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Regularities in ordered ternary semigroups

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Abstract. We present various types of regularities in ordered ternary semigroups and describe connections between these regularities.

1. Preliminaries

A nonempty set S is called a *ternary semigroup* if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$. For any x, y, z in a ternary semigroup S, we will write xyz instead of $[xyz]$.

Sioson [8] introduced the concept of regularities in n-ary semigroups. Dudek and Groździńska [2] gave characterizations of a regular n-ary semigroup using its j-ideals. As a special case of a regular n-ary semigroup, a regular ternary semigroup was studied by Santiago and Sri Bala [7]. Connections between ternary and binary semigroups were firstly studied in [3].

An *ordered ternary semigroup* $(S, [\,], \leqslant)$ is a ternary semigroup $(S, [\,])$ together with a partial order relation $\leqslant$ on S which is compatible with the ternary operation, i.e.,

$$x \leqslant y \Rightarrow xuv \leqslant yuv, \quad uxv \leqslant uyv, \quad uvx \leqslant uvy$$

for all $x, y, u, v \in S$.

Ordered ternary semigroups have been studied by many authors (see, e.g., [4], [5], [6]). Daddi and Pawar [1] introduced the concepts of ordered quasi-ideals and ordered bi-ideals in ordered ternary semigroups and characterized a regular ordered ternary semigroup using its ordered ideals.

Throughout this paper, we write S for an ordered ternary semigroup, unless specify otherwise.

Let A, B, C be nonempty subsets of S. We denote

$$(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\},$$

and note that $A \subseteq (A]$, $(A] = ((A]]$, $(A](B](C] \subseteq (ABC]$, $(A]BC \subseteq (ABC]$, $A(B]C \subseteq (ABC]$, $AB(C] \subseteq (ABC]$, $(A \cup B] = (A] \cup (B]$ and $A \subseteq B$ implies $(A] \subseteq (B]$.

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A nonempty subset I of S is called an *ordered left* (resp. *right*, *lateral*) *ideal* of S if $SSI \subseteq I$ (resp. $ISS \subseteq I$, $SIS \subseteq I$) and $(I] = I$.

If I is an *ordered left*, *right* and *lateral ideal* of S, then it is called an *ordered ideal* of S.

A nonempty subset Q of S is called an *ordered quasi-ideal* of S if $(QSS] \cap (SQS] \cap (SSQ] \subseteq Q$, $(QSS] \cap (SSQSS] \cap (SSQ] \subseteq Q$ and $(Q] = Q$.

A nonempty subset B of S is called an *ordered bi-ideal* of S if $BBB \subseteq B$, $BSBSB \subseteq B$ and $(B] = B$.

An *ordered quasi-ideal* (*bi-ideal*) Q of S is called *semiprime* if $\emptyset \neq A \subseteq S$, $A^3 \subseteq Q$ implies $A \subseteq Q$.

For a nonempty set A of S, we denote by $L(A), R(A), M(A), I(A), Q(A)$ and $B(A)$ the ordered left ideal, the ordered right ideal, the ordered lateral ideal, the ordered ideal, the ordered quasi-ideal and the ordered bi-ideal of S generated by A, respectively.

Lemma 1.1. (cf.[1]) *Let A be a nonempty subset of S. Then*

(i) $L(A) = (A \cup SSA]$,

(ii) $R(A) = (A \cup ASS]$,

(iii) $M(A) = (A \cup SAS \cup SSASS]$,

(iv) $I(A) = (A \cup SSA \cup ASS \cup SAS \cup SSASS]$,

(v) $B(A) = (A \cup AAA \cup ASASA]$,

(vi) $Q(A) = (A \cup SSA] \cap (A \cup SAS \cup SSASS] \cap (A \cup ASS]$.

In particular case, for $a \in S$, we write $L(a), R(a), M(a), I(a), Q(a)$ and $B(a)$ instead of $L(\{a\}), R(\{a\}), M(\{a\}), I(\{a\}), Q(\{a\})$ and $B(\{a\})$, respectively.

2. Regularities in ordered ternary semigroups

An ordered ternary semigroup S is called *regular*, if each its element is *regular*, i.e., for each $a \in S$ there exists $x \in S$ such that $a \leqslant axa$.

We note that S is called *regular* if and only if for each $a \in S$ there exist $x, y \in S$ such that $a \leqslant axaya$.

