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On ordered semigroups without nilpotent ideals

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Abstract. We characterize those left, right and two-sided ideals of ordered semigroups without nilpotent that contain at least one minimal left and at least one minimal right ideal.

1. Introduction

A semigroup $(S, \cdot)$ with a partial order $\leqslant$ that is *compatible* with the semigroup operation, i.e.,

$$\text{for any } a, b, c \in S,\ a \leqslant b \text{ implies } ac \leqslant bc \text{ and } ca \leqslant cb,$$

is called a *partially ordered semigroup* (or simply an *ordered semigroup*) (see [1], [4], [5], [6]).

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. If A is a nonempty subset of S, then $(A] = \{x \in S \mid x \leqslant a \text{ for some } a \in A\}$.

It is routine matters to verify that for subsets A and B of S the following hold: (1) $A \subseteq B$ implies $(A] \subseteq (B]$, (2) $((A]] = (A]$, (3) $(A](B] \subseteq (AB]$, (4) $((A]B] = (A(B]] = ((A](B]] = (AB]$, (5) $(A] \cup (B] = (A \cup B]$.

For further information we refer to [9].

In [7], Kehayopulu introduced the concepts of left, right and two-sided ideals of an ordered semigroup as follows.

Let $(S, \cdot, \leqslant)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp., *right*) *ideal* of an ordered semigroup S if satisfies:

(i) $SA \subseteq A$ (resp., $AS \subseteq A$),

(ii) for any $x \in A$ and $y \in S$, if $y \leqslant x$ then $y \in A$, or equivalently, $A = (A]$.

If A is both a left ideal and a right ideal of S, then A is called a *two-sided ideal* (or an *ideal*) of S. Note that if A and B are two-sided ideals of S, then $(BAB]$ and $(AB]$ are two-sided ideals of S. Intersection of two-sided ideals of S is a two-sided ideal of S if it is nonempty. Union of two-sided ideals of S is a two-sided ideal of S. For a nonempty subset A of S, we denote by $(A)_l$ the left ideal of S generated by A. We have $(A)_l = (A \cup SA]$. Similarly, $(A)_r = (A \cup AS]$ and $(A) = (A \cup SA \cup AS \cup SAS]$ are the right and two-sided generated by A, respectively.

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The *zero* of an ordered semigroup $(S, \cdot, \leqslant)$ is element of S, usually denote by 0, such that $0 \leqslant a$ and $0a = a0 = 0$ for all $a \in S$. A left ideal L of S is called *simple* if for any left ideal $L' \subseteq L$ implies $L' = L$. The notions of simple right and two-sided ideals are defined similarly. Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. An element a of S with zero is called *nilpotent* if there exists an element $n \in \mathbb{N}$ such that $a^n = 0$. A two-sided ideal A of S is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $A^n = \{0\}$ (see [8]).

An ordered semigroup $(S, \cdot \leqslant)$ is *without nilpotent ideals* if S contains zero element but no nilpotent left or right ideal $\neq \{0\}$. A left (resp., right, two-sided) ideal of an ordered semigroup $(S, \cdot \leqslant)$ will be called *minimal left ideal* of S if it is $\neq \{0\}$ and contains no proper subset $\neq \{0\}$ which is also left (resp., right, two-sided) ideal of S.

In [2], Clifford initiated the study of semigroups with at least one left and at least one right minimal ideals. Later, he was in turn extension to the semigroup having a zero element of the same results (without order) in [3].

The purpose of the present paper is to extend results of A.H. Clifford [3] to ordered semigroups.

2. Two-sided ideals containing minimal left ideals

Theorem 2.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals. Then any minimal two-sided ideal of S is simple.*

Proof. Let M be a minimal two-sided ideal of S. Assume that $B \neq \{0\}$ is a proper two-sided ideal of M . Since $(MBM]$ is a two-sided ideal of S and $(MBM] \subseteq M$, therefore $(MBM] = \{0\}$ because M is a minimal left ideal. We get

$$(MB]^2 = (MB](MB] \subseteq (MBMB] = ((MBM]B] = (0B] = \{0\}.$$

Clearly $(MB]$ is a left ideal. This means that $(MB]$ is a nilpotent left ideal. By assumption we have $(MB] = \{0\}$. Since $(SBS]$ is a two-sided ideal and $(SBS] \subseteq (SMS] \subseteq M$, we have either $(SBS] = \{0\}$ or $(SBS] = M$.
Case 1. $(SBS] = M$. We obtain that,

$$(SB]^2 = (SB](SB] \subseteq (SBSB] = ((SBS]B] = (MB] = \{0\}.$$

Case 2. $(SBS] = \{0\}$. We obtain that,

$$(SB]^2 = (SB](SB] \subseteq (SBSB] = ((SBS]B] = (\{0\}B] = \{0\}.$$

This shows that $(SB]^2 = \{0\}$. Thus $(SB]$ is nilpotent. Hence $(SB] = \{0\}$. Similarly, we have $(BS] = \{0\}$. This shows $(SB] \subseteq B$ and $(BS] \subseteq B$, so that B is a two-sided ideal of S. This is a contradiction. □

