered semigroup S is completely regular if and only if for every $a \in S$, $B(a) = B(a^2)$.*

Corollary 3.19. *Let S be an completely regular ordered semigroup. Then every bi-ideal of S is principal bi-ideal generated by some ordered idempotent.*

Then we have the following corollary which follows from Theorem 3.11.

Corollary 3.20. *Let S be an completely regular ordered semigroup. Then every bi-ideal of S is minimal.*

In the following theorem completely regular ordered semigroups are characterized by the minimality of their bi-ideals.

Theorem 3.21. *Let S be a regular ordered semigroup. Then S is completely regular if and only if S is union of its bi-ideals.*

Proof. First suppose that S is completely regular. Since S is regular we have that $\mathcal{H} = \beta$, by Lemma 3.6. Then by Theorem 2.4, S is union of $\beta-$classes and so S is union of minimal bi-ideals, by Theorem 3.8. Therefore from Corollary 3.20, S is union of bi-ideals of S.

Conversely, assume that S is union of its minimal bi-ideals. Then S is union of its t-simple ordered subsemigroups, by Theorem 3.15 and Theorem 3.16. Hence by Theorem 2.4, S is completely regular. □

Acknowledgements. I express my deepest gratitude to the editor in chief of the journal Professor Wieslaw A. Dudek for communicating the paper and to the referee of the paper for their important valuable comments.

References

[1] **A.K. Bhuniya and K. Hansda**, *On completely regular and Clifford ordered semigroups*, arxiv:1701.01282v1

[2] **Y. Cao**, *Characterizations of regular ordered semigroup by quasi-ideals*, Vietnam J. Math. **30** (2002), 239–250.

[3] **K. Hansda**, *Bi-ideals in Clifford ordered semigroups*, Discuss. Math. Gen. Alg. and Appl. **33** (2013), 73–84.

[4] **K. Hansda**, *Regularity of subsemigroups generated by ordered idempotents*, Quasigroups and Related Systems **22** (2014), 217–222.

[5] **N. Kehayopulu**, *Note on Green's relation in ordered semigroups*, Math. Japonica **36** (1991), 211–214.

[6] **N. Kehayopulu**, *On completely regular poe-semigroups*, Math. Japonica **37** (1992), 123–130.

[7] **N. Kehayopulu**, *On regular duo ordered semigroups*, Math. Japonica **37** (1992), 535–540.

[8] **N. Kehayopulu**, *On completely regular ordered semigroups*, Scinetiae Math. **1** (1998), 27-2.

[9] **N. Kehayopulu, J. S. Ponizovskii and M. Tsingelis**, *Bi-ideals in ordered semigroups and ordered groups*, J. Math. Sci. **112** (2002), 4353–4354.

[10] **N. Kehayopulu**, *Ideals and Green's relations in ordered semigroups*, Intern. J. Math. Math. Sci. Article ID 61286 (2006), 1–8.

[11] **T. Saito**, *Regular elements in an ordered semigroups*, Pacific J. Math. **13** (1963), 263–295.

[12] **X. Xu and J. Ma**, *A note on minimal bi-ideal in ordered semigroups*, Southeast Asian Bull. Math. **27** (2003), 149–154.

Received November 9, 2017
Revised December 23, 2018

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