Lemma 2.2. (cf. [1]) *The following statements are equivalent:*

(i) *S is regular,*

(ii) *$A \subseteq (ASA]$ for any $A \subseteq S$,*

(iii) *$a \in (aSa]$ for any $a \in S$,*

(iv) *$A \subseteq (ASASA]$ for any $A \subseteq S$,*

(v) *$a \in (aSaSa]$ for any $a \in S$.*

Definition 2.3. An ordered ternary semigroup S is called *left* (*right*) *regular*, if each its element is *left* (*right*) *regular* , i.e., for each $a \in S$ there exists $x \in S$ such that $a \leqslant xaa$ ($a \leqslant aax$).

Note that S is *left* (*right*) *regular* if and only if for each $a \in S$ there exist $x, y \in S$ such that $a \leqslant xyaaa$ ($a \leqslant aaaxy$).

Lemma 2.4. *The following statements are equivalent:*

(i) *S is left* (*resp. right*) *regular,*

(ii) $A \subseteq (SAA]$ (*resp.* $A \subseteq (AAS]$) *for any* $A \subseteq S$,

(iii) $a \in (Saa]$ (*resp.* $a \in (aaS]$) *for any* $a \in S$,

(iv) $A \subseteq (SSAAA]$ (*resp.* $A \subseteq (AAASS]$) *for any* $A \subseteq S$,

(v) $a \in (SSaaa]$ (*resp.* $a \in (aaaSS]$) *for any* $a \in S$.

Theorem 2.5. *S is both left regular and right regular ordered ternary semigroup if and only if every ordered quasi-ideal of S is semiprime.*

Proof. Let S be both left regular and right regular and $\emptyset \neq A \subseteq S$. Let Q be an ordered quasi-ideal of S such that $A^3 \subseteq Q$. By Lemma 2.4,

$$A \subseteq (AAS] \subseteq ((A](AAS](S]] \subseteq (AAASS] \subseteq (QSS],$$

$$A \subseteq (SAA] \subseteq ((S](SAA](A]] \subseteq (SSAAA] \subseteq (SSQ],$$

$$A \subseteq (AAS] \subseteq ((SAA](A](S]] \subseteq (SAAAS] \subseteq (SQS].$$

Hence, $A \subseteq (QSS] \cap (SQS] \cap (SSQ] \subseteq Q$.

Conversely, assume that every ordered quasi-ideal of S is semiprime and $\emptyset \neq A \subseteq S$. We have $A^3 \subseteq Q(A^3) = (A^3 \cup SSA^3] \cap (A^3 \cup SA^3S \cup SSA^3SS] \cap (A^3 \cup A^3SS]$. By assumption, $A \subseteq (A^3 \cup SSA^3] \cap (A^3 \cup SA^3S \cup SSA^3SS] \cap (A^3 \cup A^3SS] \subseteq (A^3 \cup SSA^3]$. Thus,

$$A^3 \subseteq (AA(A^3 \cup SSA^3] \cup SSA^3] \subseteq ((A^5 \cup AASSA^3] \cup (SSA^3]] \subseteq (SSA^3]$$

and then $A \subseteq (A^3 \cup SSA^3] \subseteq ((SSA^3] \cup SSA^3] \subseteq (SSA^3] \subseteq (SAA]$. Similarly, we have $A \subseteq (AAS]$. By Lemma 2.4, S is both left regular and right regular ordered ternary semigroup. $\square$

Definition 2.6. An ordered ternary semigroup S is called *intra-regular*, if each its element is *intra-regular*, i.e., for each $a \in S$ there exist $x, y \in S$ such that $a \leqslant xa^3y$.

Note that S is *intra-regular* if and only if for each $a \in S$ there exist $w, x, y, z \in S$ such that $a \leqslant wxa^3yz$.

Lemma 2.7. *The following statements are equivalent:*

(*i*) *S is intra-regular,*

(*ii*) $A \subseteq (SA^3S]$ *for any* $A \subseteq S$,

(*iii*) $a \in (Sa^3S]$ *for any* $a \in S$,

(*iv*) $A \subseteq (SSA^3SS]$ *for any* $A \subseteq S$,

(*v*) $a \in (SSa^3SS]$ *for any* $a \in S$.

Theorem 2.8. *The following statements are equivalent:*

(*i*) *S is intra-regular,*

(*ii*) $L \cap X \cap R \subseteq (LXR]$ *for any ordered left ideal L, ordered right ideal R and* $\emptyset \neq X \subseteq S$.

