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Computing maximal and minimal subgroups with respect to a given property in certain finite groups

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Abstract. Suppose G is a finite group and $max(G)$, $nmax(G)$, $snmax(G)$, $maxn(G)$ and $minn(G)$ are denoted the number of maximal, normal maximal, self-normalizing maximal, maximal normal and minimal normal subgroups of G, respectively. The aim of this paper is to compute these numbers for certain classes of finite groups.

1. Introduction

The *Asymptotic Group Theory* is an active part of finite groups. In group theory it is usual to compute some numerical invariants related to a given group, but in asymptotic group theory we study the asymptotic behaviors of these invariants, see [10] for details. In this work, we will consider a set $\mathcal{L}(n)$ containing six groups of orders $2n, 4n, 6n, 8n, 2^n$ and 2^n, respectively. Some numeric invariants of these groups like the number of maximal and minimal subgroups will be computed. Such type of problems is studied nowadays in a new branch of group theory named *subgroup growth*. It is possible to use our calculations for solving some problems in probabilistic group theory as computing the possibility of generating one of these groups by two or more elements.

Let G be a finite group. A subgroup H of G is called self-normalizing in G if the normalizer of H in G, $N_G(H)$, is equal to H itself. It is clear that a maximal subgroup of G is either normal subgroup or self-normalizing. The notations $Max(G)$, $NMax(G)$, $SNMax(G)$, $MaxN(G)$, $MinN(G)$, $NMin(G)$ and $Syl(G)$ stand for the set of all maximal subgroups, normal maximal subgroups, self-normalizing maximal subgroups, maximal normal subgroups, minimal normal subgroups, normal minimal subgroups and Sylow subgroups of G, respectively. We also define $max(G) = |Max(G)|$, $nmax(G) = |NMax(G)|$, $snmax(G) = |SNMax(G)|$, $maxn(G) = |MaxN(G)|$, $minn(G) = |MinN(G)|$, $nmin(G) = |NMin(G)|$ and $syl(G) = |Syl(G)|$. The intersection of all normal maximal subgroups and all

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self-normalizing maximal subgroups of G are denoted by $R(G)$ and $L(G)$, respectively. Obviously, $R(G)$, $L(G)$ and the Frattini subgroup $\phi(G)$ are characteristic subgroups of G and $\phi(G)$ can be written as the intersection of $L(G)$ and $R(G)$.

The set of all prime factors of a positive integer n is denoted by $\pi(n)$. It is convention to write $\pi(G)$ as $\pi(|G|)$. Caviour [5, Theorem], proved that the number of subgroups in the dihedral group of order n is $d(n) + \sigma(n)$, where $d(n)$ and $\sigma(n)$ denote the number of divisors of n and its summation, respectively. Calhoun [4] generalized this result to the class of groups which can be formed as cyclic extensions of cyclic groups. Lauderdale [8], proved that if G is a finite non-cyclic group, then $max(G) \geqslant |\pi(G)| + p$, where p is either the smallest prime in $\pi(G)$ or the smallest prime in $\pi(G)$ such that G has a non-cyclic Sylow $p-$subgroup and some examples showing that this bound is best possible was given.

Tărnăuceanu [12] computed the number of some types of subgroups of finite abelian groups. In [13], he continued his work and presented the concept of normality degree of a finite group. This quantity measures the probability of a random subgroup to be normal. He obtained explicit formulas for some particular classes of finite groups. For the sake of completeness, we mention here a result of [3] which is crucial in this paper.

Lemma 1.1. *Let G be a finite group. Then,*

- $\acute{G} \leqslant R(G)$, *where* $\acute{G} = [G, G]$ *is the derived group of* G,
- $Z(G) \leqslant L(G)$,
- $L(G)$ *is nilpotent,*
- $\frac{L(G)}{\phi(G)} = Z(\frac{G}{\phi(G)})$.

We now give the presentation of the dihedral group D_{2n}, dicyclic group T_{4n}, semi-dihedral group SD_{2^n} and three groups U_{6n}, V_{8n} and $H(n)$ that will be used later. The groups U_{6n} and V_{8n} were presented for first time in the famous book of James and Liebeck [9]. In this book, the group V_{8n} was studied in the case that n is odd. We refer to [6], for the main properties of the group V_{8n}, when n is an even positive integer. Note that a dicyclic group is an extension of the cyclic group of order 2 by a cyclic group of order $2n$, but the extension is not split.

$$\begin{aligned}
T_{4n} &= \langle a, b \mid a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\
U_{6n} &= \langle a, b \mid a^{2n} = b^3 = e,\ bab = a \rangle, \\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b \rangle, \\
SD_{2^n} &= \langle a, b \mid a^{2^{n-1}} = b^2 = e, b^{-1}ab = a^{-1+2^{n-2}} \rangle, \\
H(n) &= \langle x, y, z \mid x^{2^{n-2}} = y^2 = z^2 = e, [x, y] = [y, z] = e, x^z = xy \rangle.
\end{aligned}$$

The group $H(n)$ was presented by Abbaspour and Behravesh in [2]. They computed the character table of this group and proved that $|H(n)| = 2^n$, $Z(H(n)) = \langle x^2 \rangle \langle y \rangle \cong Z_{2^{n-3}} \times Z_2$ and $H(n)' = \langle y \rangle$ has order 2.

