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On (semi)topological hoops

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Abstract. Hoops are naturally ordered commutative residuated integral monoids, introduced by Bosbach in [6, 7], that BL-algebras are particular cases of hoops. Now, in this paper, we introduce the concept of (semi)topological hoop and we get some related results. Then we derive here conditions that imply a hoop to be a semitopological or a topological hoop and we study some properties of them. Specially, we show that in a hoop A, if $(A, \rightarrow, \mathcal{T})$ is a semitopological hoop and $\{1\}$ is an open set or A is bounded and satisfies the double negation property, then $(A, \mathcal{T})$ is a topological hoop. Finally, we construct a discrete topology on quotient hoops, under suitable conditions.

1. Introduction

Algebra and topology, the two fundamental domains of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra studies all kinds of operations and provides a basis for algorithms and calculations. In applications, in higher level domains of mathematics, such as functional analysis, dynamical systems, representation theory, and others, topology and algebra come in contact most naturally. Many of the most important objects of mathematics represent a blend of algebraic and of topological structures. Topological function spaces and linear topological spaces in general, topological groups and topological fields, transformation groups, topological lattices are objects of this kind. Very often an algebraic structure and a topology come naturally together; this is the case when they are both determined by the nature of the elements of the set considered. The rules that describe the relationship between a topology and algebraic operation are almost always transparent and natural the operation has to be continuous, jointly continuous, jointly or separately. In the 20th century many topologists and algebraists have contributed to the topological algebra. Some outstanding mathematicians were involved, among them Dieudonné, Pontryagin, Weyl. Hoops are naturally ordered commutative residuated integral monoids, introduced by Bosbach in [6, 7]. In the last years, the hoops theory have enriched with deep structure theorems [1, 2, 3, 4, 5, 6, 7, 12]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops ([2], Corollary 2.10) one obtains an elegant short proof of the completeness theorem for

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the propositional basic logic ([2], Theorem 3.8), introduced by Hájek in [9]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. Now, in this paper, we introduce the concept of (semi)topological hoops and we bring some useful examples of them.

2. Preliminaries

In this section, we gather some basic notions relevant to hoop which will need in the next sections.

A *hoop* is an algebraic structure $(A, \odot, \to, 1)$ of type $(2, 2, 0)$ such that, for all $x, y, z \in A$:
$(HP1)$ $(A, \odot, 1)$ is a commutative monoid.
$(HP2)$ $x \to x = 1$.
$(HP3)$ $(x \odot y) \to z = x \to (y \to z)$.
$(HP4)$ $x \odot (x \to y) = y \odot (y \to x)$.

On a hoop A we define $x \leqslant y$ if and only if $x \to y = 1$. Then " $\leqslant$ " is a partial order on A. A hoop A is *bounded* if, for all $x \in A$, there is an element $0 \in A$ such that $0 \leqslant x$. Let A be a bounded hoop. For all $x \in A$, we define a negation " $'$ " on A by, $x' = x \to 0$. If $(x')' = x$, for all $x \in A$, then the bounded hoop A is said to have the *double negation property*, or (DNP) for short. Finally, we let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$ (cf. [2]).

Example 2.1. (cf. [8]) (i) Let $\mathbf{G} = (G, +, -, 0, \vee, \wedge)$ be an ℓ-group and $0 \leqslant u \in G$. Suppose that operations $\odot$ and $\to$ on $G[u] = [0, u]$ are defined as follows:

$$x \odot y = (x - u + y) \vee 0 \quad , \quad x \to y = (y - x + u) \wedge 0.$$

Then by routine calculations we can see that $\mathbf{G[u]} = (G[u], \odot, \to, u)$ is a hoop.
(ii) Let $A = \{0, a, b, c, d, 1\}$ and operations $\odot$ and $\to$ on A are defined as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Then with these operations A is a bounded hoop with (DNP).

The following proposition provides some properties of hoops.

Proposition 2.2. (cf. [6, 7]) *Let A be a hoop. Then, for all $x, y, z \in A$, the following conditions hold:*

(i) $(A, \leqslant)$ *is a meet-semilattice with* $x \wedge y = x \odot (x \to y)$.
(ii) $x \odot y \leqslant z$ *if and only if* $x \leqslant y \to z$.
(iii) $x \odot y \leqslant x, y$.
(iv) $x \leqslant y \to x$.
(v) $x \to 1 = 1$.
(vi) $1 \to x = x$.
(vii) $x \leqslant y \to (x \odot y)$.
(viii) $x \odot (x \to y) \leqslant y$.
(ix) $x \leqslant (x \to y) \to y$.
(x) $x \leqslant y$ *implies* $x \odot z \leqslant y \odot z$.
(xi) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$.
(xii) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$.
(xiii) $(x \to y) \leqslant (y \to z) \to (x \to z)$.

Proposition 2.3. (cf. [8]) *Let A be a bounded hoop. Then, for all $x, y \in A$, the following conditions hold:*

(i) $1' = 0$ *and* $0' = 1$.
(ii) $x \leqslant x''$.
(iii) $x \odot x' = 0$.
(iv) $x''' = x'$.
(v) $x' \leqslant x \to y$.
(vi) *If* $x = x''$, *then* $x \to y = y' \to x'$.
(vii) $x = x''$ *if and only if* $(x \to y) \to y = (y \to x) \to x$.

Proposition 2.4. (cf. [8]) *Let A be a hoop and for any $x, y \in A$, we define,*

$$x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x)$$

Then, for all $x, y, z \in A$, the following conditions are equivalent:

(i) $\sqcup$ *is associative operation on* A,
(ii) $x \leqslant y$ *implies* $x \sqcup z \leqslant y \sqcup z$,
(iii) $x \sqcup (y \wedge z) \leqslant (x \sqcup y) \wedge (x \sqcup z)$,
(iv) $\sqcup$ *is the join operation on* A.