Proof. $(\Rightarrow)$: Let $a \in L \cap X \cap R$. Since S is intra-regular, there exist $w, x, y, z \in S$ $a \leqslant wxaaayz \in LaR \subseteq LXR \subseteq (LXR]$. Hence, $L \cap X \cap R \subseteq (LXR]$.
$(\Leftarrow)$: Let $a \in S$. By assumption and Lemma 1.1,

$$\begin{aligned} a \in L(a) \cap \{a\} \cap R(a) &\subseteq (L(a)\{a\}R(a)] \subseteq ((a \cup SSa](a](a \cup aSS]] \\ &\subseteq (a^3] \cup (a^3SS] \cup (SSa^3] \cup (SSa^3SS]. \end{aligned}$$

Case 1: $a \in (a^3]$; $a \leqslant aaa \leqslant aaaaa \leqslant aaaaaaa \in SSa^3SS$.
Case 2: $a \in (aaaSS]$; there exist $x, y \in S$, $a \leqslant aaaxy \leqslant aa(aaaxy)xy \in SSa^3SS$.
Case 3: $a \in (SSaaa]$; there exist $x, y \in S$, $a \leqslant xyaaa \leqslant xy(xyaaa)aa \in SSa^3SS$.
Case 4: $a \in (SSaaaSS]$; it is obvious. By Lemma 2.7, S is intra-regular. □

Definition 2.9. An ordered ternary semigroup S is called *completely regular*, if it is regular, left regular and right regular.

Lemma 2.10. *The following statements are equivalent:*

(*i*) *S is completely regular,*

(*ii*) $A \subseteq (A^3SASA^3]$ *for any* $A \subseteq S$,

(*iii*) $a \in (a^3SaSa^3]$ *for any* $a \in S$.

Theorem 2.11. *S is completely regular if and only if every ordered quasi-ideal of S is completely regular.*

Proof. Assume that S is completely regular. Let Q be an ordered quasi-ideal of S and $\emptyset \neq A \subseteq Q$. By Lemma 2.10,

$$\begin{aligned} A \subseteq (A^3SASA^3] &\subseteq ((A](A^3SASA^3](ASASA](A^3SASA^3](A]] \\ &\subseteq ((A](A^3SASA^3](ASA](A^3SASA^3](A]] \\ &\subseteq (A^3(ASASA)A(A(ASAAAAS)ASA)A^3] \\ &\subseteq (A^3(QSQSQ)A(QSQSQ)A^3] \\ &\subseteq (A^3QAQA^3]. \end{aligned}$$

By Lemma 2.10, Q is completely regular.
The conversely is clear because S itself is an ordered quasi-ideal. □

Theorem 2.12. *S is completely regular if and only if every ordered bi-ideal of S is semiprime.*

Proof. Assume that S is completely regular and $\emptyset \neq A \subseteq S$. Let B be an ordered bi-ideal of S and $A^3 \subseteq B$. By Lemma 2.10 and Lemma 2.4,

$$A \subseteq (A^3SASA^3] \subseteq (BSASB] \subseteq (BS(SSAAA]SB] \subseteq (BSBSB] \subseteq (B] = B.$$

Hence, every ordered bi-ideal of S is semiprime.

Conversely, assume that every ordered bi-ideal of S is semiprime. Let $\emptyset \neq A \subseteq S$. First we show that $(A^3SASA^3]$ is an ordered bi-ideal of S. Thus,

$$\begin{aligned}(A^3SASA^3]S(A^3SASA^3]S(A^3SASA^3] &\subseteq (A^3SASA^3SA^3SASA^3SA^3SASA^3] \\ &= (A^3(SASA^3SA^3S)A(SA^3SA^3SAS)A^3] \\ &\subseteq (A^3SASA^3].\end{aligned}$$

Clearly, $((A^3SASA^3]] = (A^3SASA^3]$. So, $(A^3SASA^3]$ is ordered bi-ideals of S. Since $A^9 \subseteq (A^3SASA^3]$, by assumption, $A^3 \subseteq (A^3SASA^3]$, and $A \subseteq (A^3SASA^3]$. By Lemma 2.10, S is completely regular. □

Now, we define the notions of a left lightly regularity and a right lightly regularity of an ordered ternary semigroups as follows.