Set $\mathcal{L}(n) = \{D_{2n}, T_{4n}, U_{6n}, V_{8n}, SD_{2^n}, H(n)\}$. The aim of this paper is to find exact expressions for the numbers $max(G)$, $min(G)$, $nmax(G)$, $nmin(G)$, $maxn(G)$ and $minn(G)$. In each case the groups $R(G)$, $L(G)$ and $\phi(G)$ will be computed, where $G \in \mathcal{L}(n)$. To do this, we assume that $|G| = 2^a p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ is the prime factorization of $|G|$, $2 < p_1 < p_2 < \dots < p_r$, and $c_0 = 1 < c_1 < c_2 < \dots < c_t$ are all odd divisors of $|G|$. Define:

$$a_{ij} = \begin{cases} 2^{j-1} & i = 1 \\ 2^{j-1}c_{i-1} & i \neq 1 \end{cases}.$$

The *order table* of G, $OT(G)$, is a matrix of size $(k+1) \times \tau(m)$, where k and m are non-negative integers with $n = |G| = 2^k m$ and $OT(G) = [a_{ij}]$. The columns and rows of this table are labeled by the powers of 2 and odd divisors of $|G|$, respectively. The order table of an arbitrary group G is recorded in Table 1.

It is easy to see that this table has exactly $d(\frac{n}{2^k})$ rows and $k+1$ columns and so there are $(k+1)d(\frac{n}{2^k})$ entries. Suppose x is a positive integer with prime factorization $x = p_1^{a_1} p_2^{a_2} ... p_s^{a_s}$. It is well-know that $\tau(x) = (a_1+1)(a_2+1)...(a_s+1)$ and $\sigma(x) = \frac{p_1^{a_1+1}-1}{p_1-1} \dots \frac{p_s^{a_s+1}-1}{p_s-1}$.

Throughout this paper our notations are standard and mainly taken from [9]. Our calculations are done with the aid of GAP [14].

Table 1. Orders of subgroups, when $|G| = n = 2^r p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$.

	j	1	2	3	...	r+1
i		1	2	4	...	2^r
1	1	1	2	4	...	2^r
2	c_1	c_1	$2c_1$	$4c_1$	...	$2^r c_1$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$i+1$	c_i	c_i	$2c_i$	$4c_i$	...	$2^r c_i$
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t+1$	c_t	c_t	$2c_t$	$4c_t$	...	$2^r c_t$

In [1], the second author investigated the structure of finite groups with exactly $m \leqslant 4$ maximal subgroups, and the structure of groups of odd order with exactly $n = |\pi(G)|$ minimal subgroups. The aim of this paper is to continue this work by computing the number of maximal, normal maximal, self-normalizing maximal, maximal normal and minimal normal subgroups of certain groups.

2. Main results

The aim of this section is to continue our earlier work [11] by computing some group theoretic parameters introduced in Section 1 for groups in $\mathcal{L}(n)$. The calculations of each group are given separately in a subsection.

2.1. Dihedral Group D_{2n}

The dihedral group D_{2n} can be presented as $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1} \rangle$. The method given for dihedral groups can be applied in other members of $\mathcal{L}(n)$ and so it is useful to present with details our method for dihedral groups. By the proof of [5, Theorem], all subgroups of dihedral group D_{2n} have one of the following forms:

1. A subgroup of $\langle a^i \rangle \simeq C_{\frac{n}{i}}$ where $i|n$,
2. A subgroup of $\langle a^i, a^j b \rangle \simeq D_{\frac{2n}{i}}$, where $i|n$ and $1 \leqslant j \leqslant i$.

These subgroups are said of type (1) and (2), respectively.

Lemma 2.2. *Suppose n, $n \geqslant 3$, is a natural number. Then,*

1. *For any prime factor p of n, $\langle a \rangle$ and $\langle a^p, a^j b \rangle$ are all maximal subgroups of D_{2n}, where $1 \leqslant j \leqslant p$.*
2. *For any prime factor p of n, $\langle a^{\frac{n}{p}} \rangle$ and $\langle a^j b \rangle$ are all minimal subgroups of D_{2n}, where p is prime.*

Proof. The proof follows from the main theorem of [5]. □

Theorem 2.3. *Suppose $n = 2^r m \geqslant 3$, where r is a non-negative integer and m is a positive integer. Then,*

1. *$max(D_{2n}) = 1+\nu(n)$, $snmax(D_{2n}) = \nu(m)$ and $maxn(D_{2n}) = nmax(D_{2n}) = 3$ when n is even. In other case, $maxn(D_{2n}) = nmax(D_{2n}) = 1$.*
2. *$min(D_{2n}) = \pi(n) + n$, $nmin(D_{2n}) = minn(D_{2n}) = \pi(n)$ and $syl(D_{2n}) = \pi(m) + m$.*
3. *$R(D_{2n}) = \langle a^2 \rangle$, $L(D_{2n}) = \langle a^{p_1 p_2 \cdots p_s} \rangle$ and $\phi(D_{2n}) = R(D_{2n}) \bigcap L(D_{2n}) = \langle a^{2p_1 p_2 \cdots p_s} \rangle$.*

Proof. By Lemma 2.2, $Max(D_{2n}) = 1 + \nu(n)$ and $Min(D_{2n}) = \pi(n) + n$. It is easy to see that for odd n, all normal subgroups are given by the group itself and all subgroups of $\langle a \rangle$. If n is even then there are two more normal subgroups given by $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$. This proves that $NMin(D_{2n}) = MinN(D_{2n}) = \pi(n)$ and $Min(D_{2n}) = \pi(n) + n$. For each finite group G, $Max(G) = NMax(G) + SNMax(G)$. Hence $SNMax(D_{2n}) = \nu(m)$ and $MaxN(D_{2n}) = NMax(D_{2n}) = 3$ when n is even. In other case, $MaxN(D_{2n}) = NMax(D_{2n}) = 1$. The last part is a consequence of (1) and (2). □

Suppose $n = 2^r m$ where m is an odd positive integer with prime factorization $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. In Table 2, the general form of the subgroup lattice of D_{2n} is given by which we can find all maximal subgroups of D_{2n}.

Table 2. The subgroup structure of D_{2n}.

