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Retractable finitely supported Cb-sets

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Abstract. A construction for retractable state-finite automata without outputs has been given by Nagy. Retractable automata are automata all whose sub automata are retracts of it, and retracts are the subobjects whose related inclusion morphism have a left inverse. Studying retracts is an important subject in different branches of mathematics as well as computer science. In this paper, following Nagy's works, we study retractable finitely supported Cb-sets. The category of finitely supported Cb-sets introduced by Pitts is equivalent to one of the presheaf categories of Bazem, Coquand, and Huber. We characterize retractable finitely supported Cb-sets as ones which have a decomposition into retractable components. We also give a description of retractable cyclic finitely supported Cb-sets. Furthermore, recalling the notion of s-separated finitely supported Cb-sets, and support maps, we construct a subcategory of finitely supported Cb-sets consisted of s-separated finitely supported Cb-sets with 2-equivariant support maps, and characterize its retractable objects.

1. Introduction

Let $\mathbb{D}$ be a countable infinite set. A permutation π on $\mathbb{D}$ is said to be *finitary* if it changes only a finite number of elements of $\mathbb{D}$. Consider the group $G = \mathrm{Perm_f}(\mathbb{D})$ of finitary permutations on $\mathbb{D}$, and take a set X with an action of G on it, that is, a G-set. An element $x \in X$ is said to have a *finite support* $C \subseteq \mathbb{D}$ if it is invariant (fixed) under the action of each element π of G which fixes all the elements of C (that is, if $\pi c = c$, for all $c \in C$, then $\pi x = x$).

A G-set X every element of which has a finite support is said to be a *nominal set*. The notion of a nominal set was introduced by Fraenkel in 1922, and developed by Mostowski in the 1930s under the name of Fraenkel-Mostowski hierarchy or briefly FM-sets. The FM-sets were used to prove the independence of the axiom of choice from the other axioms (in the classical Zermelo-Fraenkel (ZF) set theory).

In 2001, Gabbay and Pitts rediscovered those sets in the context of name abstraction. They called them nominal sets, and applied this notion to properly model the syntax of formal systems involving variable binding operations (see [5]). Nominal techniques have also been used in game theory [1], in logic ([4], [9]), in domain theory [11], and in proof theory [12].

In [10], Pitts generalized the notion of nominal sets, by first adding two elements 0,1 to $\mathbb{D}$, then generalizing the notion of a finitary permutation to *finite*

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substitution, and considering the monoid Cb instead of the group G. Then he defined the notion of a support for Cb-sets, sets with an action of Cb on them, and invented the notion of *finitely supported* Cb-sets, as a generalization of nominal sets.

On the other hand, an equivariant map of a Cb-set X onto a sub Cb-set Y of X is called retraction if it leaves the elements of Y fixed. A Cb-set X is called retractable, if for every sub Cb-set Y, there exists a retraction of X onto Y. The notion of retractable plays a crucial role in many areas of mathematics, such as homological algebra, topological spaces, ordered algebraic structures, etc.

The main contribution of this paper is at giving a characterization of retractable finitely supported Cb-sets. In [8], Nagy showed that every retractable cyclic state-finite automaton has a sub automaton with no proper sub automaton called minimal automaton and then in Theorem 2 of [8], he characterized retractable state-finite automata without outputs. We found that every retractable cyclic finitely supported Cb-set has a unique fix-simple sub Cb-set with a unique zero element. In [3], we introduced fix-simple finitely supported Cb-sets with a unique zero element as finitely supported Cb-sets with no proper non-singleton sub Cb-sets. In fact, our fix-simple finitely supported Cb-sets with unique zero palys the role of Nagy's minimal automaton. In Section 4, in Theorem 4.12, by the same scheme of Nagy but different in details and proofs, we characterize retractable finitely supported Cb-sets.

In the following, to have a better scenery of the structure of this paper, we bring a summary of the results of each section. After a brief introduction in Section 1, we bring the basic notions and results about M-sets, sets with an action of a monoid M, and the monoid Cb in Section 2, needed in this paper. Then Section 3 is about retractions of M-sets and a description of decomposable finitely supported Cb-sets is given. Section 4 is devoted to retractable finitely supported Cb-sets and we characterize them. In Section 5, a subcategory of finitely supported Cb-sets is introduced, and its retractable objects are characterized.

2. Preliminaries

This section has devoted to give some basic notions needed in this paper. For more information one can see [2, 3, 7, 10].