Definition 2.13. An ordered ternary semigroup S is called *left* (*right*) *lightly regular*, if each its element is *left* (*light*) *lightly regular*), i.e., for each $a \in S$ there exist $x, y, z \in S$ such that $a \leqslant xyaza$ ($a \leqslant axayz$).

Lemma 2.14. *The following statements are equivalent:*

- (i) *S is left* (*resp. right*) *lightly regular,*
- (ii) *$A \subseteq (SSASA]$* (*resp. $A \subseteq (ASASS]$*) *for any $A \subseteq S$,*
- (iii) *$a \in (SSaSa]$* (*resp. $a \in (aSaSS]$*) *for any $a \in S$.*

Theorem 2.15. *The following statements are equivalent:*

- (i) *S is left lightly regular,*
- (ii) *$R \cap M \cap L \subseteq (SSRML]$ for any ordered left ideal L, ordered right ideal R and ordered lateral ideal M of S,*
- (iii) *$L \subseteq (LSL]$ for any ordered left ideal L of S,*
- (iv) *$L \cap M \subseteq (LML]$ for any ordered left ideal L and ordered lateral ideal M of S.*

Proof. $(i) \Leftrightarrow (ii)$: Let L, R and M be an ordered left ideal, an ordered right ideal and an ordered lateral ideal of S, respectively and $a \in R \cap M \cap L$. Since S is left lightly regular, there exist $x, y, z \in S$ such that $a \leqslant xyaza \leqslant xyaz(xyaza) = xy(azx)(yaz)a \in SSRML$. Hence, $R \cap M \cap L \subseteq (SSRML]$.

Conversely, let $\emptyset \neq A \subseteq S$. Then $A \subseteq R(A) \cap M(A) \cap L(A)$. By assumption and Lemma 1.1,

$$\begin{aligned} A &\subseteq R(A) \cap M(A) \cap L(A) \subseteq (SSR(A)M(A)L(A)] \\ &= ((S](S](A \cup ASS](A \cup SAS \cup SSASS](A \cup SSA]] \\ &\subseteq (S^2A^3 \cup S^2A^2S^2A \cup S^2ASASA \cup S^2ASAS^3A \cup S^2AS^2AS^2A \\ &\quad \cup S^2AS^2AS^4A \cup S^2AS^2A^2 \cup S^2AS^2AS^2A \cup S^2AS^3ASA \\ &\quad \cup S^2AS^3AS^3A \cup S^2AS^4AS^2A \cup S^2AS^4AS^4A] \\ &\subseteq (SSASA]. \end{aligned}$$

By Lemma 2.14, S is left lightly regular.

$(i) \Rightarrow (iv)$: Let L and M be an ordered left ideal and an ordered lateral ideal of S and $a \in L \cap M$. Since S is left lightly regular, there exist $x, y, z \in S$ such that $a \leqslant xyaza \leqslant xy(xyaza)za = (xyxya)(zaz)a \in LML$. Hence, $L \cap M \subseteq (LML]$.

$(iv) \Rightarrow (iii)$: It is clear because S itself is an ordered lateral ideal of S

$(iii) \Rightarrow (i)$: Let $a \in S$. Then $a \in L(a)$. By assumption and Lemma 1.1,

$$\begin{aligned} a \in L(a) &\subseteq (L(a)SL(a)] = ((a \cup SSa](S](a \cup SSa]] \\ &\subseteq (aSa \cup aSSSa \cup SSaSa \cup SSaSSSa] \\ &= (aSa] \cup (aSSSa] \cup (SSaSa] \cup (SSaSSSa]. \end{aligned}$$

Case 1: $a \in (aSa]$; there exists $x \in S$, $a \leqslant axa \leqslant ax(axa) \in SSaSa$.
Case 2: $a \in (aSSSa]$; there exist $x, y, z \in S$, $a \leqslant axyza \leqslant axyz(axyza) \in SSaSa$.
Case 3: $a \in (SSaSa]$; it is obvious.
Case 4: $a \in (SSaSSSa]$; it is obvious, since $(SSaSSSa] \subseteq (SSaSa]$.
Thus, S is left lightly regular. □

The next theorem can be similarly proved as Theorem 2.15.