Subgroups	Max	Min	$NMax$	$SNMax$	$MinN$	Syl_p
Type 1	$\langle a \rangle$	$\langle a^{\frac{n}{p}} \rangle$	$\langle a \rangle$	$-$	$\langle a^{\frac{n}{p}} \rangle$	$\langle a^{\frac{n}{p_i^{\alpha_i}}} \rangle$
Type 2	$\langle a^{p_i}, a^j b \rangle$	$\langle a^j b \rangle$	$\langle a^2, a^j b \rangle$	$\langle a^{p_i}, a^j b \rangle$	$-$	$\langle a^m, a^j b \rangle$

Here $p_i \neq 2$ for all i.

j	1	2	$\cdots$	r	$r+1$	$r+2$
c_i	1	2	$\cdots$	2^{r-1}	2^r	2^{r+1}
1		$\langle a^{\frac{n}{2}} \rangle, \langle a^j b \rangle$				$Syl_2(D_{2n})$
p_i	$\langle a^{\frac{n}{p_i}} \rangle$					
$\vdots$						
$\prod_{1 \leqslant k \leqslant s} p_k^{\alpha_k - 1}$				$\phi(D_{2n})$	$L(D_{2n})$	
$\vdots$						
$p_i^{\alpha_i}$	Syl_{p_i}					
$\vdots$						
$\frac{m}{p_i}$						$\langle a^{p_i}, a^j b \rangle$
$\vdots$						
m				$R(D_{2n})$	$\langle a \rangle \langle a^2, a^j b \rangle$	

The summation of all primes p such that $p|n$ is denoted by $\nu(n)$, i.e. $\nu(n) = \sum_{p \in \pi(n)} p$.

2.2. Dicyclic Groups T_{4n}

We recall that the dicyclic group T_{4n} can be presented as

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

In this subsection the maximal, normal maximal, self-normalizing maximal, maximal normal, minimal normal and Sylow subgroups of dicyclic group T_{4n} are computed. Suppose $n = 2^r p_1^{a_1}.p_2^{a_2} \ldots p_s^{a_s}$. Since each element of T_{4n} has the form $a^i b^j$, $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant 3$, it can see that an arbitrary subgroup of the dicyclic group T_{4n} has one of the following forms (see [11, 2.2]):

1. A subgroup of $\langle a^i \rangle$, $i|2n$;
2. A subgroup of $\langle a^i, a^j b \rangle$, where $i|n$ and $1 \leqslant j \leqslant i$.

Lemma 2.4. *The following hold:*

1. *All maximal subgroups of form the dicyclic group T_{4n} have the form $\langle a\rangle$ or $\langle a^p, a^j b\rangle$, where $1 \leqslant j \leqslant p$ and $p \in \pi(n)$;*
2. *Every minimal subgroup of form the dicyclic group T_{4n} has the form $\langle a^{\frac{n}{p}}\rangle$, where $p \in \pi(2n)$.*

Proof. It is clear that $\langle a^i\rangle$, $i|n$, is maximal if and only if $i = 1$. We now consider the subgroups of the form $\langle a^i, a^j b\rangle$, where $i|n$ and $1 \leqslant j \leqslant i$. If p is a prime divisor of i then obviously $\langle a^i, a^j b\rangle \leqslant \langle a^p, a^j b\rangle$ and since $\langle a^i, a^j b\rangle$ is maximal, i has to be prime. The second statement is a direct consequence of the presentation of dicyclic groups. $\square$

Table 3. The subgroup structure of T_{4n}.

Type	Max	Min	$NMax$	$SNMax$	$MinN$	Syl_p
$\langle a^i\rangle$	$\langle a\rangle$	$\langle a^{\frac{2n}{p}}\rangle$	$\langle a\rangle$	-	$\langle a^{\frac{2n}{p_i}}\rangle$	$\langle a^{\frac{2n}{p_i^{\alpha_i}}}\rangle$
$\langle a^i, a^j b\rangle$	$\langle a^p, a^j b\rangle$	-	$\langle a^2, a^j b\rangle$	$\langle a^{p_i}, a^j b\rangle$	-	$\langle a^m, a^j b\rangle$

Here, $p_i \neq 2$ for all i.

j	1	2	$\cdots$	$r+1$	$r+2$	$r+3$
c_i	1	2	$\cdots$	2^r	2^{r+1}	2^{r+2}
1		$\boxed{\langle a^{\frac{2n}{2}}\rangle}$				$\boxed{Syl_2}$
p_i	$\boxed{\langle a^{\frac{2n}{p_i}}\rangle}$					
$\vdots$						
$\prod_{1\leqslant k\leqslant s} p_k^{\alpha_k-1}$				$\boxed{\phi(T_{4n})}$	$\boxed{L(T_{4n})}$	
$\vdots$						
$p_i^{\alpha_i}$	$\boxed{Syl_{p_i}}$					
$\vdots$						
$\frac{m}{p_i}$						$\boxed{\langle a^{p_i}, a^j b\rangle}$
$\vdots$						
m				$\boxed{R(T_{4n})}$	$\boxed{\langle a\rangle, \langle a^2, a^j b\rangle}$	

Theorem 2.5. *Suppose $n \geqslant 2$ is a positive integer. Then,*

1. $max(T_{4n}) = \nu(n) + 1$ *and* $snmax(T_{4n}) = \nu(n)$. *If n is even then* $maxn(T_{4n}) = nmax(T_{4n}) = 3$, *and* $maxn(T_{4n}) = nmax(T_{4n}) = 1$, *otherwise.*
2. $minn(T_{4n}) = nmin(T_{4n}) = min(T_{4n}) = \pi(2n)$ *and* $syl(T_{4n}) = \pi(n) + n$.
3. $R(T_{4n}) = \bigcap NMax(T_{4n}) = \langle a^2\rangle$ *and* $L(T_{4n}) = \bigcap SNMax(T_{4n}) = \langle a^{p_1p_2\cdots p_s}\rangle$, *where $n = 2^r p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$ is the prime factorization of n. Moreover,* $\phi(T_{4n}) = \langle a^{2p_1p_2\cdots p_s}\rangle$