2.1. M-sets

A (left) *M-set* for a monoid M with identity e is a set X equipped with a map $M \times X \to X, (m, x) \mapsto mx$, called an *action* of M on X, such that $ex = x$ and $m(m'x) = (mm')x$, for all $x \in X$ and $m, m' \in M$. An *equivariant map* from an M-set X to an M-set Y is a map $f : X \to Y$ with $f(mx) = mf(x)$, for all $x \in X, m \in M$.

An element x of an M-set X is called a *zero* (or a *fixed*) element if $mx = x$, for all $m \in M$. We denote the set of all zero elements of an M-set X by $Z(X)$.

The M-set X all of whose elements are zero is called a *discrete* M-set, or an M-set with *identity action*.

A subset Y of an M-set X is a *sub M-set* (or *M-subset*) of Y if for all $m \in M$ and $y \in Y$ we have $my \in Y$. The subset $Z(X)$ of X is in fact a sub M-set.

An M-set X is said to be *zero-decomposable* if there exists a collection $\{X_i\}_{i\in I}$ of sub M-sets of X such that $X = \bigcup_{i\in I} X_i$, and $X_i \cap X_j = \{\theta\} \in Z(X)$ or $X_i \cap X_j = \emptyset$, for all $i \neq j$. In this case, we say X has a *zero-decomposition* of X_i's and call X_i's the components of X.

Note. If for all $i \neq j$ we have $X_i \cap X_j = \emptyset$, then we call X *decomposable*.

2.2. The monoid Cb

Let $\mathbb{D}$ be an infinite countable set, whose elements are sometimes called *directions* (*atomic names* or *data values*) and $\mathrm{Perm}\mathbb{D}$ be the group of all permutations (bijection maps) on $\mathbb{D}$. A permutation $\pi \in \mathrm{Perm}\mathbb{D}$ is said to be *finitary* if the set $\{d \in \mathbb{D} \mid \pi(d) \neq d\}$ is finite. Clearly the set $\mathrm{Perm}_{\mathrm{f}}\mathbb{D}$ of all finitary permutations is a subgroup of $\mathrm{Perm}\mathbb{D}$.

Also, we take $2 = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

Definition 2.1. (a) A finite *substitution* is a map $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$ for which $\mathrm{Dom}_{\mathrm{f}}\sigma = \{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite.

(b) A finite substitution satisfies *injectivity condition*, if

$$(\forall d, d' \in \mathbb{D}),\ \sigma(d) = \sigma(d') \notin 2 \Rightarrow d = d'.$$

(c) If $d \in \mathbb{D}$ and $b \in 2$, we write (b/d) for the finite substitution which maps d to b, and is the identity mapping on all the other elements of $\mathbb{D}$. Each (b/d) is called a *basic substitution*.

(d) If $d, d' \in \mathbb{D}$ then we write $(d\,d')$ for the finite substitution that transposes d and d', and keeps fixed all other elements. Each $(d\,d')$ is called a *transposition substitution*.

Definition 2.2. (a) Let Cb be the monoid whose elements are finite substitutions satisfying injectivity condition, with the monoid operation given by $\sigma \cdot \sigma' = \hat{\sigma}\sigma'$, where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined the same as σ. The identity element of Cb is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup 2$.

(b) Take S to be the subsemigroup of Cb generated by basic substitutions. The members of S are of the form $\delta = (b_1/d_1) \cdots (b_k/d_k) \in S$ for some $d_i \in \mathbb{D}$ and $b_i \in 2$, and we denote the set $\{d_1, \cdots, d_k\}$ by $\mathbb{D}_\delta$.

Remark 2.3. (1) Notice that each finite permutation π on $\mathbb{D}$, can be considered as a finite substitution $\iota \circ \pi : \mathbb{D} \to \mathbb{D} \cup 2$. Doing so, throughout this paper, we consider the group $\mathrm{Perm}_{\mathrm{f}}\mathbb{D}$ as a submonoid of Cb, and denote $\iota \circ \pi$ with the same notation π.

(2) Let $\delta \in S$, and $\pi \in \mathrm{Perm_f}(\mathbb{D})$. Then, $\pi\delta = \delta'\pi$, and $\delta\pi = \pi\delta''$, where $\mathbb{D}_{\delta'} = \{\pi d : d \in \mathbb{D}_\delta\}$, and $\mathbb{D}_{\delta''} = \{\pi^{-1} d : d \in \mathbb{D}_\delta\}$.

(3) Let $d \neq d' \in \mathbb{D}$ and $b, b' \in 2$. Then

$$(b/d)(b'/d') = (b'/d')(b/d).$$

But, $(1/d)(0/d) = (0/d)$ and $(0/d)(1/d) = (1/d)$, and hence $(1/d)(0/d) \neq (0/d)(1/d)$.