A hoop A is called a $\sqcup$-*hoop*, if $\sqcup$ is a join operation on A.

Remark 2.5. (cf. [8]) $\sqcup$-hoop $(A, \sqcup, \wedge)$ is a distributive lattice.

Let A be a hoop. A non-empty subset F of A is called a *filter* of A if,
$(F1)$ $x, y \in F$ implies $x \odot y \in F$.
$(F2)$ $x \leqslant y$ and $x \in F$ imply $y \in F$, for any $x, y \in A$.

We use $\mathcal{F}(A)$ to denote the set of all filters of A. Clearly, $1 \in F$, for all $F \in \mathcal{F}(A)$. $F \in \mathcal{F}(A)$ is called a *proper filter* if $F \neq A$. It can be easily seen that, if A is a bounded hoop, then a filter is proper if and only if it does not contain 0 (cf. [8]).

Proposition 2.6. (cf. [8]) *Let A be a hoop and F be a non-empty subset of A. Then $F \in \mathcal{F}(A)$ if and only if $1 \in F$ and if, for any $x, y \in A$, $x \in F$ and $x \to y \in F$, then $y \in F$.*

Let A be a hoop and $F \in \mathcal{F}(A)$. We define a binary relation $\sim_F$ on A by $x \sim_F y$ if and only if $x \to y$, $y \to x \in F$, for any $x, y \in A$. Then $\sim_F$ is a congruence relation on A. Let $A/F = \{\overline{x} \mid x \in A\}$, where $\overline{x} = \{y \in A \mid x \sim_F y\}$. Then the binary relation $\leqslant$ on A/F defined by:

$$\overline{x} \leqslant \overline{y} \text{ if and only if } x \to y \in F,$$

is a partial order on A/F (cf. [9]). Thus $(A/F, \otimes, \rightsquigarrow, 1_{A/F})$ is a hoop, where for any $x, y \in A$:

$$1_{A/F} = \overline{1}, \quad \overline{x} \otimes \overline{y} = \overline{x \odot y}, \quad \overline{x} \rightsquigarrow \overline{y} = \overline{x \to y}.$$

In the follows, we recall some definitions of topological spaces.

A set X with a family $\mathcal{T}$ of its subsets is called a *topological space*, denoted by $(X, \mathcal{T})$, if $X, \emptyset \in \mathcal{T}$ and $\mathcal{T}$ is closed under a finite intersection and arbitrary union. The members of $\mathcal{T}$ are called *open sets* of X and the complement of $U \in \mathcal{T}$, that is U^c, is said to be a *closed set*. If B is a subset of X, the smallest closed set containing B is called the *closure* of B and denoted by $\overline{B}$. A subfamily $\{U_\alpha\}$ of $\mathcal{T}$ is said to be a *base* of U if for any $x \in U \in \mathcal{T}$, there exists an α such that $x \in U_\alpha \subseteq U$, or equivalently, each $U \in \mathcal{T}$ is the union of members of $\{U_\alpha\}$. A subset P of a topological space $(X, \mathcal{T})$ is said to be a *neighborhood* of $x \in X$ if there exists an open set U such that $x \in U \subseteq P$. A topological space X is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise, X is said to be *connected* (cf. [10, 11]).

Let $(A, *)$ be an algebra of type 2 and $\mathcal{T}$ be a topology on A. Then $\mathcal{A} = (A, *, \mathcal{T})$ is called:

• *left (right) topological algebra* if for each $a \in A$, the map $l_a \colon A \to A (r_a \colon A \to A)$ is defined by $x \to a * x (x \to x * a)$ is continuous, or equivalently, for any $x \in A$, and any open neighborhood U of $a * x (x * a)$, there exists an open neighborhood V of x such that $a * V \subseteq U (V * a \subseteq U)$. In this case we also call that the operation $*$ is continuous in the second (first) variable.

• *semitopological algebra* if $\mathcal{A}$ is a right and left topological algebra. In this case we also call that the operation $*$ is continuous in each variable separately.

• *topological algebra* if the operation $*$ is continuous, or equivalently, if for any $x, y \in A$ and any open neighborhood W of $x*y$, there exist two open neighborhoods U and V of x and y, respectively, such that $U * V \subseteq W$ (cf. [11]).

Proposition 2.7. (cf. [11]) *Let $(A, *)$ be a commutative algebra of type 2 and $\mathcal{T}$ be a topology on A. Then, right and left topological algebras are equivalent. Moreover, $(A, *, \mathcal{T})$ is a semitopological algebra if and only if it is right or left topological algebra.*

Let A be a non-empty set, $\{*_i\}_{i \in I}$ be a family of operations of type 2 on A and $\mathcal{T}$ be a topology on A. Then:

(*i*) $(A, \{*_i\}_{i\in I}, \mathcal{T})$ is a *right(left) topological algebra* if for any $i \in I$, $(A, *_i, \mathcal{T})$ is a right (left) topological algebra,

(*ii*) $(A, \{*_i\}_{i\in I}, \mathcal{T})$ is a *(semi)topological algebra* if for all $i \in I$, $(A, *_i, \mathcal{T})$ is a (semi)topological algebra (cf. [11]).

Note: From now one, A is a hoop and $\mathcal{T}$ is a topology on A.

3. (Semi)topological hoop

In this section we define the notions of (semi)topological hoop and state and prove some related results.

Definition 3.1. Let $(A, \{*_i\}, \mathcal{T})$, where $\{*_i\} \subseteq \{\odot, \rightarrow\}$, be a (semi)topological algebra. Then $(A, \{*_i\}, \mathcal{T})$ is called a *(semi)topological hoop*. Moreover, we say $(A, \mathcal{T})$ is a *(semi)topological* hoop if $(A, \odot, \rightarrow, \mathcal{T})$ is a (semi)topological hoop.