Proof. By Lemma 2.4, the group T_{4n} has exactly one subgroup of index 2 and p_i subgroups of the form $\langle a^{p_i}, a^j b\rangle$, for each positive integer i with $1 \leqslant i \leqslant s$. Therefore, $max(T_{4n}) = 1 + \nu(n)$. Among all maximal subgroups of T_{4n}, the only normal subgroup is $\langle a\rangle$ and so $snmax(T_{4n}) = \nu(n)$. If n is odd then $\langle a\rangle$ is the unique normal maximal subgroup of T_{4n} which is also maximal normal in T_{4n}. Thus $maxn(T_{4n}) = nmax(T_{4n}) = 1$. If n is even then $\langle a\rangle$, $\langle a^2, ab\rangle$ and $\langle a^2, a^2 b\rangle$ are only maximal normal (normal maximal) subgroups of T_{4n} and so $maxn(T_{4n}) = nmax(T_{4n}) = 3$. This completes the proof of Part (1).

Since $\langle a\rangle$ is a characteristic subgroup of T_{4n}, all subgroups of $\langle a\rangle$ is normal in G. Hence the number of minimal, minimal normal and normal minimal subgroups of T_{4n} are equal to $pi(2n)$, as desired. If p is odd prime then all Sylow $p-$subgroups of G are contained in $\langle a\rangle$ and there are n Sylow $2-$subgroup in the form of $\langle a^n, a^j b\rangle$, where $1 \leqslant j \leqslant n$. Therefore, $syl(T_{4n}) = \pi(n) + n$, which completes the Part (2).

The proof of Part (3) follows from our discussion given in Part (1), □

2.3. Group U_{6n}

The aim of this subsection is to compute the same invariants as Subsection 2.1 for the group U_{6n} of order $6n$, where $n = 2^r 3^k p_1^{\alpha_1} \ldots p_s^{\alpha_s}$. The presentation of this group was given in Section 1. Note that this group has four types of subgroups as follows (see [11, 2.3]):

(a) A subgroup of $G_1 = \langle a^i\rangle$, where $i|2n$;

(b) A subgroup of $G_2 = \langle a^i, b\rangle$, where $i|2n$;

(c) A subgroup of $G_3 = \langle a^i b\rangle$, where $i|2n$ and $2 \cdot 3^k \nmid i$.

(d) A subgroup of $G_4 = \langle a^i b^2\rangle$, where $i|2n$ and $2 \cdot 3^k \nmid i$.

Note that $\langle a^i, b\rangle = \langle a^i, b^2\rangle$ and if $n = 2^r \cdot 3^k \cdot m$, $6 \nmid m$ and $2 \cdot 3^k | i$ then $(a^{2\cdot 3^k})^{2^r m} = a^{2^{r+1}\cdot 3^k \cdot m} b^{2^r m} = a^{2n} b^{2^r \cdot m} = b^{2^r \cdot m} = b$ or b^{-1}. Therefore, $\langle a^i b\rangle = \langle a^i, b\rangle = \langle a^i, b^2\rangle = \langle a^i b^2\rangle$. This proves that in this special case, the subgroups of types (b), (c) and (d) are the same. This is the reason that in cases (c) and (d) we force the condition that $2 \cdot 3^k \nmid i$.

Theorem 2.6. *The following hold:*

1. *All maximal subgroups of U_{6n} have one of the forms $\langle a^p, b\rangle$ or $\langle ab^j\rangle$, where p is a prime divisor of n and $1 \leqslant j \leqslant 3$. In particular, $max(U_{6n}) = \pi(2n) + 3$.*
2. *All minimal subgroups of U_{6n} have one of the forms $\langle a^n b^j\rangle$ or $\langle a^{\frac{2n}{p}}\rangle$, where p is a prime divisor of n and $1 \leqslant j \leqslant 3$. In particular,*

$$min(U_{6n}) = \begin{cases} \pi(2n) + 5 \text{ if } r = 0 \\ \pi(2n) + 3 \text{ if } r = k = 0 \text{ or } r, k > 0 \\ \pi(2n) + 1 \text{ if } k = 0 \end{cases}.$$

3. *$maxn(U_{6n}) = nmax(U_{6n}) = \pi(2n)$ and $snmax(U_{6n}) = 3$.*
4. *$nmin(U_{6n}) = minn(U_{6n}) = \pi(2n) + 1$.*

Proof. Since U_{6n} is metacyclic, it is supersolvable. Now by Huppert't theorem, all maximal subgroups of U_{6n} have index p, for a prime number p. Therefore, we have to count all subgroups of index a prime in the group U_{6n}. It is easy to see that there is a unique maximal subgroup of type (a) which has index 3. There are also two other maximal subgroups $\langle ab^j\rangle$, $j = 1, 2$ of index 3. Note that all subgroups of the form $\langle a^p, b\rangle$, p is prime, have index p and so they are maximal in U_{6n}. Finally, if p is a prime divisor of n then $\langle a^p, b\rangle$ is a maximal subgroup different from those maximal subgroups of index 3. Therefore, $max(U_{6n}) = \pi(2n) + 3$.

Suppose $n = 2^r3^kp_1^{\alpha_1}\ldots p_s^{\alpha_s}$. To count the number of minimal subgroup of U_{6n}, we consider three cases that ($r = 0$ and $k > 0$), ($k = 0$ and $r > 0$) or ($r = k = 0$ or $r, k > 0$).