Theorem 2.4. [3] *For the monoid Cb, we have*

$$Cb = \mathrm{Perm_f}(\mathbb{D})S^\iota,$$

where $S^\iota = S \cup \{\iota\}$.

2.3. Finitely supported Cb-sets

In this subsection, basic notions about finitely supported Cb-sets which is needed in the sequel are given, some of which [3, 10].

The following definition introduces the notion of a, so called, *support*, which is the central notion to define finitely supported Cb-sets.

Definition 2.5. (a) Suppose X is a Cb-set. A subset $C \subseteq \mathbb{D}$ *supports* an element x of X if, for every $\sigma, \sigma' \in Cb$,

$$(\sigma(c) = \sigma'(c), (\forall c \in C)) \Rightarrow \sigma x = \sigma' x$$

If there is a finite (possibly empty) support C then we say that x is *finitely supported.*

(b) A Cb-set X whose all elements have finite supports, is called a *finitely supported Cb-set.*

We denote the category of all Cb-sets with equivariant maps between them by Cb-**Set**, and its full subcategory of all finitely supported Cb-sets by $(Cb\text{-}\mathbf{Set})_{\mathrm{fs}}$.

Remark 2.6. Let X be a Cb-set and $x \in X$.

(1) If X is finitely supported, then the set $\{d \in \mathbb{D} \mid (0/d)x \neq x\}$ is in fact the least finite support of x. From now on, we call the least finite support for x *the support* for x, and denote it by $\mathrm{supp}\, x$.

(2) x is a zero element if and only if $\mathrm{supp}\, x = \emptyset$ if and only if $\delta\, x = x$, for all $\delta \in S$.

Example 2.7. (1) The set $\mathbb{D} \cup 2$ is a finitely supported Cb-set, with the *canonical action* given by evaluation; that is,

$$\forall \sigma \in Cb,\ x \in \mathbb{D} \cup 2,\ \sigma x = \hat{\sigma}(x),$$

in which $\hat{\sigma}$ is defined as in Definition 2.2(a). Also, for each $d \in \mathbb{D}$, $\mathrm{supp}\, d = \{d\}$, and $\mathrm{supp}\, 0 = \mathrm{supp}\, 1 = \emptyset$, since both of $0, 1$ are zero elements.

(2) The set $\mathbb{D}\cup\{\mathsf{0}\}$ is a finitely supported Cb-set with the action is given by

$$\forall\sigma\in Cb,\ x\in\mathbb{D}\cup\{\mathsf{0}\},\ \sigma x=\hat{\sigma}(x).$$

Also, for each $d\in\mathbb{D}$, $\operatorname{supp} d=\{d\}$, and $\operatorname{supp}\mathsf{0}=\emptyset$, since $\mathsf{0}$ is a zero element.

(3) All discrete Cb-sets are clearly finitely supported Cb-sets, because of Remark 2.6(2).

Remark 2.8. [3] (1) Every finitely supported Cb-set has a zero element.

(2) Every finite finitely supported Cb-set is discrete.

Lemma 2.9. [3] *Let X be a non-empty finitely supported Cb-set, and $x\in X$. Then*

(i) $\delta\, x=x$ *if and only if* $\mathbb{D}_{\delta}\cap\operatorname{supp} x=\emptyset$.

(ii) *If* $\delta\in S$, *then* $\operatorname{supp}\delta x\subseteq\operatorname{supp} x\setminus\mathbb{D}_{\delta}$.

(iii) *For* $\pi\in\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$, *we have* $\operatorname{supp}\pi x=\pi\operatorname{supp} x$. *In particular,*

$$|\operatorname{supp}\pi x|=|\pi\operatorname{supp} x|=|\operatorname{supp} x|.$$

Remark 2.10. [3] For a finitely supported Cb-set X and $x\in X$, we have

$$S_x\doteq\{\delta\in S\mid\delta x=x\},\quad S'_x\doteq S\setminus S_x=\{\delta\in S\mid\delta x\neq x\},$$

which they are two subsemigroups of S.

The following lemma is useful in Theorem 2.14.

Lemma 2.11. *Let X be a finitely supported Cb-set, and x a non-zero element of X. Then, S'_x is an ideal of S.*

Proof. Suppose $\delta\in S$ and $\delta_1\in S'_x$. We show that $\delta\delta_1,\delta_1\delta\in S'_x$. Notice that, since $\delta_1\in S'_x$, we get $\delta_1 x\neq x$ and so using part (i) of Lemma 2.9, $\mathbb{D}_{\delta_1}\cap\operatorname{supp} x\neq\emptyset$. On the other hand, since $\mathbb{D}_{\delta_1\delta}=\mathbb{D}_{\delta\delta_1}=\mathbb{D}_{\delta_1}\cup\mathbb{D}_{\delta}$, we get $\mathbb{D}_{\delta\delta_1}\cap\operatorname{supp} x\neq\emptyset$ and $\mathbb{D}_{\delta_1\delta}\cap\operatorname{supp} x\neq\emptyset$. Thus $\delta_1\delta x\neq x$ and $\delta\delta_1 x\neq x$ which means $\delta_1\delta,\delta\delta_1\in S'_x$. $\square$

Definition 2.12. A cyclic finitely supported Cb-set X is a finitely supported Cb-set which is generated by only one element. That means, it is of the form $Cb\,x$, for some $x\in X$.