Theorem 2.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with zero. If L is a minimal left ideal of S, then for each $a \in S$, $(La]$ is also a minimal left ideal of S or $(La] = \{0\}$.*

Proof. Let $a \in S$. It is clear that $(La]$ is a left ideal of S. Assume that $(La] \neq \{0\}$ and there exists a left ideal $L' \neq \{0\}$ of S such that $L' \subset (La]$. Now let A be the set of all elements x in L such that $xa \leqslant y$ for some $y \in L'$ and $(A] = A$. We shall show that A is a left ideal of S. Clearly $A \neq \{0\}$ because $L' \neq \{0\}$. Let $s \in S$. We have

$$sxa = s(xa) \leqslant sy \in SL' \subseteq L'.$$

This shows that $sxa \in L'$. We have $SA \subseteq A$. Thus A is a left ideal of S. Since L is a minimal left ideal of S, $A = L$. So, $(La] = (Aa] \subseteq L'$. This is a contradiction. Therefore $(La]$ does not contain any proper left ideal. □

Theorem 2.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup with zero. Let A be a two-sided ideal of S containing at least one minimal left ideal of S. Then the sum of all minimal left ideals of S contained in A is a two-sided ideal of S. In particular, if A is minimal, then it is a sum of minimal left ideals of S.*

Proof. Let $B = \bigcup_{i \in I} L_i$ be the sum of all minimal left ideals of S contained in A. It is clear that B is a left ideal of S. We claim that B is a right ideal of S. Now let $s \in S$. Then $Bs = (\bigcup_{i \in I} L_i)s$. By Theorem 2.2, $(L_i s]$ is a minimal left ideal of S or $(L_i s] = \{0\}$ for all $i \in I$. So,

$$Bs = (\bigcup_{i \in I} L_i)s = \bigcup_{i \in I} L_i s \subseteq \bigcup_{i \in I} (L_i s] \subseteq B.$$

This shows that B is a two-sided ideal of S. Consequently, if A minimal, then $A = B$. □

Theorem 2.4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals. Let M be a minimal two-sided ideal of S containing at least one minimal left ideal of S. Then every left ideal of M is a left ideal of S.*

Proof. Firstly we shall show that for any minimal left ideal L of S contained in M is a minimal left ideal of M. For this, let L be a minimal left ideal of S contained in M. Suppose that $L' \neq \{0\}$ is a left ideal of M contained in L. Since $(ML']$ is a left ideal of S and $(ML'] \subseteq (ML] \subseteq (SL] \subseteq L$. Then $(ML'] = \{0\}$ or $(ML'] = L$.
Case 1. $(ML'] = L$. Then $L = (ML'] \subseteq L'$, so that $L = L'$.
Case 2. $(ML'] = \{0\}$. Since $(SL']$ is a left ideal of S and $(SL'] \subseteq (SL] \subseteq L$. We get $(SL'] \subseteq L$, whence $(SL'] = L$ or $(SL'] = \{0\}$. If $(SL'] = L$, then

$$L^2 = (SL']^2 = (SL'](SL'] \subseteq (SL'SL'] \subseteq (SMSL'] \subseteq (ML'] = \{0\}.$$

This is a contradiction to the assumption that S contains no nilpotent ideal. Thus $(SL'] = \{0\}$. Then $SL' \subseteq (SL'] \subseteq L'$, whence L' is a left ideal of S. By the minimality of L this implies $L' = L$. Thus L is a minimal left ideal of M.

Now let L_M be any left ideal of M. If $L_M = \{0\}$ then L_M is a left ideal of S. Suppose that $L_M \neq \{0\}$. By Theorem 2.3, M is the sum of all minimal left ideals of S contained in it. Therefore each $a \neq 0$ in L_M belong to some minimal left ideal L_S of S contained in M. Hence $L_M \cap L_S \neq \{0\}$ since $L_M \cap L_S$ is a left ideal of M contained in L_S and L_S is a minimal left ideal of M. This proves that $L_M \cap L_S = L_S$, whence $L_S \subseteq L_M$. Hence L_M is the sum of all minimal left ideal L_S of S contained in M such that $L_M \cap L_S \neq \{0\}$. Since, the sum of left ideals of S is a left ideal, L_M is a left ideal of S. □

3. Two-sided ideals containing minimal ideals

Theorem 3.1. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. Then for each minimal left ideal of S contained in M corresponds at least one minimal right ideal R of S contained in M such that $(LR] = M$ and $(RL] \neq \{0\}$.*

Proof. Let L be a minimal left ideal of S contained in M. Clearly $(ML]$ is a left ideal contained in L. Then $(ML] = L$ or $(ML] = \{0\}$. If $(ML] = \{0\}$ then we get $L^2 \subseteq (ML] = \{0\}$, which contradicts to the assumption that S contains no nilpotent ideal. Thus $(ML] = L$. Since $(LM]$ is a two-sided ideal of S contained in M, so $(LM] = M$ or $(LM] = \{0\}$. Similarly, if $(LM] = \{0\}$ then $L^2 \subseteq (LM] = \{0\}$, which is impossible. This implies $(LM] = M$.