Note: Let $U, V \subseteq A$. Then we define $U \odot V$, $U \rightarrow V$ and $U \times V$ as follows:

$$U \odot V = \{x \odot y \mid x \in U, y \in V\}, \qquad U \rightarrow V = \{x \rightarrow y \mid x \in U, y \in V\}.$$

Example 3.2. (*i*) Every hoop with the discrete topology is a topological hoop.

(*ii*) Let $A = \{0, a, b, 1\}$ be a set. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then A with these operations and the topology $\mathcal{T} = \{\emptyset, \{0\}, \{1, a, b\}, A\}$ is a bounded topological hoop.

Note: We know that, any topological hoop is always a semitopological hoop. In the following example we show that every semitopological hoop is not a topological hoop, in general.

Example 3.3. Let $A = \{0, a, b, 1\}$ be a set. Define the operations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then A with these operations and the topology $\mathcal{T} = \{\emptyset, \{1, b\}, \{1, a, b\}, A\}$ is a semitopological hoop, but it is not a topological hoop. Because $0 \rightarrow 0 = 1 \in \{1, b\}$ and $A \rightarrow A = A$ and it is clear that $A \not\subseteq \{1, b\}$.

Example 3.4. Let $\odot$ and $\to$ on the real unit interval $A = [0,1]$ be defined as follows:

$$x \odot y = \min\{x, y\} \quad \text{and} \quad x \to y = \begin{cases} 1 & x \leqslant y \\ y & otherwise \end{cases}$$

Then A with these operations is a bounded hoop.

Now, let $\mathcal{T}$ be a topology on A with the base $B = \{(a,b] \cap A \mid a, b \in \mathbb{R}\}$. Then $V = (a, 0] \cap A = \{0\}$ for $a < 0$, and so $\{0\}$ is an open neighborhood of 0.

We prove $(A, \odot, \mathcal{T})$ is a topological hoop. For this, let $x, y \in A$ and $U \in \mathcal{T}$ such that $x \odot y \in U$.

Case 1: Let $x = y = 0$. Then $\{0\}$ is an open neighborhood of 0 and $x \odot y \in \{0\} \odot \{0\} \subseteq U$.

Case 2: Let $x = 0$ and $0 \neq y$. Then $x \odot y = 0 \in U$. Since $\{0\}$ is an open neighborhood of 0 and $y \in (0, y]$, we have $x \odot y \in \{0\} \odot (0, y] = \{0\} \subseteq U$.

Case 3: Let $0 \neq x = y$. Then $x \odot x = x \in U$. Hence $(0, x] \cap U$ is an open neighborhood of x such that $x \odot x \in ((0, x] \cap U) \odot ((0, x] \cap U) \subseteq U$.

Case 4: Let $x < y$. Then $x \odot y = x \in U$. Since $x \in (0, x] \cap U \in \mathcal{T}$ and $y \in (x, y] \in \mathcal{T}$, we obtain $x \odot y \in ((0, x] \cap U) \odot (x, y] = (0, x] \cap U \subseteq U$.

Case 5: Let $x > y$. Then $x \odot y = y \in U$. Since $x \in (y, x]$ and $y \in (0, y] \cap U$, $x \odot y \in (y, x] \odot ((0, y] \cap U) = (0, y] \cap U \subseteq U$.

Hence, $(A, \odot, \mathcal{T})$ is a topological hoop. Now, we prove that $(A, \to, \mathcal{T})$ is not a topological hoop. For this, we consider $1/2 \to 1/2 = 1 \in (1/2, 1]$. Let $a \in \mathbb{R}$ and $(a, 1/2]$ be a neighborhood of $1/2$. Suppose $b = (a + 1/2)/2$. Then $b \in (a, 1/2]$, and so $b < 1/2$. Hence, $1/2 \to b = b \notin (1/2, 1]$.

Proposition 3.5. *Let $x^2 = x$, for all $x \in A$. Then there exists a topology $\mathcal{T}$ on A such that $\odot$ is continuous.*

Proof. Let $a \in A$. Define $A_a = \{x \in A \mid x \odot a = a\}$. Clearly, $a \in A_a$. We prove that $A_a \in \mathcal{F}(A)$. For this, let $x, y \in A_a$. Then $x \odot a = y \odot a = a$. By (HP1),

$$(x \odot y) \odot a = x \odot (y \odot a) = x \odot a = a.$$

Hence, $x \odot y \in A_a$. Also, suppose $x \leqslant y$ and $x \in A_a$, for some $x, y \in A$. Then by Proposition 2.2(iii) and (x), we have $a = x \odot a \leqslant y \odot a \leqslant a$. Thus, $y \odot a = a$. Hence, $y \in A_a$, and so $A_a \in \mathcal{F}(A)$, for all $a \in A$. Let $B = \{A_a \mid a \in A\}$. Suppose $a \in A_x \cap A_y$ and z be an arbitrary element of A_a. Then

$$z \odot x = z \odot (a \odot x) = (z \odot a) \odot x = a \odot x = x$$

and

$$z \odot y = z \odot (a \odot y) = (z \odot a) \odot y = a \odot y = y.$$

Thus, $z \in A_x \cap A_y$, and so B is a basis. Let $\mathcal{T}$ be a topology generated by B. We prove that $\odot$ is continuous. Let $x, y \in A$. Then $x \odot y \in A_{x \odot y}$. Since $x \in A_x$ and $y \in A_y$, it is enough to prove that $A_x \odot A_y \subseteq A_{x \odot y}$. Let $\alpha \in A_x \odot A_y$. Then there exist $a \in A_x$ and $b \in A_y$ such that $\alpha = a \odot b$. Since $a \in A_x$ and $b \in A_y$, $a \odot x = x$ and $b \odot y = y$, respectively. Thus, by (HP1),

$$\alpha \odot (x \odot y) = (a \odot b) \odot (x \odot y) = (a \odot x) \odot (b \odot y) = x \odot y.$$