Remark 2.13. [3] If Cbx is a non-singleton cyclic finitely supported Cb-set, then

$$Cbx=\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})S'_x x\cup\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x,\quad \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})S'_x x\cap\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})x=\emptyset.$$

Theorem 2.14. *Let Cbx be a non-singleton cyclic finitely supported Cb-set. Then,*

(i) $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})S'_x x$ *is a sub Cb-set of Cbx.*

(ii) *If* $\operatorname{supp} x=\{d_1,\cdots,d_k\}$, *then* $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})S'_x x=\bigcup_{i=1}^{k}Cb(b_i/d_i)x$.

Proof. First, notice that, since Cbx is non-singleton, we get that $\operatorname{supp} x \neq \emptyset$. So, for all $d \in \operatorname{supp} x$, we have $(0/d)x \in S'_x x$ which means that $\mathrm{Perm_f}(\mathbb{D})S'_x x$ is a non-empty set.

(i) Let $\pi_1\delta_1 x \in \mathrm{Perm_f}(\mathbb{D})S'_x x$ and $\sigma \in Cb$. Then, by Theorem 2.4, we have $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$ with $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$. If $\sigma \in \mathrm{Perm_f}(\mathbb{D})$, then $\sigma\pi_1\delta_1 x \in \mathrm{Perm_f}(\mathbb{D})S'_x x$. Let $\sigma = \pi\delta$. Then, applying Remark 2.3(2) and Lemma 2.11, we get that

$$\sigma\pi_1\delta_1 x = \pi\delta\pi_1\delta_1 x = \pi\pi_1\delta'\delta_1 x \in \mathrm{Perm_f}(\mathbb{D})S'_x x.$$

(ii) If $d \in \operatorname{supp} x$, then by Lemma 2.9(i), $(b/d) \in S'_x$, and so applying (i), $Cb(b/d)x \subseteq \mathrm{Perm_f}(\mathbb{D})S'_x x$. Thus, $\bigcup_{i=1}^k Cb(b_i/d_i)x \subseteq \mathrm{Perm_f}(\mathbb{D})S'_x x$.

To prove the reverse inclusion, let $a \in \mathrm{Perm_f}(\mathbb{D})S'_x x$. Then, there exist $\delta \in S'_x$ and $\pi \in \mathrm{Perm_f}(\mathbb{D})$ with $a = \pi\delta x$. Since $\delta \in S'_x$, by Lemma 2.9(i), we get that $\mathbb{D}_\delta \cap \operatorname{supp} x \neq \emptyset$. Let $d \in \operatorname{supp} x \cap \mathbb{D}_\delta$. Then, $\delta x = \delta_1(b/d)x$ where $\delta_1 \in S$ and $b \in 2$. Thus, $Cb\delta x \subseteq Cb(b/d)x$ which means that $\mathrm{Perm_f}(\mathbb{D})S'_x x \subseteq \bigcup_{i=1}^k Cb(b_i/d_i)x$. □

3. Retractions of finitely supported Cb-sets

In this section, we show that a retract of an indecomposable M-set is indecomposable. Theorem 3.6 gives a characterization of retracts of a decomposable finitely supported Cb-set. As a result of this theorem, for finding retractions of a decomposable finitely supported Cb-set, it is sufficient to obtain retractions of its indecomposable sub Cb-sets.

Definition 3.1. Let Y be a (finitely supported) M-set and X a sub M-set of it. Then, X is called *a retract* of Y if there exists an equivariant map $g : Y \to X$, called retraction, such that $g(x) = x$, for all $x \in X$.