Now by the left-right dual Theorem 2.3, M is the sum of all minimal right ideals of S contained in M. We have $(LR] \neq \{0\}$ for some right ideal R of S contained in M, since if $(LR] = \{0\}$ for every R, then $(LM] = \{0\}$, which contradicts to $(LM] = M$. If $(RL] = \{0\}$, then we have

$$L = (ML] = ((LR]L] = ((LR](L]] = (LRL] = (L(RL]] = (L\{0\}] = \{0\}.$$

This is a contradiction. Hence $(RL] \neq \{0\}$. □

Theorem 3.2. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. Let L and R are minimal left and right ideals of S contained in M such that $(LR] = M$ and $RL \neq \{0\}$. Then for each a in $(R \cap L) \setminus \{0\}$ and $b \in (RL]$ we have $ax \leqslant b$, $ya \leqslant b$, for some $x, y \in (RL]$.*

Proof. Suppose that $a \in (R \cap L) \setminus \{0\}$ and $b \in (RL]$. Then $(Ma]$ is a left ideal of S contained in L, since $a \in L$. Hence $(Ma] = L$ or $(Ma] = \{0\}$. If $Ma = \{0\}$ then $(Ma]^2 \subseteq (MaMa] \subseteq (Ma] = \{0\}$, which is impossible by the assumption on S. Thus $(Ma] = L$. Since $(aR]$ is a right ideal of S contained in R, we have $(aR] = R$ or $(aR] = \{0\}$. If $(aR] = \{0\}$, then by hypothesis, we have $M = (LR] = (MaR] = (M(aR]] = \{0\}$, which is a contradiction to our assumption. This proves that $(aR] = R$. Then $(a(RL]] = (aRL] = ((aR]L] = (RL]$. This implies $ax \leqslant b$ for some $x \in RL$. Dually $(aM]$ is a right ideal contained in R. Then the case $(aM] = \{0\}$ is excluded as above, whence $(aM] = R$. If $(La] = \{0\}$ then we have $M = (LR] = (L(aM]] = (LaM] = (\{0\}M] = \{0\}$, this is a contradiction. Thus

$(La] = L$, and hence $(RL] = (R(La]] = (RLa] = ((RL]a]$. Hence $ya \leqslant b$, for some $y \in (RL]$. □

Corollary 3.3. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. Let L be minimal left ideal of S contained in M, then for each $a \in L$ there exists an element $e \in L \setminus \{0\}$ such that $a \leqslant ea$.*

Proof. Let $a \in L$. If $a = 0$ then $a \leqslant ea$. Now let $a \neq 0$. Let $(a)_l = (a \cup La]$. We know that $(a)_l$ is a minimal left ideal of S. Since $(La]$ is a left ideal such that $(La] \subseteq (a \cup La]$. This implies $(La] = (a \cup La]$ or $(La] = \{0\}$ because $(a)_l$ is a minimal left ideal. By the proof of Theorem 3.2, we have $(La] \neq \{0\}$. We conclude that $(La] = (a \cup La]$. Hence $a \leqslant ea$ for some $e \in L \setminus \{0\}$. Hence, the theorem is proved. □

The following theorem is proved dually:

Corollary 3.4. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. Let R be minimal right ideal of S contained in M, then for each $a \in R$ there exists an element $f \neq 0$ in R such that $a \leqslant af$.*

Corollary 3.5. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. If R and L are minimal left and right ideal of S contained in M, then there exists $e \neq 0$ in $(RL]$ such that $e \leqslant e^2$.*

Proof. Since $(RL](RL] \subseteq (R(LRL)] \subseteq (RL]$, so that $(RL]$ is closed. By Theorem 3.1, we have $(RL] \neq \{0\}$. Now let $a \in (RL]$ be such that $a \neq 0$. Since $(RL] \subseteq R \cap L$, implies $a \in L$ and $a \in R$, by the proof of Theorem 3.2, we have $(aR] = R$ and $(La] = L$. Thus $(RL] = ((aR](La]] = (aRLa] = (a(RL]a]$. Hence $a \leqslant axa$ for some $x \in (RL]$. Since $(RL]$ is closed, $ax \in (RL]$. This shows that $ax \leqslant (axa)x = (ax)(ax) = (ax)^2$ as required. □

Corollary 3.6. *Let $(S, \cdot, \leqslant)$ be an ordered semigroup without nilpotent ideals and M be a minimal two-sided ideal of S containing at least one minimal left and at least one minimal right ideal of S. Let R and L be minimal left and right ideal of S contained in M. Let e be an element in $(RL]$ such that $e \neq 0$ and $e \leqslant e^2$ then $R = (eS]$, $L = (Se]$*

Proof. It is clear that $(eS]$ is a right ideal of S. We observe that $(eS] \subseteq ((RL]S] = (RLS] \subseteq R$. Then $(eS]$ is contained in R. Since $0 \neq e \leqslant e^2 \in eS$, we have $(eS] \neq \{0\}$ By the minimality of R, this implies $(eS] = R$. Similarly, we obtain $(Se] = L$. □

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