Theorem 2.16. *The following statements are equivalent:*

(i) *S is right lightly regular,*

(ii) *$R \cap M \cap L \subseteq (RMLSS]$ for any ordered left ideal L, ordered right ideal R and ordered lateral ideal M of S,*

(iii) *$R \subseteq (RSR]$ for any ordered right ideal L of S,*

(iv) *$R \cap M \subseteq (RMR]$ for any ordered right ideal R, ordered lateral ideal M of S.*

Definition 2.17. An ordered ternary semigroup S is called *generalized regular*, if each its element is *generalized regular*, i.e., for each $a \in S$ there exist w, x, y, z such that $a \leqslant wxayz$.

Lemma 2.18. *The following statements are equivalent:*

(i) *S is generalized regular,*

(ii) *$A \subseteq (SSASS]$ for any $A \subseteq S$,*

(iii) *$a \in (SSaSS]$ for any $a \in S$.*

Theorem 2.19. *The following statements are equivalent:*

(i) *S is generalized regular,*

(ii) *$L \subseteq (SSLSS]$ for any ordered left ideal L of S,*

(iii) *$R \subseteq (SSRSS]$ for any ordered right ideal R of S,*

(iv) *$M \subseteq (SSMSS]$ for any ordered lateral ideal M of S,*

(v) *$I \subseteq (SSISS]$ for any ordered ideal I of S.*

Proof. $(i) \Leftrightarrow (v)$: Let I be an ordered ideal of S. By Lemma 2.18, $I \subseteq (SSISS]$. Conversely, let $a \in S$. Then $a \in I(a)$. By assumption and Lemma 1.1,

$$\begin{aligned} a \in I(a) &\subseteq (SSI(a)SS] \subseteq ((S](S](a \cup SSa \cup aSS \cup SaS \cup SSaSS](S](S]] \\ &\subseteq (S^2aS^2 \cup S^4aS^2 \cup S^2aS^4 \cup S^3aS^3 \cup S^4aS^4] \\ &= (S^2aS^2] \cup (S^4aS^2] \cup (S^2aS^4] \cup (S^3aS^3] \cup (S^4aS^4]. \end{aligned}$$

CASE 1: $a \in (S^2aS^2]$; it is obvious.
CASE 2: $a \in (S^4aS^2]$; it is obvious, since $(S^4aS^2] \subseteq (S^2aS^2]$.
CASE 3: $a \in (S^2aS^4]$; it is obvious, since $(S^2aS^4] \subseteq (S^2aS^2]$.
CASE 4: $a \in (S^3aS^3]$; there exist $u, v, w, x, y, z \in S$, $a \leqslant uvwaxyz \leqslant uvw(uvwaxyz)xyz = (uvw)(uvw)a(xyz)(xyz) \in SSaSS$.
CASE 5: $a \in (S^4aS^4]$; it is obvious, since $(S^4aS^4] \subseteq (S^2aS^2]$.

Thus, S is generalized regular.
$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ Can be proved similarly. □

3. Connections between regularities

The proof of following proposition is not difficult.

Proposition 3.1. *Let S be an ordered ternary semigroup.*

(i) *If S is completely regular, then it is regular, left regular and right regular.*

(*ii*) *If* S *is left or right regular, then it is intra-regular.*

(*iii*) *If* S *is left* (*resp. right*) *regular, then it is left* (*resp. right*) *lightly regular.*

(*iv*) *If* S *is regular, then it is left and right lightly regular.*

(*v*) *If* S *is intra-regular or left lightly regular or right lightly regular, then it is generalized regular.*

Now, we give examples to show that the converses statements are not true.

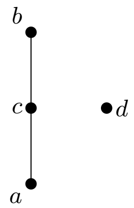
Example 3.2. Let $S = \{a, b, c, d\}$. A ternary operation $[\ ]$ on S and the figure of a partial order relation $\leqslant$ on S are as follows:

$[\ ]$	a	b	c	d
aa	a	a	a	d
ab	a	a	a	d
ac	a	a	a	d
ad	d	d	d	d

$[\ ]$	a	b	c	d
ba	b	b	b	d
bb	b	b	b	d
bc	b	b	b	d
bd	d	d	d	d

$[\ ]$	a	b	c	d
ca	a	a	a	d
cb	a	a	a	d
cc	a	a	a	d
cd	d	d	d	d

$[\ ]$	a	b	c	d
da	d	d	d	d
db	d	d	d	d
dc	d	d	d	d
dd	d	d	d	d



It is clear that a, b, d are left lightly regular. Since $c \in (SScSc] = S$, S is left lightly regular. However, S is neither regular nor right lightly regular because $c \notin (cSc] = \{a, d\} = (cScSS]$.