Lemma 3.2. ([7], Lemma I.5.36) *Let X be an indecomposable M-set, and $\varphi : X \to Y$ an equivariant map. Then, $\varphi(X)$ is an indecomposable sub M-set of Y.*

Proposition 3.3. *A retract of an indecomposable M-set is indecomposable.*

Proof. Let Y be a retract of an indecomposable M-set X. Then, there exists a retraction $\varphi : X \to Y$. We show that Y is indecomposable. On the contrary, suppose $Y = Y_1 \cup Y_2$ is a decomposition of Y. Since X is indecomposable, by Lemma 3.2, $\varphi(X)$ is indecomposable. So, $\varphi(X) \subseteq Y_1$ or $\varphi(X) \subseteq Y_2$. Assume $\varphi(X) \subseteq Y_1$. Since φ is a retraction and $Y \subseteq X$, we get that

$$Y = \varphi(Y) \subseteq \varphi(X) \subseteq Y_1,$$

which is impossible. Similarly, the case $\varphi(X) \subseteq Y_2$ is impossible. Thus, Y is indecomposable. □

Theorem 3.4. ([7], Theorem I.5.10) *Every M-set has a decomposition into indecomposable sub M-sets.*

Remark 3.5. Let X be a finitely supported Cb-set and Y a sub Cb-set of X. Then, by Theorem 3.4, X has a decomposition into its indecomposable sub Cb-sets. Take $X = \bigcup_\alpha X_\alpha$. Then,

$$Y = Y \cap X = Y \cap (\bigcup_\alpha X_\alpha) = \bigcup_\alpha (Y \cap X_\alpha) = \bigcup_\alpha Y_\alpha,$$

where $Y_\alpha = Y \cap X_\alpha$.

Theorem 3.6. *Let X be a decomposable finitely supported Cb-set, and Y a sub Cb-set of it considered in* Remark 3.5. *Then, Y is a retract of X if and only if*

$$\forall\alpha \ \ (Y_\alpha \neq \emptyset \Rightarrow Y_\alpha \ \ \textit{is a retract of} \ \ X_\alpha).$$

Proof. Suppose $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\alpha Y_\alpha$. Let $\varphi : X \to Y$ be a retraction. Then, $\varphi|_{X_\alpha} : X_\alpha \to Y$ is an equivariant map. Suppose $Y_\alpha \neq \emptyset$. Now, since $Y_\alpha \subseteq X_\alpha$ and φ is a retraction, we get $Y_\alpha \subseteq \varphi(X_\alpha)$. On the other hand, by Lemma 3.2, $\varphi|_{X_\alpha}(X_\alpha) = \varphi(X_\alpha)$ is indecomposable, and so, $\varphi(X_\alpha) = Y_\alpha$. Therefore, $\varphi|_{X_\alpha} : X_\alpha \to Y_\alpha$ is a retraction.

To prove the other part, let Y be a sub Cb-set of X. Then, we show that Y is a retract of X. If $Y_\alpha \neq \emptyset$, then since Y_α is a retract of X_α, we get a retraction $\varphi_\alpha : X_\alpha \to Y_\alpha$. Now, the assignment $\varphi : X \to Y$ defined by

$$\varphi(x) = \begin{cases} \varphi_\alpha(x), & \text{if} \quad x \in X_\alpha \text{ and } Y_\alpha \neq \emptyset \\ \theta \in Y, & \text{if} \quad x \in X_\alpha \text{ and } Y_\alpha = \emptyset \end{cases}$$

is a retraction. □

4. Retractable finitely supported Cb-sets

In this section, we study retractable finitely supported Cb-sets. Discrete finitely supported Cb-sets are retractable. So, we focus on non-discrete finitely supported Cb-sets. As a result of Lemma 4.3, a retractable indecomposable finitely supported Cb-set has a unique zero element. In Theorem 4.12, we give a characterization of a non-discrete retractable finitely supported Cb-set.

Definition 4.1. Let X be a (finitely supported) M-set. Then, X is called *retractable* if every non-empty sub M-set of X is a retract of it.

Remark 4.2. (1) Every sub M-set of a retractable M-set is retractable.

(2) Retracts of a cyclic M-set are cyclic. This is because, if A is a retract of Mx, then there exists a retraction $\varphi : Mx \to A$. Notice that, since φ is surjective, we get $\varphi(Mx) = A$. On the other hand, since φ is equivariant, we get that $\varphi(Mx) = M\varphi(x)$. Therefore, $A = M\varphi(x)$ which means that A is cyclic.