Hence, $\alpha \in A_{x \odot y}$. Therefore, $\odot$ is continuous. $\square$

Proposition 3.6. *Let A be bounded with (DNP). Then $(A, \to, \mathcal{T})$ is a semitopological hoop if and only if $(A, \odot, ', \mathcal{T})$ is a semitopological hoop.*

Proof. $(\Rightarrow)$ Let $(A, \to, \mathcal{T})$ be a semitopological hoop. It is clear that $'$ is continuous. Now, we prove that $\odot$ is continuous in the second variable. Let $x \odot y \in U \in \mathcal{T}$. Since A has (DNP), by (HP3)

$$x \odot y = (x \odot y)'' = ((x \odot y) \to 0) \to 0 = (x \to (y \to 0)) \to 0 = (x \to y')',$$

hence $(x \to y')' \in U$. Since $'$ is continuous, there exists $V \in \mathcal{T}$, such that $x \to y' \in V$ and $V' \subseteq U$. Also, since $\to$ is continuous in the second variable, there exists $W \in \mathcal{T}$, such that $y' \in W$ and $x \to y' \in x \to W \subseteq V$. Again, since $'$ is continuous, there is $Q \in \mathcal{T}$ such that $y \in Q$ and $y' \in Q' \subseteq W$. Now, $Q \in \mathcal{T}$ is an open neighborhood of $y \in Q$ and $x \odot y \in x \odot Q \subseteq U$, because if $z \in Q$, then

$$x \odot z = (x \to z')' \in (x \to Q')' \subseteq (x \to W)' \subseteq V' \subseteq U.$$

Since the operator $\odot$ is commutative, $\odot$ is continuous in each variable. Hence, $(A, \odot, ', \mathcal{T})$ is a semitopological hoop.

$(\Leftarrow)$ Let $(A, \odot, ', \mathcal{T})$ be a semitopological hoop. We prove that $(A, \to, \mathcal{T})$ is a semitopological hoop. For this, we prove that $\to$ is continuous in two variables. At first, we show that $\to$ is continuous in the second variable. Let $x \to y \in U \in \mathcal{T}$. Since A has (DNP), by (HP3),

$$(x \odot y')' = x \to y'' = x \to y \in U$$

Since $'$ is continuous, there exists an open neighborhood V of $x \odot y'$ such that $V' \subseteq U$. Also, since $\odot$ is continuous in the second variable, there exists an open neighborhood W of y' such that $x \odot y' \in x \odot W \subseteq V$. Again, since $'$ is continuous, there is $Q \in \mathcal{T}$, such that $y \in Q$ and $Q' \subseteq W$. Now, Q is an open neighborhood of y such that $x \to y \in x \to Q \subseteq U$, because if $z \in Q$, then

$$x \to z = (x \odot z')' \in (x \odot Q')' \subseteq (x \odot W)' \subseteq V' \subseteq U.$$

Now, we prove that $\to$ is continuous in the first variable. For this, let $x \to y \in U \in \mathcal{T}$. Then $x \to y = (x \odot y')' \in U$. Since $'$ is continuous, there is $V \in \mathcal{T}$ such that $x \odot y' \in V$ and $V' \subseteq U$. Since $\odot$ is continuous in the first variable, there exists $Q \in \mathcal{T}$, $x \in Q$ and $x \odot y' \in Q \odot y' \subseteq V$. Thus, Q is an open neighborhood of x such that

$$x \to y = (x \odot y')' \in (Q \odot y')' \subseteq V' \subseteq U$$

Hence, $\to$ is continuous in the first variable. $\square$

Theorem 3.7. *Let A be bounded with (DNP). If $(A, \rightarrow, \mathcal{T})$ is a topological hoop, then $(A, \mathcal{T})$ is a topological hoop.*

Proof. Let $\rightarrow$ be continuous. Then the maps $'$ and $f : A\times A \hookrightarrow A\times A$ by $f(x, y) = (x, y')$, both, are continuous. Since for each $x, y \in A$, $x \odot y = (x \rightarrow y')'$, we get that $\odot$ is the composite of continuous maps f, $\rightarrow$ and $'$. Hence $\odot$ is continuous. □

For an arbitrary element $a \in A$ we define the subset

$$V(a) = \{x \in A \mid x \rightarrow a, a \rightarrow x \in V\}.$$

Theorem 3.8. *There is a nontrivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological hoop.*

Proof. Let

$$\mathcal{T} = \{U \subseteq A \mid \text{for every } a \in U, \text{there exists } F \in \mathcal{F}(A) \text{ such that } F(a) \subseteq U\}.$$

Suppose $\{U_i : i \in I\}$ is a collection of members of $\mathcal{T}$. For any $x \in \bigcup U_i$, there are $F \in \mathcal{F}(A)$ and $j \in I$ such that $F(x) \subseteq U_j \subseteq \bigcup U_i$. Hence $\bigcup U_i \in \mathcal{T}$. On the other hand, for any $x \in \bigcap U_i$ and any $i \in I$, there are $F_i \in \mathcal{F}(A)$ such that $x \in F_i(x) \subseteq U_i$. Let $F = \bigcap F_i$. Then $x \in F(x) \subseteq \bigcap U_i$. Hence $\bigcap U_i \in \mathcal{F}(A)$. Thus, $\mathcal{T}$ is a topology on A. Let $F \in \mathcal{F}(A)$, $x \in A$ and $y \in F(x)$. If $z \in F(y)$, then $z \rightarrow y$ and $y \rightarrow z$, both, are in F. Since $y \rightarrow x$ and $x \rightarrow y$, both, are in F, we get that $z \rightarrow x \in F$ and $x \rightarrow z \in F$. Hence $F(y) \subseteq F(x)$ and so $F(x)$ is in $\mathcal{T}$. Therefore, $\mathcal{T}$ is nontrivial topology. Let $* \in \{\odot, \rightarrow\}$, $F \in \mathcal{F}(A)$ and $x, y \in A$. Since $F(x) = \overline{x}$ and $F(y) = \overline{y}$, $F(x * y) = F(x) * F(y)$. This proves that $*$ is continuous. □

Corollary 3.9. *Let $\mathcal{T}$ be as in Theorem 3.8 and $X \subseteq A$. Then:*
- (i) *for each $F \in \mathcal{F}(A)$, $F(X)$ is an open and closed subset of A. Moreover, each filter is an open and closed set,*
- (ii) $\overline{X} = \bigcap\{F(X) \mid F \in \mathcal{F}(A)\}$.