1. ($r = 0$ and $k > 0$). In this case, U_{6n} has a unique subgroup $\langle a^{\frac{2n}{p_t}}\rangle$ of order p_t, $1 \leqslant t \leqslant s$. On the other hand, all subgroups $\langle a^n\rangle$, $\langle a^nb\rangle$ and $\langle a^nb^2\rangle$ are different subgroups of order 2 and there is no other subgroups of order two in U_{6n}. Finally, $\langle b\rangle$, $\langle a^{\frac{2n}{3}}b\rangle$ and $\langle a^{\frac{2n}{3}}b^2\rangle$ are only minimal subgroups of order 3. Hence $Min(U_{6n}) = \pi(2n) + 5$.
2. ($k = 0$ and $r > 0$). In this case we have a unique minimal subgroup of order p, p is a prime divisor of $6n$. Thus, $Min(U_{6n}) = \pi(2n) + 1$
3. ($r = k = 0$ or $r, k > 0$). If $r = k = 0$ then we have three minimal subgroups $\langle a^n\rangle$, $\langle a^nb\rangle$ and $\langle a^nb^2\rangle$ of order two and for each prime divisor p of $3n$, we have a unique minimal subgroup of order p. So, $Min(U_{6n}) = \pi(2n) + 3$.

Thus, our calculations in cases (1), (2) and (3) show that

$$Min(U_{6n}) = \begin{cases} \pi(2n) + 5 \ \textit{if}\ r = 0 \\ \pi(2n) + 3 \ \textit{if}\ r = k = 0 \text{ or } r, k > 0 \\ \pi(2n) + 1 \ \textit{if}\ k = 0 \end{cases},$$

which completes Part (2).

To prove (3), we note that all maximal subgroups of the form $\langle a^p, b\rangle$, p is a prime divisor of $2n$, are normal and three maximal subgroups $\langle a\rangle$, $\langle ab\rangle$ and $\langle ab^2\rangle$ are conjugate and so $maxn(U_{6n}) = nmax(U_{6n}) = \pi(2n)$. This also proves that $snmax(U_{6n}) = \pi(2n) + 3 - \pi(2n) = 3$.

Finally, by our discussion on the minimal subgroups of U_{6n}, we can see that $nmin(U_{6n}) = minn(U_{6n}) = \pi(2n) + 1$ which completes the proof. $\square$

Corollary 2.7. *The following hold:*

1. $syl(U_{6n}) = \pi(\frac{n}{2^r}) + 3$,
2. $R(U_{6n}) = \bigcap NMax(U_{6n}) = \langle a^{6p_1p_2\cdots p_s}, b\rangle$,
3. $L(U_{6n}) = \bigcap SNMax(U_{6n}) = \langle a^2\rangle = C_n$,
4. $\phi(U_{6n}) = R(U_{6n}) \bigcap L(U_{6n}) = \langle a^{23p_1p_2\cdots p_s}\rangle = C_{2^r3^{k-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_s^{\alpha_s-1}}$.

Proof. By our discussion in Theorem 2.6, one can easily see that $\langle a^{\frac{n}{2^r}}\rangle$, $\langle a^{\frac{n}{2^r}}b\rangle$ and $\langle a^{\frac{n}{2^r}}b^2\rangle$ are three Sylow 2−subgroups of U_{6n} and all Sylow p−subgroups of U_{6n}, p is an odd prime, are normal and so $syl(U_{6n}) = \pi(\frac{n}{2^r}) + 3$ which proves Part (1). Other parts are also straightforward consequences of Theorem 2.6. $\square$

All of given properties of the group U_{6n} are recorded in Table 4.

Table 4. The subgroup structure of U_{6n}.

Subgroups Types	Max	Min	$NMax$	$SNMax$	$MinN$	Syl_p
$\langle a^i\rangle$	$\langle a\rangle$	$\langle a^{\frac{2n}{p}}\rangle$	$-$	$\langle a\rangle$	$\langle a^{\frac{2n}{p}}\rangle$	$\langle a^{\frac{2n}{p_i^{\alpha_i}}}\rangle$
$\langle a^i b\rangle$	$\langle ab\rangle$	$\langle a^{\frac{2n}{3}} b\rangle$	$-$	$\langle ab\rangle$	$-$	$\langle a^{3^k m} b\rangle$
$\langle a^i b^2\rangle$	$\langle ab^2\rangle$	$\langle a^{\frac{2n}{3}} b^2\rangle$	$-$	$\langle ab^2\rangle$	$-$	$\langle a^{3^k m} b^2\rangle$
$\langle a^i, b\rangle$	$\langle a^p, b\rangle$	$\langle a^{2n}, b\rangle$	$\langle a^p, b\rangle$	$-$	$\langle a^{2n}, b\rangle$	$-$

j	1	2	$\cdots$	r	$r+1$	$r+2$
c_i	1	2	$\cdots$	2^{r-1}	2^r	2^{r+1}
1		$\boxed{\langle a^n\rangle}$				$\boxed{Syl_2(U_{6n})}$
3	$\boxed{\langle b\rangle, \langle a^{\frac{2n}{3}}\rangle, \langle a^{\frac{2n}{3}} b^{\pm}\rangle}$					
p_i	$\boxed{\langle a^{\frac{2n}{p_i}}\rangle}$					
$\vdots$						
$3^{k-1}\prod_{1\leqslant k\leqslant s} p_k^{\alpha_k-1}$					$\boxed{\phi(U_{6n})}$	
$\vdots$						
$3^k\prod_{1\leqslant k\leqslant s} p_k^{\alpha_k-1}$						$\boxed{R(U_{6n})}$
$\vdots$						
3^{k+1}	$\boxed{Syl_3}$					
$\vdots$						
$p_i^{\alpha_i}$	$\boxed{Syl_{p_i}}$					
$\vdots$						
$\frac{3^{k+1}m}{p_i}$						$\boxed{\langle a^{p_i}, b\rangle}$
$\vdots$						
$3^k m$					$\boxed{L(U_{6n})}$	$\boxed{\langle a\rangle, \langle ab^{\pm}\rangle, \langle a^3, b\rangle}$
$3^{k+1}m$					$\boxed{\langle a^2, b\rangle}$	

2.4. Group V_{8n}

We recall that the group V_{8n} can be represented as $\langle a,b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle$. Suppose $n = 2^r p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$. Then clearly $|V_{8n}| = 2^{r+3} p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$. Based on calculations given [11], there are nine different types for the subgroups of V_{8n}. In what follows, these types together with the number of each type are given in.