Lemma 4.3. *Let X be an indecomposable retractable M-set with $Z(X) \neq \emptyset$. Then, X has a unique zero element.*

Proof. If $\theta_1 \neq \theta_2 \in Z(X)$, then the sub M-set $\{\theta_1, \theta_2\}$ is a retract of X, and so, there exists a retraction $\varphi : X \to \{\theta_1, \theta_2\}$. Notice that, since X is indecomposable, by Lemma 3.2, $\varphi(X)$ is indecomposable, and so, $\varphi(X) = \theta_1$ or $\varphi(X) = \theta_2$. If $\varphi(X) = \theta_1$, then $\theta_2 = \varphi(\theta_2) = \theta_1$ which is a contradiction. Similarly, $\varphi(X) = \theta_2$ is impossible. □

Corollary 4.4. *A retractable indecomposable finitely supported Cb-set has a unique zero element.*

Proof. It follows by Remark 2.8(1) and Lemma 4.3. □

In characterizing retractable finitely supported Cb-sets, we apply the notion of fix-simple finitely supported Cb-sets with unique zero element introduced and characterized in [3]. A fix-simple finitely supported Cb-set with a unique zero element has no proper non-singleton sub Cb-sets. We called them θ-simple where θ is a notation for a zero element.

First, we recall needed facts of [3]

Theorem 4.5. [3] *For a non-discrete finitely supported Cb-set X with a unique zero element θ, the followings are equivalent*:

(i) *X is θ-simple*;

(ii) *X is a cyclic finitely supported Cb-set of the form of* $\mathrm{Perm_f}(\mathbb{D})\, x \cup \{\theta\}$, *for some non-zero element $x \in X$. Furthermore, $(b/d)x = \theta$, for all $d \in \operatorname{supp} x$.*

Remark 4.6. [3] Let X be an infinite finitely supported Cb-set with a unique zero element θ, and $x \in X$. Then,

(1) X has a θ-simple sub Cb-set.

(2) If X is simple, then X is θ-simple.

(3) If $X = Cbx$ is cyclic with $|\operatorname{supp} x| = 1$, then X simple.

(4) X is simple if and only if X is θ-simple, and $\operatorname{supp} x \neq \operatorname{supp} x'$, for all non-zero elements $x \neq x'$.

As a result of Theorem 4.5, we get the following corollary.

Corollary 4.7. *All θ-simple (simple) finitely supported Cb-sets are retractable.*

Lemma 4.8. *A retractable non-singleton cyclic finitely supported Cb-set has a unique θ-simple sub Cb-set.*

Proof. Let $X = Cbx$ be retractable with a non-zero element x. Then, by Corollary 4.4, X has a unique zero element θ. Also, by Remark 4.6(1), X has a θ-simple sub Cb-set. Suppose X has two θ-simple sub Cb-sets X_1 and X_2. Applying Theorem 4.5, we get that $X_1 = \mathrm{Perm_f}(\mathbb{D})x_1 \cup \{\theta\}$ and $X_2 = \mathrm{Perm_f}(\mathbb{D})x_2 \cup \{\theta\}$. Since X is retractable, by Remark 4.2, $X_1 \cup X_2$ is a retract of X, and so is cyclic. Therefore, $X_1 = X_2$. □

Proposition 4.9. *Suppose that X is a non-discrete retractable finitely supported Cb-set. Also, suppose $\{B_i\}_{i\in I}$ is the collection of all distinct θ-simple sub Cb-sets of X. Take $X_i = \bigcup_{x\in X}\{Cbx : B_i \subseteq Cbx\}$. Then,*

(i) *every X_i is a retracteble sub Cb-set of X.*

(ii) *every X_i is indecomposable, and has a unique zero element.*

(iii) *for all $i \neq j$, $X_i \cap X_j = \emptyset$ or $X_i \cap X_j = \{\theta\}$.*

(iv) *$X = \bigcup_{i\in I} X_i$.*

Proof. (i) Let $x \in X_i$ and $\sigma \in Cb$. Then, we show that $\sigma x \in X_i$. Notice that $B_i \subseteq Cbx$. Since X is retractable, by Remark 4.2(1), Cbx is retractable, and so, by Lemma 4.8, Cbx has a unique θ-simple sub Cb-set B_i. Also, since $Cb\sigma x \subseteq Cbx$, we get that $B_i \subseteq Cb\sigma x$, and so, $\sigma x \in X_i$. Now, applying Remark 4.2(1), X_i's are retractable.

(ii) Since $\bigcap\{Cbx : B_i \subseteq Cbx\} = B_i$, we get X_i is indecomposable. Now, since X is retractable, by Remark 4.2(1), X_i is retractable, and so, by Lemma 4.3, has a unique zero element.

(iii) Let $x \in X_i \cap X_j$ with $x \neq \theta$. Then, $Cbx \subseteq X_i \cap X_j$ and so $B_i, B_j \subseteq Cbx$ which contradicts Lemma 4.8 that states Cbx has a unique θ-simple sub Cb-set.