Proof. (i). Let $F \in \mathcal{F}(A)$, and $y \in \overline{F(X)}$. Then, $F(y) \cap F(X) \neq \emptyset$. Hence there is $x \in X$, such that $F(y) = F(x)$ and so $y \in F(x) \subseteq F(X)$. Therefore, $F(X)$ is closed. But $F(X)$ is open because it is a union of open sets.

(ii). Let $X \subseteq A$ and $x \in \overline{X}$. Since for all $F \in \mathcal{F}(A)$, $x \rightarrow x = 1 \in F$, we have $x \in F(x)$, and so $x \in \bigcap\{F(X) \mid F \in \mathcal{F}(A)\}$.

Conversely, let $x \in \bigcap\{F(X) \mid F \in \mathcal{F}(A)\}$. Then, for all $F \in \mathcal{F}(A)$, $x \in F(X)$. Since $F(X) = \bigcup_{a\in X} F(a)$, there exists $b \in X$ such that $x \in F(b)$. Moreover, since $x \rightarrow b \in F$ and $b \rightarrow x \in F$, we have $b \in F(x) \cap X$. Hence, $x \in \overline{X}$. □

Theorem 3.10. *Let Ω be a family of nonempty subsets of A such that Ω is closed under intersection and for each $x, y \in A$ and $V \in \Omega$,*
- (i) *if $x \in V$ and $x \leqslant y$, then $y \in V$,*
- (ii) *if $x \in V$, then there exists $U \in \Omega$ such that $U(x) \subseteq V$,*
- (iii) *there exists $W \in \Omega$ such that $W(x) \subseteq V$, for any $x \in W$ or equivalently,*

$W(W) \subseteq V$.

Then there is a nontrivial topology $\mathcal{T}$ on A such that $(A, \mathcal{T})$ is a topological hoop.

Proof. It is easy to prove that $\mathcal{F}(A) \subseteq \Omega$. Let

$$\mathcal{T} = \{O \subseteq A \mid \text{for every } a \in O, \text{there exists } V \in \Omega \text{ such that } V(a) \subseteq O\}.$$

Firstly, we prove that $\mathcal{T}$ is closed under union and intersection. For this let $\{O_i : i \in I\} \subseteq \mathcal{T}$. Then, for every $a \in \bigcup O_i$, there exist $i \in I$ and $V \in \Omega$ such that $a \in V(a) \subseteq O_i \subseteq \bigcup O_i$. Hence $\mathcal{T}$ is closed under union. For any $a \in \bigcap O_i$ and any $i \in I$, there exists $V_i \in \Omega$ such that $a \in V_i(a) \subseteq O_i$. Put $V = \bigcap V_i$, then $V(a) \subseteq \bigcap V_i(a) \subseteq \bigcap O_i$ and so $\mathcal{T}$ is closed under intersection. Hence, $\mathcal{T}$ is a topology on A. Now, we prove that for each $V \in \Omega$ and $a \in A$, $V(a)$ is an open set. Let $a \in A$, $V \in \Omega$ and $x \in V(a)$. Then, $x \to a, a \to x \in V$. By (ii), there exist U_1 and $U_2 \in \Omega$ such that $U_1(a \to x) \subseteq V$ and $U_2(x \to a) \subseteq V$. Put $W = U_1 \cap U_2 \in \Omega$. If $y \in W(x)$, then $x \to y$ and $y \to x \in W$. By Proposition 2.2$(xiii)$,

$$x \to y \leqslant (y \to a) \to (x \to a), \qquad y \to x \leqslant (x \to a) \to (y \to a).$$

By (i),

$$(y \to a) \to (x \to a) \in W, \qquad (x \to a) \to (y \to a) \in W$$

Thus,

$$y \to a \in W(x \to a) \subseteq (U_1 \cap U_2)(x \to a) \subseteq U_2(x \to a) \subseteq V.$$

By the similar way, we can see that $a \to y \in V$. Then obviously, $W(x) \subseteq V(a)$. Hence, $V(a)$ is an open set and $\mathcal{T}$ is a nontrivial topology. Clearly, the set $B = \{V(a) : V \in \Omega, a \in A\}$ is a base for $\mathcal{T}$.