1. The cyclic subgroups $G_1(i) = \langle a^i\rangle$, where $i \mid 2n$. For each divisor i of $2n$, there exists exactly one subgroup of this type and so we obtain $\tau(2n)$ cyclic subgroups contained in $\langle a\rangle$.
2. The cyclic subgroups $G_2(i) = \langle a^i b^2\rangle$, $i|n$. All the subgroups in this case are different from those are given in part (1). On the other hand, for each divisor i of n we will

have a unique cyclic subgroup of this form. Thus, we find exactly $\tau(n)$ subgroups of the form G_2.

3. The subgroups $G_3(i) = \langle a^i, b^2\rangle$, where $i|2n$. A similar argument as Part (1) shows that there are $\tau(2n)$ subgroups in this form and all such subgroups are different from those are given in parts (1) and (2). Thus, the number of such subgroups are $2\tau(2n) + \tau(n)$.
4. The subgroups $G_4(i,j) = \langle a^i, a^jb\rangle$, $i|2n$, $1 \leqslant j \leqslant i$, i is even and j is odd. In this case, it is easy to see that $\langle a^i, a^jb\rangle = \langle a^u, a^vb\rangle$ if and only if $i = u$ and $j = v$. Since i is even, all divisors of $2n$ are $2n$, $\frac{2n}{2}$, $\ldots$, 2 and since i is odd there are $\sum_{d|n} d = \sigma(n)$ subgroups in this form. .
5. The subgroups $G_5(i,j) = \langle a^i, a^jb^3\rangle$, $i|2n$, $1 \leqslant j \leqslant i$, i is even and j is odd. In this case, it is easy to see that $\langle a^i, a^jb^3\rangle = \langle a^u, a^vb^3\rangle$ if and only if $i = u$ and $j = v$. Since i is even, all divisors of $2n$ are $2n$, $\frac{2n}{2}$, $\ldots$, 2 and since i is odd there are $\sum_{d|n} d = \sigma(n)$ subgroups in this form.
6. The subgroups $G_6(i,j) = \langle a^ib^2, a^jb\rangle$, $i|n, 1 \leqslant j \leqslant i$ and i is even. In this case, it is easy to see that $n + \frac{n}{2} + \cdots + 2 = 2[\frac{n}{2} + \frac{n}{4} + \cdots + 1] = 2\sigma(\frac{n}{2})$. So, there are $2\sigma(\frac{n}{2})$ subgroups in the form of G_6.
7. The subgroups $G_7(i,j) = \langle a^i, b^2, a^jb\rangle$, $i|2n$, $1 \leqslant j \leqslant i$ and i, j are even. The number of these subgroups are the same as the number of subgroups in part (4).
8. The subgroups $G_8(i,j) = \langle a^i, b^2, a^jb\rangle$, $i|2n$, $1 \leqslant j \leqslant i$, i is even and j is odd. Then there are the same number of subgroups as in the part (8), i.e. there are $\sigma(n)$ subgroups in the form of G_8.
9. The subgroups $G_9(i,j) = \langle a^i, b^2, a^jb\rangle$, $i|2n, 1 \leqslant j \leqslant i$ and i is odd. In this case, $2n = 2^{r+1}p_1^{\alpha_1}\cdots p_s^{\alpha_s}$ and so $\frac{n}{2^r}$ is odd. So, there are $\sum_{d|(\frac{n}{2^r})} d = \sigma(\frac{n}{2^r})$ subgroups in this form.

Theorem 2.8. *The following hold:*

1. *All maximal subgroup of* V_{8n} *have the forms* $\langle a, b^2\rangle$ *or* $\langle a^p, a^jb, b^2\rangle$, *where* p *is a prime divisor of* n *and* $1 \leqslant j \leqslant p$. *In particular,* $max(V_{8n}) = \nu(2n) + 1$.
2. *All minimal subgroup of* V_{8n} *have the forms* $\langle a^nb^2\rangle$, $\langle a^{2n}, b^2\rangle$, $\langle a^{\frac{2n}{p}}\rangle$, $p \in \pi(2n)$, *and* $\langle a^{2n}, a^jb^k\rangle$, *where* $1 \leqslant j \leqslant 2n$ *and* $2 \mid j, k = 1, 3$. *In particular,* $min(V_{8n}) = \pi(2n) + 2(n+1)$.
3. $maxn(V_{8n}) = nmax(V_{8n}) = 3$.
4. $nmin(V_{8n}) = minn(V_{8n}) = \pi(2n) + 2$.
5. $snmax(V_{8n}) = \nu(m)$, *where* $n = 2^rm$.