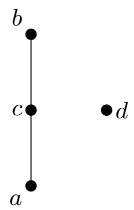
Example 3.3. Let $S = \{a, b, c, d\}$. A ternary operation $[\ ]$ on S and the figure of a partial order relation $\leqslant$ on S are as follows:

$[\ ]$	a	b	c	d
aa	a	b	a	d
ab	a	b	a	d
ac	a	b	a	d
ad	d	d	d	d

$[\ ]$	a	b	c	d
ba	a	b	a	d
bb	a	b	a	d
bc	a	b	a	d
bd	d	d	d	d

$[\ ]$	a	b	c	d
ca	a	b	a	d
cb	a	b	a	d
cc	a	b	a	d
cd	d	d	d	d

$[\ ]$	a	b	c	d
da	d	d	d	d
db	d	d	d	d
dc	d	d	d	d
dd	d	d	d	d



It is clear that a, b, d are right lightly regular. Since $c \in (cScSS] = S$, S is right lightly regular. However, S is neither regular nor left lightly regular because $c \notin (cSc] = \{a, d\} = (SScSc]$.

Example 3.4. Let $S = \{a, b, c, d, e, f\}$. A ternary operation $[\ ]$ on S is as follows:

[ ]	a	b	c	d	e	f
aa	a	a	a	a	e	a
ab	a	a	a	a	e	a
ac	a	a	a	a	e	a
ad	a	a	a	a	e	a
ae	e	e	e	e	e	e
af	a	a	a	a	e	a

[ ]	a	b	c	d	e	f
ba	a	a	a	a	e	a
bb	a	b	a	d	e	a
bc	a	a	a	a	e	a
bd	a	b	d	d	e	b
be	e	e	e	e	e	e
bf	a	a	a	a	e	a

[ ]	a	b	c	d	e	f
ca	a	a	a	a	e	a
cb	a	f	a	c	e	a
cc	a	f	c	c	e	f
cd	a	f	c	c	e	f
ce	e	e	e	e	e	e
cf	a	f	a	c	e	a

[ ]	a	b	c	d	e	f
da	a	a	a	a	e	a
db	a	b	a	d	e	a
dc	a	b	d	d	e	b
dd	a	b	d	d	e	b
de	e	e	e	e	e	e
df	a	b	a	d	e	a

[ ]	a	b	c	d	e	f
ea	e	e	e	e	e	e
eb	e	e	e	e	e	e
ec	e	e	e	e	e	e
ed	e	e	e	e	e	e
ee	e	e	e	e	e	e
ef	e	e	e	e	e	e

[ ]	a	b	c	d	e	f
fa	a	a	a	a	e	a
fb	a	f	a	c	e	a
fc	a	a	a	a	e	a
fd	a	f	c	c	e	f
fe	e	e	e	e	e	e
ff	a	a	a	a	e	a

Define a partial order relation $\leqslant$ on S by $\leqslant := \{(x, x) \mid x \in S\}$. It is clear a, b, c, d, e are regular. Since $f \in (fSf] = \{a, e, f\}$, S is regular. So, S is left lightly regular. However, S is neither left regular nor intra-regular because $f \notin (Sff] = \{a, e\} = (SSf^3SS]$.

Example 3.5. Let $S = \{a, b, c, d, e, f\}$. A ternary operation $[\ ]$ on S is as follows:

[ ]	a	b	c	d	e	f
aa	a	a	a	a	e	a
ab	a	a	a	a	e	a
ac	a	a	a	a	e	a
ad	a	a	a	a	e	a
ae	e	e	e	e	e	e
af	a	a	a	a	e	a

[ ]	a	b	c	d	e	f
ba	a	a	a	a	e	a
bb	a	b	d	b	e	f
bc	a	a	f	b	e	a
bd	a	b	d	b	e	f
be	e	e	e	e	e	e
bf	a	a	f	b	e	a

[ ]	a	b	c	d	e	f
ca	a	a	a	a	e	a
cb	a	a	a	a	e	a
cc	a	a	c	d	e	a
cd	a	d	c	d	e	c
ce	e	e	e	e	e	e
cf	a	a	a	a	e	a

[ ]	a	b	c	d	e	f
da	a	a	a	a	e	a
db	a	d	c	d	e	c
dc	a	a	c	d	e	a
dd	a	d	c	d	e	c
de	e	e	e	e	e	e
df	a	a	c	d	e	a