(iv) To prove the non-trivial part, let $x \in X$. Then, since X is retractable, by Remark 4.2(1), Cbx is retractable. Applying Lemma 4.8, there exists a unique θ-simple sub Cb-set B_i with $B_i \subseteq Cbx$. Thus by the assumption $x \in X_i$. □

Lemma 4.10. *Let X be a finitely supported Cb-set with a zero-decomposition of retractable components. Then, X is retractable.*

Proof. Suppose $X = \bigcup X_i$ is a zero-decomposition of retractable finitely supported Cb-sets X_i. Let Y be a sub Cb-set of X. Then, we show that Y is a retract of X. Take $Y_i = Y \cap X_i$. Notice that Y_i is a (possibly empty) sub Cb-set of Y. If $Y_i \neq \emptyset$, then since X_i is retractable, we get a retraction $\varphi_i : X_i \to Y_i$. Now, the assignment $\varphi : X \to Y$ defined by

$$\varphi(x) = \begin{cases} \varphi_i(x), & \text{if} \quad x \in X_i \text{ and } Y_i \neq \emptyset \\ \theta \in Y, & \text{if} \quad x \in X_i \text{ and } Y_i = \emptyset \end{cases}$$

is a retraction. □

Corollary 4.11. *Disjoint union of two retractable finitely supported Cb-sets is retractable.*

In the following theorem, we give a charatcterization of retractable finitely supported Cb-sets.

Theorem 4.12. *Let X be a finitely supported Cb-set. Then, X is retractable if and only if X has a zero-decomposition of retractable components.*

Proof. If X is discrete, then X is retractable and has a zero-decomposition of retractable components. Suppose X is non-discrete and retractable. Also, suppose $\{B_i\}_{i\in I}$ is the collection of all distinct θ-simple sub Cb-sets of X which exist by Lemma 4.8. Take $X_i = \bigcup_{x\in X}\{Cbx : B_i \subseteq Cbx\}$. Then, by Proposition 4.9, $X = \bigcup X_i$ is a zero-decomposition of retractable X_i.

The other part holds by Lemma 4.10. □

The following lemma is needed in Theorem 4.14 which gives a necessary condition for a cyclic finitely supported Cb-set to be retractable.

Lemma 4.13. *If Cbx is a non-singleton retractable cyclic finitely supported Cb-set, then there exists $d \in \operatorname{supp} x$ with* $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb(b/d)x$, *where $b \in 2$.*

Proof. Let $\operatorname{supp} x = \{d_1, \cdots, d_k\}$. Then, applying Theorem 2.14(ii), we get that $\mathrm{Perm_f}(\mathbb{D})S'_x x = \bigcup_{i=1}^k Cb(b_i/d_i)x$. Since $\mathrm{Perm_f}(\mathbb{D})S'_x x$ is a sub Cb-set of Cbx, and Cbx is retractable, by Remark 4.2, we get that $\mathrm{Perm_f}(\mathbb{D})S'_x x$ is cyclic. So, there exists $a \in \mathrm{Perm_f}(\mathbb{D})S'_x x$ with $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cba$. Since $a \in \mathrm{Perm_f}(\mathbb{D})S'_x x$, there exist $i = 1, \cdots, k$ and $\sigma \in Cb$ with $a = \sigma(b_i/d_i)x$. Applying Theorem 2.4, $\sigma \in \mathrm{Perm_f}(\mathbb{D})$ or $\sigma = \pi\delta$, where $\pi \in \mathrm{Perm_f}(\mathbb{D})$ and $\delta \in S$. If $\sigma = \pi\delta$ and $\delta \in S'_{(b_i/d_i)}$, then $Cba = Cb\delta(b_i/d_i)x$ which is a proper sub Cb-set of $Cb(b_i/d_i)x$. Thus $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb\delta(b_i/d_i)x$, and so $Cb(b_i/d_i)x \subseteq Cb\delta(b_i/d_i)x$ which is a contradiction. Therefore, $\sigma = \pi$ or $\sigma = \pi\delta$ with $\delta \in S_{(b_i/d_i)}$, and hence, we get that $Cba = Cb(b_i/d_i)x$. □

In Theorem 4.14, we give a description of a retractable cyclic finitely supported Cb-set.

Theorem 4.14. *Suppose Cbx is a cyclic finitely supported Cb-set. Also, suppose* $\operatorname{supp} x = \{d_1, \cdots, d_k\}$. *If Cbx is retractable, then*

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \bigcup_{i=1}^{l} \mathrm{Perm_f}(\mathbb{D})(b_i/d_i)\cdots(b_1/d_1)x \cup \{\theta\},$$

where $l \in \{1, \cdots, k\}$ and $d_j \in \operatorname{supp}(b_{j-1}/d_{j-1})\cdots(b_1/d_1)x$, for all $j = 2, \cdots, l$.