Now we prove that $(A, \mathcal{T})$ is a topological hoop. At first, we show that $\odot$ is continuous. Let $x \odot y \in O \in \mathcal{T}$. Consider $V \in \Omega$ such that $V(x \odot y) \subseteq O$. By (i), $1 \in V$, so $x \odot y \in V(x \odot y)$. By (iii), there is $W \in \Omega$ such that $W(W) \subseteq V$. Let $u \in W(x)$ and $v \in W(y)$. Then $u \to x, x \to u, v \to y$ and $y \to v$, all, belong to W. By Proposition 2.2(iv), $(x \to u) \leqslant [(x \odot y) \to (u \odot v)] \to (x \to u)$ and by (i), $[(x \odot y) \to (u \odot v)] \to (x \to u) \in W$. On the other hand, we have

$$\begin{aligned}(x \to u) \to ((x \odot y) \to (u \odot v)) &= (x \to u) \to [x \to (y \to (u \odot v))], \text{ by (HP3)} \\ &= [x \odot (x \to u)] \to [y \to (u \odot v)], \text{ by Prop. 2.2} \\ &\geqslant u \to [y \to (u \odot v)], \text{ by Prop. 2.2} \\ &\geqslant y \to v.\end{aligned}$$

Since $W \in \Omega$ and $y \to v \in W$, by (i), $(x \to u) \to ((x \odot y) \to (u \odot v)) \in W$. Thus, $(x \odot y) \to (u \odot v) \in W(x \to u) \subseteq W(W) \subseteq V$. Hence, $(x \odot y) \to (u \odot v) \in V$. By the similar way, we have $(u \odot v) \to (x \odot y) \in V$. Therefore, $W(x) \odot W(y) \subseteq V(x \odot y)$. This proves that $\odot$ is continuous.

Now, we prove that $\to$ is continuous. Let $x \to y \in V(x \to y)$. By (iii), there is $W \in \Omega$ such that $W(W) \subseteq V$. Let $u \in W(x)$ and $v \in W(y)$. Then $u \to x, x \to u, v \to y$ and $y \to v \in W$. By (HP3) we have,

$$\begin{aligned}(v \to y) \to ((u \to v) \to (x \to y)) &= [(u \to v) \odot (v \to y)] \to (x \to y), \text{ by Prop. 2.2}\\ &\geq (u \to y) \to (x \to y), \text{ by (HP3)}\\ &= x \to ((u \to y) \to y), \text{ by Prop. 2.2}\\ &\geq x \to u\end{aligned}$$

Since $W \in \Omega$ and $x \to u \in W$, by (i), $(v \to y) \to ((u \to v) \to (x \to y)) \in W$. Also, by Proposition 2.2(iv), $v \to y \leqslant ((u \to v) \to (x \to y)) \to (v \to y)$. Again, since $W \in \Omega$ and $v \to y \in W$, by (i), $((u \to v) \to (x \to y)) \to (v \to y) \in W$. Thus, $(u \to v) \to (x \to y) \in W(v \to y) \subseteq W(W) \subseteq V$. This implies that $(u \to v) \to (x \to y) \in V$. By the similar way, we have $(x \to y) \to (u \to v) \in V$. Therefore, $W(x) \to W(y) \subseteq V(x \to y)$ which implies that $\to$ is continuous. □

Corollary 3.11. *Let $\mathcal{T}$ be the topology in Theorem 3.10 and $X \subseteq A$. Then:*
 (i) *for each $V \in \Omega$, $V(X)$ is an open and closed subset of A,*
 (ii) $\overline{X} = \bigcap\{F(X) \mid F \in \mathcal{F}(A)\}$.

Proof. (i). Let $V \in \Omega$ and $y \in \overline{V(X)}$. Then there exists a net $\{y_i : i \in I\}$ which convergence to y. Since $\to$ is continuos, the nets $\{y_i \to y\}$ and $\{y \to y_i\}$, both, convergence to 1. Since $1 \in V$, $y_i \to y$ and $y \to y_i$, both, are in V, for some $i \in I$. Hence $y \in V(y_i) \subseteq V(X)$. Therefore, $V(X)$ is closed. But it is open because it is the union of open sets.
 (ii). The proof is similar to the proof of Corollary 3.9(ii). □

Proposition 3.12. *If $(A, \mathcal{T})$ is a topological hoop, then $(A, \wedge, \mathcal{T})$ is a topological hoop.*

Proof. Let $f : A \times A \to A \times A$ by $f(x, y) = (x, x \to y)$, for all $x, y \in A$. Since $(A, \to, \mathcal{T})$ is a topological hoop, f is continuous. Also, by Proposition 2.2(i),

$$\wedge(x, y) = x \wedge y = x \odot (x \to y) = (\odot \circ f)(x, y).$$

Since $\odot$ and f are continuous, $\wedge$ is continuous. Therefore, $(A, \wedge, \mathcal{T})$ is a topological hoop. □

Proposition 3.13. *Let A be a $\sqcup$-hoop and $\mathcal{T}$ be a topology on A. Then:*
 (i) *if $(A, \wedge, \to, \mathcal{T})$ is a topological hoop, then $(A, \sqcup, \mathcal{T})$ is a topological hoop,*
 (ii) *if A has (DNP) and $(A, \to, \mathcal{T})$ is a topological hoop, then $(A, \sqcup, \mathcal{T})$ is a topological hoop.*

Proof. (i). Let $f : A \times A \to A$ is defined by $f(x, y) = (x \to y) \to y$ and $g : A \times A \to A$ by $g(x, y) = (y \to x) \to x$, for all $x, y \in A$. Since $(A, \to, \mathcal{T})$ is a topological hoop, f and g are continuous. Also, define $f \wedge g : A \times A \to A$ by

$(f \wedge g)(x,y) = f(x,y) \wedge g(x,y)$, for all $x, y \in A$. Since $(A, \to, \mathcal{T})$ is a topological hoop, by Proposition 3.12, $\wedge$ is continuous. Then $f \wedge g$ is continuous. Moreover,

$$\sqcup(x,y) = x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x) = f(x,y) \wedge g(x,y) = (f \wedge g)(x,y).$$

Hence, $\sqcup = f \wedge g$ is continuous.