Proof. It is clear that among subgroups of type (3) there is a unique maximal subgroup of index two as $\langle a, b^2\rangle$. Among all subgroups of type (9), the cases that $i \in \pi(2n)$ lead to maximal subgroups of V_{8n}. Since such subgroups have the structure $\langle a^i, b^2, a^jb\rangle$, $1 \leqslant j \leqslant i$, $mix(V_{8n}) = \nu(2n) + 1$. Moreover, we don't have more maximal subgroups among the nine classes of subgroups of V_{8n}. This completes the proof of Part (1). To prove (2), it is enough to note that only the subgroups of the first and fourth types have prime order. Among the subgroups of the first types, $\langle a^{\frac{2n}{p}}\rangle$, $p \in \pi(2n)$, $\langle a^{2n}, b^2\rangle$ and

$\langle a^n b^2\rangle$ have prime order. On the other hand, the subgroups $\langle a^{2n}, a^j b^k\rangle$, where $1 \leqslant j \leqslant 2n$ and $2 \mid j, k = 1, 3$ have prime orders and these are all minimal subgroups of V_{8n}. Thus $min(V_{8n}) = \pi(2n) + 2(n + 1)$, as desired. Moreover, only the minimal subgroups $\langle a^{\frac{2n}{p}}\rangle$, $p \in \pi(2n)$, $\langle b^2\rangle$ and $\langle a^n b^2\rangle$ are normal and so $nmin(V_{8n}) = minn(V_{8n}) = \pi(2n) + 2$. Finally, $snmax(V_{8n}) = max(V_{8n}) - nmax(V_{8n}) = \nu(m) + 3 - 3 = \nu(m)$. □

The following corollary is an immediate consequence of definition and Theorem 2.8.

Corollary 2.9. *The following hold:*

1. $Syl(V_{8n}) = \begin{cases} \pi(m) + m & \text{if } n = 2^r m, m > 1, r \geqslant 0, \\ m & \text{if } n = 2^r, r \geqslant 0. \end{cases}$
2. $R(V_{8n}) = \bigcap NMax(V_{8n}) = \langle a^2, b^2\rangle$.
3. $L(V_{8n}) = \bigcap S - NMax(V_{8n}) = \langle a^{p_1 p_2 \cdots p_s}, b^2\rangle = \langle a^{p_1 p_2 \cdots p_s}, b^2\rangle$.
4. $\phi(V_{8n}) = R(V_{8n}) \bigcap L(V_{8n}) = \langle a^{2p_1 p_2 \cdots p_s}, b^2\rangle$.

Information regarding the group V_{8n} given in Theorem 2.8 are recorded in Table 5.

Table 5. The subgroup structure of V_{6n}.

Types	Max	Min	$NMax$	$S - NMax$	$MinN$	Syl_p
$\langle a^i\rangle$	–	$\langle a^{\frac{2n}{p}}\rangle$	–	–	$\langle a^{\frac{2n}{p}}\rangle$	$\langle a^{\frac{2n}{p_i^{\alpha_i}}}\rangle$
$\langle a^i b^2\rangle$	–	$\langle a^n b^2\rangle$	–	–	$\langle a^n b^2\rangle$	–
$\langle a^i, b^2\rangle$	$\langle a, b^2\rangle$	$\langle a^{2n}, b^2\rangle$	$\langle a, b^2\rangle$	–	$\langle a^{2n}, b^2\rangle$	–
$\langle a^i, a^j b\rangle$	–	$\langle a^{2n}, a^j b\rangle$	–	–	–	–
$\langle a^i, a^j b^3\rangle$	–	$\langle a^{2n}, a^j b^3\rangle$	–	–	–	–
$\langle a^i, b^2, a^j b\rangle$	$\langle a^2, b^2, b\rangle$	–	$\langle a^2, b^2, b\rangle$	–	–	–
$\langle a^i, b^2, a^j b\rangle$	$\langle a^2, b^2, ab\rangle$	–	$\langle a^2, b^2, ab\rangle$	–	–	–
$\langle a^i, b^2, a^j b\rangle$	$\langle a^{p^o}, b^2, a^j b\rangle$	–	–	$\langle a^{p^o}, b^2, a^j b\rangle$	–	$\langle a^m, b^2, a^j b\rangle$

j	1	2	$\cdots$	$r+2$	$r+3$	$r+4$
c_i	1	2	$\cdots$	2^{r+1}	2^{r+2}	2^{r+3}
1		$\boxed{\langle a^n\rangle, \langle a^j b^{\pm}\rangle, \langle b^2\rangle, \langle a^n b^2\rangle}$				$\boxed{Syl_2}$
p_i	$\boxed{\langle a^{\frac{2n}{p_i}}\rangle}$					
$\vdots$						
$\prod_{1\leqslant k\leqslant s} p_k^{\alpha_k - 1}$				$\boxed{\phi(V_{8n})}$	$\boxed{L(V_{8n})}$	
$\vdots$						
$p_i^{\alpha_i}$	$\boxed{Syl_{p_i}}$					
$\vdots$						
$\frac{m}{p_i}$						$\boxed{\langle a^{p_i}, b^2, a^j b\rangle}$
$\vdots$						
m				$\boxed{R(V_{8n})}$	$\boxed{\langle a, b^2\rangle, \langle a^2, b^2, a^j b\rangle}$	

2.5. Semi-dihedral group SD_{2^n} and $2-$group the H_n

In this subsection the subgroup structure of two $2-$groups SD_{2^n}, $n \geqslant 4$, and the group $H(n)$ are investigated. The semi-dihedral group SD_{2^n} has two types of subgroups and the group $H(n)$ has eleven types of subgroups. The types of subgroups for the semi-dihedral group are as follows:

1. The subgroups of $G_1(i) = \langle a^i \rangle$, $i|2^{n-1}$. In this case, for each i there exists exactly one subgroup of the given form and so we obtain $\tau(2^{n-1}) = n$ subgroups.
2. The subgroups $G_2(i,j) = \langle a^i, a^j b\rangle$, where $i|2^{n-1}$ and $1 \leqslant j \leqslant i$. In this case, there are $\sigma(n)$ subgroups of the form G_2.

Note that all maximal subgroups of a $p-$group is normal and have index p. On the other hand, all maximal subgroups of the group SD_{2^n} are known and have the from $\langle a\rangle$ and $\langle a^2, a^j b\rangle, 1 \leqslant j \leqslant 2$. In particular, $Max(SD_{2^n}) = 3$. Moreover, a $2-$group has the same number of maximal normal and normal maximal subgroups. Thus, $maxn(SD_{2^n}) = nmax(SD_{2^n}) = 3$.