[ ]	a	b	c	d	e	f
ea	e	e	e	e	e	e
eb	e	e	e	e	e	e
ec	e	e	e	e	e	e
ed	e	e	e	e	e	e
ee	e	e	e	e	e	e
ef	e	e	e	e	e	e

[ ]	a	b	c	d	e	f
fa	a	a	a	a	e	a
fb	a	a	a	a	e	a
fc	a	a	f	b	e	a
fd	a	b	f	b	e	f
fe	e	e	e	e	e	e
ff	a	a	a	a	e	a

Define a partial order relation $\leqslant$ on S by $\leqslant := \{(x, x) | x \in S\}$. It is clear a, b, c, d, e are regular. Since $f \in (fSf] = \{a, e, f\}$, S is regular. So, S is right lightly regular. However, S is neither right regular nor intra-regular because $f \notin (ffS] = \{a, e\} = (SSf^3SS]$.

Example 3.6. Let $S = \{a, b, c, d, e\}$. A ternary operation $[\ ]$ on S and the figure of a partial order relation $\leqslant$ on S are as follows:

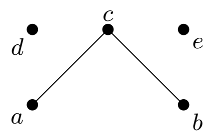
[ ]	a	b	c	d	e
aa	b	b	b	b	e
ab	b	b	b	b	e
ac	b	b	b	b	e
ad	b	b	b	b	e
ae	e	e	e	e	e

[ ]	a	b	c	d	e
ba	b	b	b	b	e
bb	b	b	b	b	e
bc	b	b	b	b	e
bd	b	b	b	b	e
be	e	e	e	e	e

[ ]	a	b	c	d	e
ca	c	c	c	c	e
cb	c	c	c	c	e
cc	c	c	c	c	e
cd	c	c	c	c	e
ce	e	e	e	e	e

[ ]	a	b	c	d	e
da	c	c	c	c	e
db	c	c	c	c	e
dc	c	c	c	c	e
dd	c	c	c	d	e
de	e	e	e	e	e

[ ]	a	b	c	d	e
ea	e	e	e	e	e
eb	e	e	e	e	e
ec	e	e	e	e	e
ed	e	e	e	e	e
ee	e	e	e	e	e



It is clear b, c, d, e are left regular. Since $a \in (Saa] = \{a, b, c, e\}$, S is left regular. So, S is intra-regular and generalized regular. However, S is neither right lightly regular nor regular because $a \notin (aSaSS] = \{b, e\} = (aSa]$.

Example 3.7. Let $S = \{a, b, c, d, e\}$. A ternary operation [ ] on S and the figure of a partial order relation $\leqslant$ on S are as follows:

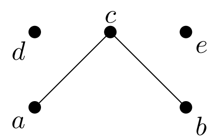
[ ]	a	b	c	d	e
aa	b	b	c	c	e
ab	b	b	c	c	e
ac	b	b	c	c	e
ad	b	b	c	c	e
ae	e	e	e	e	e

[ ]	a	b	c	d	e
ba	b	b	c	c	e
bb	b	b	c	c	e
bc	b	b	c	c	e
bd	b	b	c	c	e
be	e	e	e	e	e

[ ]	a	b	c	d	e
ca	b	b	c	c	e
cb	b	b	c	c	e
cc	b	b	c	c	e
cd	b	b	c	c	e
ce	e	e	e	e	e

[ ]	a	b	c	d	e
da	b	b	c	c	e
db	b	b	c	c	e
dc	b	b	c	c	e
dd	b	b	c	d	e
de	e	e	e	e	e

[ ]	a	b	c	d	e
ea	e	e	e	e	e
eb	e	e	e	e	e
ec	e	e	e	e	e
ed	e	e	e	e	e
ee	e	e	e	e	e



It is clear b, c, d, e are right regular. Since $a \in (aaS] = \{a, b, c, e\}$, S is right regular. So, S is intra-regular and generalized regular. However, S is neither left lightly regular nor regular because $a \notin (SSaSa] = \{b, e\} = (aSa]$.