(ii). Let $x, y \in A$, $U \in \mathcal{T}$ and $x \sqcup y \in U$. Since A has (DNP), by Proposition 2.3(vii), $x \sqcup y = (x \to y) \to y$. Moreover, since $(A, \to, \mathcal{T})$ is a topological hoop, $\sqcup$ is continuous. $\square$

Theorem 3.14. *Let $\mathcal{T}$ be a topology on A and $h : A^3 \to A^2$ is defined by $h(a,b,c) = (a \to b, b \to c)$, for all $a, b, c \in A$. If $\{1\}$ is an open set and h is continuous, then $(A, \mathcal{T})$ is a topological hoop.*

Proof. Let $a \in A$ and $h_a(b) = (a \to b, b \to a)$. Since h is continuous, h_a is continuous. Now, since $\{1\}$ is open, $\{1\} \times \{1\}$ is open in A^2. On the other hand,

$$\begin{aligned} h_a^{-1}(1,1) &= \{b \in A \mid h_a(b) = (1,1)\} = \{b \in A \mid (a \to b, b \to a) = (1,1)\} \\ &= \{b \in A \mid a \to b = 1, b \to a = 1\} = \{b \in A \mid b = a\} = \{a\}. \end{aligned}$$

Hence, $\{a\}$ is an open set and $\mathcal{T}$ is a discrete topology. Therefore, $(A, \mathcal{T})$ is a topological hoop. $\square$

Theorem 3.15. *Let $(A, \to, \mathcal{T})$ be a semitopological hoop. If $\{1\}$ is an open set, then $(A, \mathcal{T})$ is a topological hoop.*

Proof. Let $\{1\}$ be an open set and $x \in A$. Since $(A, \mathcal{T})$ is a semitopological hoop and $x \to x = 1 \in \{1\}$, there is an open sets U such that $x \in U$, $x \to U = 1$ and $U \to x = \{1\}$, which implies that $U = \{x\}$. Hence $\mathcal{T}$ is a discrete topology on A and so $(A, \mathcal{T})$ is a topological hoop. $\square$

Proposition 3.16. *Let $(A, \to, \mathcal{T})$ be a topological hoop and $F \in \mathcal{F}(A)$. Then:*

(i) *if 1 is an interior point of F, then F is an open set,*

(ii) *if F is an open set, then F is closed,*

(iii) *if A is connected, then A has no open proper filter.*

Proof. Let $(A, \to, \mathcal{T})$ be a topological hoop and $F \in \mathcal{F}(A)$.

(i). Suppose $x \in F$. Since 1 is an interior point of F, there exists $U \in \mathcal{T}$ such that $x \to x = 1 \in U \subseteq F$. Since $\to$ is continuous, there exists $V \in \mathcal{T}$ such that $x \in V$ and $V \to V \subseteq F$. Now, for all $y \in V$, we have $x \to y \in V \to V \subseteq F$, and so $x \to y \in F$. Since $F \in \mathcal{F}(A)$ and $x \in F$, by Proposition 2.6, $y \in F$. Thus, $y \in V \subseteq F$ which implies that F is an open set.

(ii). Let F be an open set. We prove that F is closed. For this, we show that F^c is an open set. Let $x \in F^c$. Then $x \notin F$. Since $x \to x = 1 \in F \in \mathcal{T}$ and $\to$ is continuous, there exists $U \in \mathcal{T}$ such that $x \in U$ and $U \to U \subseteq F$. Now, we prove that $U \subseteq F^c$. For this, let $U \cap F \neq \emptyset$. Then there is $y \in U \cap F$ such that

$y \to U \subseteq F$. So, for all $z \in U$, $y \to z \in F$. Since $F \in \mathcal{F}(A)$, by Proposition 2.6, $z \in F$, and so $U \subseteq F$. Thus, $x \in F$, which is a contradiction. Then $U \cap F = \emptyset$. Hence, $x \in U \subseteq F^c$ shows that F^c is an open set and so F is closed.

(iii). Suppose F is an open filter of A. Then by (ii), F is closed. Since A is connected, we have $A = F$. $\square$

A topological space A is called *totally disconnected*, if every connected subset $X \subseteq A$ is either empty or a singleton. A subset X of A is called a *component subspace*, if it is the maximal connected subspace (cf. [11]).

Proposition 3.17. *Let $(A, \to, \mathcal{T})$ be a semitopological hoop. Then A is totally disconnected if and only if every its connected subset containing* 1 *consists just* 1.

Proof. $(\Rightarrow)$ Suppose A is totally disconnected and $X \subseteq A$ is a connected of 1. Then it is clear that $X = \{1\}$.

$(\Leftarrow)$ Let D be a connected subset of A and $x \in D$. Then by (HP2), $1 \in (D \to x) \cap (x \to D)$. Since $(A, \to, \mathcal{T})$ is a semitopological hoop and D is connected, it is clear that $x \to D$ and $D \to x$ are connected. By assumption, $D \to x = \{1\}$ and $x \to D = \{1\}$ and so $D = \{x\}$. Therefore, A is totally disconnected. $\square$

Proposition 3.18. *Let $(A, \mathcal{T})$ be a topological hoop and $C \subseteq A$ be a component of 1 which contains all connected subset of A. Then C is a filter of A.*

Proof. Let $a \in C$. Since $(A, \odot, \mathcal{T})$ is a topological hoop, $a \odot C$ is a connected subset of A. Since $a \in C \cap (a \odot C)$, the set $C \cup (a \odot C)$ is a connected subset of A which contains 1. By assumption, $C \cup (a \odot C) \subseteq C$, and so $a \odot C \subseteq C$. Hence, $C \odot C \subseteq C$. Now, suppose that $x \leqslant y$ and $x \in C$, for some $x, y \in A$. Then $x \wedge y = x \in C$. Thus, $x = x \wedge y \in C \wedge y$. Since $(A, \mathcal{T})$ is a topological hoop, by Proposition 3.12, $(A, \wedge, \mathcal{T})$ is a topological hoop. Thus, $C \wedge y$ is a connected set, and so $C \wedge y \subseteq C$. Hence, $y = 1 \wedge y \in C \wedge y \subseteq C$, and so $y \in C$. Therefore, $C \in \mathcal{F}(A)$. $\square$

Let A be a hoop and $F \in \mathcal{F}(A)$. In the preliminary, we saw that A/F is a quotient hoop and $\pi_F : A \to A/F$ is a canonical epimorphism. Let $\mathcal{T}$ be a topology on A and U be a subset of A/F. Then we say that U is an open subset of A/F if and only if $\pi_F^{-1}(U)$ is an open subset of A. Now, if we consider

$$\overline{\mathcal{T}} = \{U \subseteq A/F \mid \pi_F^{-1}(U) \in \mathcal{T}\}$$

then it is easy to show that $\overline{\mathcal{T}}$ is a topology on A/F. This topology on A/F is called *the quotient topology induced by π_F*. It is well known that it is the largest topology on A/F making π_F continuous.