Theorem 2.10. *The following hold:*

1. *All minimal subgroups of the group SD_{2^n} have the from $\langle a^{2^{n-2}}\rangle$ and $\langle a^{2^{n-1}}, a^j b\rangle$, $1 \leqslant j \leqslant 2^{n-1}$, where $2|j$. In particular, $min(SD_{2^n}) = 2^{n-2}+1$.*
2. *$nmin(SD_{2^n}) = minn(SD_{2^n}) = 1$.*
3. *$R(SD_{2^n}) = \bigcap NMax(SD_{2^n}) = \langle a^2\rangle$ and $\phi(SD_{2^n}) = R(SD_{2^n})$.*

Table 6. The structure of SD_{2^n}.

type subgroups	Max	Min	$NMax$	$MinN$	Syl_p
$\langle a^i\rangle$	$\langle a\rangle$	$\langle a^{2^{n-2}}\rangle$	$\langle a\rangle$	$\langle a^{2^{n-2}}\rangle$	
$\langle a^i, a^j b\rangle$	$\langle a^2, a^j b\rangle$	$\langle a^{2^{n-1}}, a^j b\rangle$	$\langle a^2, a^j b\rangle$	$-$	$\langle a, b\rangle$

j	1	2	$\cdots$	$r-1$	r	$r+1$
c_i	1	2	$\cdots$	2^{r-2}	2^{r-1}	2^r
1		$Min(SD_{2n})$		$R(SD_{2n})$	$Max(SD_{2n})$	Syl_2

We recall that the group $H(n)$ can be presented as

$$H(n) = \langle a, b, c | a^{2^{n-2}} = b^2 = c^2 = e | [a,b] = [b,c] = e, a^c = ab\rangle.$$

where $n \geqslant 4$. The types of subgroups of $H(n)$ are as follows:

1. Subgroups $G_1(i) = \langle a^i\rangle, i|2^{n-2}$.
2. Subgroups $G_2(i) = \langle a^i, b\rangle, i|2^{n-2}$.
3. Subgroups $G_3(i) = \langle a^i, c\rangle, i|2^{n-2}$.
4. Subgroups $G_4(i) = \langle a^i, bc\rangle, i|2^{n-2}$.
5. Subgroups $G_5(i) = \langle a^i b\rangle, i|2^{n-3}$.
6. Subgroups $G_6(i) = \langle a^i c\rangle, i|2^{n-3}$.

7. Subgroups $G_7(i) = \langle a^i bc\rangle, i|2^{n-3}$.
8. Subgroups $G_8(i) = \langle a^i b, a^i c\rangle, i|2^{n-3}$.
9. Subgroups $G_9(i) = \langle a^i b, a^i bc\rangle, i|2^{n-3}$.
10. Subgroups $G_{10}(i) = \langle a^i c, a^i bc\rangle, i|2^{n-3}$.
11. Subgroups $G_{11}(i) = \langle a^i, b, c\rangle, i|2^{n-2}$.

A similar calculations as other groups shows that the number of subgroups of a given order in $H(n)$ satisfies all information given in Table 7.

Theorem 2.11. *The following hold*

1. *All maximal subgroups of $H(n)$ have the from $\langle a, b\rangle$, $\langle a, c\rangle$ or $\langle a^2, b, c\rangle$. In particular, $Max(H_n) = 3$.*
2. *All minimal subgroup of $H(n)$ have the from $\langle b\rangle$, $\langle c\rangle$, $\langle bc\rangle$, $\langle a^{2^{n-3}} b\rangle$, $\langle a^{2^{n-3}} c\rangle$, $\langle a^{2^{n-3}} bc\rangle$ or $\langle a^{2^{n-3}}\rangle$. In particular, $Min(H_n) = 7$.*
3. *$MaxN(H_n) = NMax(H_n) = 3$.*
4. *$NMin(H_n) = MinN(H_n) = 3$.*
5. *$Syl_p(H_n) = 1$.*
6. *$R(H_n) = \bigcap NMax(H_n) = \langle a^2, bc\rangle$ and $\phi(H_n) = R(H_n)$.*

Table 7. The subgroup structure of H_n.

type subgroups	Max	Min	$NMax$	$MinN$	Syl_p
$\langle a^i\rangle$	$-$	$\langle a^{2^{n-3}}\rangle$	$-$	$\langle a^{2^{n-3}}\rangle$	$-$
$\langle a^i, b\rangle$	$\langle a, b\rangle$	$\langle a^{2^{n-2}}, b\rangle$	$\langle a, b\rangle$	$\langle a^{2^{n-2}}, b\rangle$	$-$
$\langle a^i, c\rangle$	$\langle a, c\rangle$	$\langle a^{2^{n-2}}, c\rangle$	$\langle a, c\rangle$	$-$	$-$
$\langle a^i, bc\rangle$	$-$	$\langle a^{2^{n-2}}, bc\rangle$	$-$	$-$	$-$
$\langle a^i b\rangle$	$-$	$\langle a^{2^{n-3}} b\rangle$	$-$	$\langle a^{2^{n-3}} b\rangle$	$-$
$\langle a^i c\rangle$	$-$	$\langle a^{2^{n-3}} c\rangle$	$-$	$-$	$-$
$\langle a^i bc\rangle$	$-$	$\langle a^{2^{n-3}} bc\rangle$	$-$	$-$	$-$
$\langle a^i, b, c\rangle$	$\langle a^2, b, c\rangle$	$-$	$\langle a^2, b, c\rangle$	$-$	$\langle a, b, c\rangle$

j	1	2	$\cdots$	$r-1$	r	$r+1$
c_i	1	2	$\cdots$	2^{r-2}	2^{r-1}	2^r
1		$Min(H_n)$		$R(H_n)$	$Max(H_n)$	$Syl_2(H_n)$

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