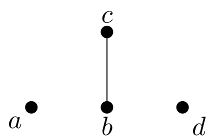
Example 3.8. Let $S = \{a, b, c, d\}$. A ternary operation [ ] on S and the figure of a partial order relation $\leqslant$ on S are as follows:

[ ]	a	b	c	d
aa	a	a	a	a
ab	a	a	a	a
ac	a	a	a	a
ad	a	a	a	a

[ ]	a	b	c	d
ba	a	a	a	a
bb	a	b	b	b
bc	a	b	b	b
bd	a	b	b	b

[ ]	a	b	c	d
ca	a	a	a	a
cb	a	b	b	b
cc	a	b	b	b
cd	a	b	b	c

[ ]	a	b	c	d
da	a	a	a	a
db	a	b	b	b
dc	a	b	b	c
dd	a	b	c	d



It is clear a, b, d are generalized regular. Since $c \in (SScSS] = \{a, b, c\}$, S is generalized regular. S is not intra-regular because $c \notin (SSc^3SS] = \{a, b\}$.

Example 3.9. Let $S = \{a, b, c, d, e, f, g\}$. A ternary operation [ ] on S and the figure of a partial order relation $\leqslant$ on S are as follows:

[ ]	a	b	c	d	e	f	g
aa	a	a	a	a	e	a	a
ab	a	a	a	a	e	a	a
ac	a	a	a	a	e	a	a
ad	a	a	a	a	e	a	a
ae	e	e	e	e	e	e	e
af	a	a	a	a	e	a	a
ag	a	a	a	a	e	a	a

[ ]	a	b	c	d	e	f	g
ba	a	a	a	a	e	a	a
bb	a	b	b	d	e	b	d
bc	a	b	b	d	e	b	d
bd	a	b	b	d	e	b	d
be	e	e	e	e	e	e	e
bf	a	b	b	d	e	b	d
bg	a	b	b	d	e	b	d

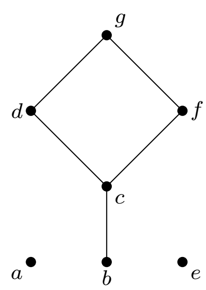
[ ]	a	b	c	d	e	f	g
ca	a	a	a	a	e	a	a
cb	a	b	b	d	e	b	d
cc	a	b	b	d	e	b	d
cd	a	b	b	d	e	b	d
ce	e	e	e	e	e	e	e
cf	a	b	b	d	e	b	d
cg	a	b	b	d	e	b	d

[ ]	a	b	c	d	e	f	g
da	a	a	a	a	e	a	a
db	a	b	b	d	e	b	d
dc	a	b	b	d	e	b	d
dd	a	b	b	d	e	b	d
de	e	e	e	e	e	e	e
df	a	b	b	d	e	b	d
dg	a	b	b	d	e	b	d

[ ]	a	b	c	d	e	f	g
ea	e	e	e	e	e	e	e
eb	e	e	e	e	e	e	e
ec	e	e	e	e	e	e	e
ed	e	e	e	e	e	e	e
ee	e	e	e	e	e	e	e
ef	e	e	e	e	e	e	e
eg	e	e	e	e	e	e	e

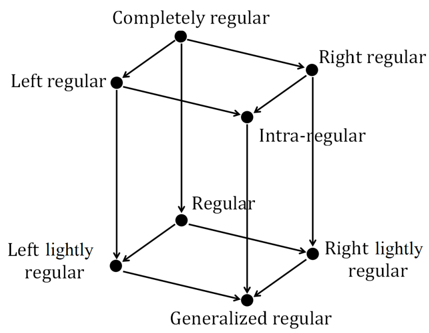
[ ]	a	b	c	d	e	f	g
fa	a	a	a	a	e	a	a
fb	a	f	f	g	e	f	g
fc	a	f	f	g	e	f	g
fd	a	f	f	g	e	f	g
fe	e	e	e	e	e	e	e
ff	a	f	f	g	e	f	g
fg	a	f	f	g	e	f	g

[ ]	a	b	c	d	e	f	g
ga	a	a	a	a	e	a	a
gb	a	f	f	g	e	f	g
gc	a	f	f	g	e	f	g
gd	a	f	f	g	e	f	g
ge	e	e	e	e	e	e	e
gf	a	f	f	g	e	f	g
gg	a	f	f	g	e	f	g



It is clear a, b, d, e, f, g are left regular. Since $c \in (Scc] = \{a, b, c, e, f\}$, S is left regular. Similarly, a, b, d, e, f, g are right regular and $c \in (ccS] = \{a, b, c, d, e\}$, S is right regular. However, S is not regular because $c \notin (cSc] = \{a, b, e\}$.

Now, we conclude the connections of the eight regularities as the figure.



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