Theorem 3.19. *Let A be a hoop and $F \in \mathcal{F}(A)$. If $(A, \mathcal{T})$ is a (semi)topological hoop and π_F is an open set, then $(A/F, \overline{\mathcal{T}})$ is a (semi)topological hoop.*

Proof. Let $(A, \mathcal{T})$ be a topological hoop, $\star \in \{\otimes, \rightsquigarrow\}$ and $\overline{x} \star \overline{y} \in V \in \overline{\mathcal{T}}$, for $\overline{x}, \overline{y} \in A/F$. Then $\overline{x * y} \in V$, for some $* \in \{\odot, \rightarrow\}$. Since π_F is continuous, $x * y \in \pi_F^{-1}(V) \in \mathcal{T}$. Since $(A, \mathcal{T})$ is a topological hoop, there exist $U, W \in \mathcal{T}$ such that $x \in U$, $y \in W$ and $x * y \in U * W \subseteq \pi_F^{-1}(V)$. Since π_F is an open map, $\pi_F(U)$ and $\pi_F(W)$ are in $\overline{\mathcal{T}}$, $\overline{x} \in \pi_F(U)$, $\overline{y} \in \pi_F(W)$ and $\overline{x} \star \overline{y} \in \pi_F(U) \star \pi_F(W) \subseteq V$. Hence, $(A/F, \star, \overline{\mathcal{T}})$ is a topological hoop. □

Proposition 3.20. *Let $(A, \mathcal{T})$ be a topological hoop and $F \in \mathcal{F}(A)$. Then:*

(*i*) *A/F has a discrete topology if and only if F is open,*

(*ii*) *if $(A, \mathcal{T})$ is a compact topological hoop, then A/F is a discrete finite topological hoop if and only if F is open.*

Proof. (*i*). Since A/F has a discrete topology, every single set such as $\{x/F\}$ is open, for any $x \in A$. Since $1 \in A$, $\{1/F\}$ is open. Since $\{1/F\} = F$, F is open. Conversely, if F is an open set, then $\{1/F\}$ is an open set, too. Since A/F is a hoop, by Theorem 3.15, A/F has a discrete topology.

(*ii*). Suppose A is compact. Since π is a continuous epimorphism, $\pi(A) = A/F$ is compact. Let F is open. Then by (*i*), A/F has a discrete topology and so every single subset is open. Moreover, since A/F is compact, A/F is equal to union of finite open subsets. Thus A/F is finite. The converse, by (*i*) is clear. □

Definition 3.21. (cf. [11]) Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A *local basis at x* is a set B of open neighborhoods of x such that for all $U \in \mathcal{T}$ if $x \in U$, then there exists $H \in B$ such that $x \in H \subseteq U$.

Lemma 3.22. *Let $F \in \mathcal{F}(A)$. If $\mathcal{T}$ is a topology on A and $\overline{\mathcal{T}}$ is the quotient topology on A/F, then for each $x \in A$, $\pi_F^{-1}(\pi_F(\overline{x})) = \overline{x}$. Moreover, if $V \in \overline{\mathcal{T}}$, then there exists $U \in \mathcal{T}$ such that $\pi_F(U) = V$.*

Proof. The proof is easy. □

Theorem 3.23. *Let $(A, \mathcal{T})$ be a semitopological hoop and $F \in \mathcal{F}(A)$. Then*

$$B = \{\pi(U * x) \mid U \in \mathcal{T}, 1 \in U, x \in A\}$$

is a local base of the space A/F at the point $x/F \in A/F$, such that $* \in \{\odot, \rightarrow\}$ and the map $\pi : A \to A/F$ is open.

Proof. Let $U \in \mathcal{T}$. Since $1 \in U$, it is clear that $x \in U * x$, for all $x \in A$. Thus, $x/F \in \pi(U * x)$. Now, suppose that $x/F \in A/F$. Then there exists $W \in \overline{\mathcal{T}}$ such that $x/F \in W$. Since W is open and π is continuous, we have $x \in \pi^{-1}(W) = O$. On the other hand, by (HP1), $x = 1 * x \in O$. Since $*$ is continuous, there exists $U \in \mathcal{T}$ such that $1 \in U$ and $x \in U * x \subseteq O$. Thus,

$$x/F \in \pi(U * x) \subseteq \pi(O) = \pi(\pi^{-1}(W)) = W$$

and so $\pi^{-1}(\pi(U * x)) \subseteq O$. By Lemma 3.22, $\pi^{-1}(\pi(U * x)) = (U * x)/F \subseteq O$. Thus, $\pi(U * x) \subseteq W$. Hence, B is a local basis. By definition of quotient topology, $\pi(U * x) = (U * x)/F = \bigcup_{y \in U * x} y/F$ and by Lemma 3.22,

$$\pi^{-1}(\pi(U * x)) = (U * x)/F = \bigcup_{y \in U * x} y/F.$$

Since $\bigcup_{y \in U * x} y/F$ is open in A and π is continuous, we get $\pi(U * x)$ is open in $\overline{\mathcal{T}}$. Therefore, π is open. □

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