Proof. Suppose Cbx is retractable. If $\mathrm{Perm_f}(\mathbb{D})S'_x x = \{\theta\}$, then by Remark 2.13 we get that $Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \{\theta\}$. Suppose there exists $\delta \in S'_x$ with $\delta x \neq \theta$. By Lemma 4.13, there exist $d \in \operatorname{supp} x$ and $b \in 2$, say $d = d_1$, $b = b_1$, with $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb(b_1/d_1)x$. So applying Remark 2.13, we have

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup Cb(b_1/d_1)x.$$

By the assumption, Cbx is retractable. So, by Remark 4.2(1), $Cb(b_1/d_1)x$ is retractable. Now, if $\mathrm{Perm_f}(\mathbb{D})S'_{(b_1/d_1)}(b_1/d_1)x = \{\theta\}$, then

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \mathrm{Perm_f}(\mathbb{D})(b_1/d_1)x \cup \{\theta\}.$$

Otherwise, we show that $\mathrm{Perm_f}(\mathbb{D})S'_{(b_1/d_1)}(b_1/d_1)x = Cb(b_2/d_2)(b_1/d_1)x$, with $d_2 \in \mathrm{supp}\,(b_1/d_1)x$. Similar to the proof of Theorem 2.14,

$$\mathrm{Perm_f}(\mathbb{D})S'_{(b_1/d_1)}(b_1/d_1)x = \bigcup_j Cb(b_j/d_j)(b_1/d_1)x,$$

where for all j, $d_j \in \mathrm{supp}\,(b_1/d_1)x$. On the other hand, $Cb(b_1/d_1)x$ is retractable, and so applying Lemma 4.13, $\mathrm{Perm_f}(\mathbb{D})S'_{(b_1/d_1)}(b_1/d_1)x$ is cyclic. Therefore, there exist $d_2 \in \mathrm{supp}\,(b_1/d_1)x$ and $b_2 \in 2$ such that

$$Cb(b_1/d_1)x = \mathrm{Perm_f}(\mathbb{D})(b_1/d_1)x \cup Cb(b_2/d_2)(b_1/d_1)x.$$

By continuing this process, we get

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \bigcup_{i=1}^{l} \mathrm{Perm_f}(\mathbb{D})(b_i/d_i)\cdots(b_1/d_1)x \cup \{\theta\},$$

where $l = 1, \cdots, k$. □

5. 2-s-separated finitely supported Cb-sets

In this section, we consider s-separated finitely supported Cb-sets with 2-equivariant support maps (briefly 2-s-separated finitely supported Cb-set) introduced in [6], and characterize retractable objects in this category.

To find retractable s-separated finitely supported Cb-sets with 2-equivariant support maps, first, in Theorem 5.8, we give a description of them. Thereafter, in Theorem 5.10, we prove that retractable s-separated finitely supported Cb-sets with 2-equivariant support maps are discrete or simple or are a disjoint union of a simple sub Cb-set and a discrete sub Cb-set. Also, we give a description of cyclic s-separated finitely supported Cb-sets with 2-equivariant support maps.

First, we recall our definitions of the support map and 2-equivariant support map of [6].

Definition 5.1. Let X be a finitely supported Cb-set, and $x \in X$. Then,

(a) the map

$$\mathrm{supp} : X \to \mathcal{P}_{\mathrm{f}}(\mathbb{D} \cup 2), x \mapsto \mathrm{supp}\, x$$

is called *the support map of* X.

(b) the support map of X is 2-*equivariant* if $\mathrm{supp}\,\sigma x = (\sigma \mathrm{supp}\, x) \setminus 2$, for all $\sigma \in Cb$.

Definition 5.2. [6] (a) A finitely supported Cb-set X is called an *stabilizer-separated* or briefly *s-separated* if $\mathrm{supp}\, x \neq \mathrm{supp}\, x'$, for all non-zero elements $x \neq x' \in X$.

(b) A finitely supported Cb-set X is called an *s-separated with* 2-*equivariant support map* or briefly 2-*s-separated* if X is s-separated and the support map of X is 2-equivariant.

Remark 5.3. Applying Definition 5.2 and Remark 4.6(4), we get that all s-separated θ-simple finitely supported Cb-sets are simple.

Lemma 5.4. [6] *Suppose X is a finitely supported Cb-set, and $x \in X$.*

(i) *Let X be s-separated and $x' \neq x$ be two non-zero elements of X. Then, $|\mathrm{supp}\, x| = |\mathrm{supp}\, x'|$ if and only if $Cbx = Cbx'$.*

(ii) *The support map of X is 2-equivariant if and only if $\mathrm{supp}\, \delta x = (\mathrm{supp}\, x) \backslash \mathbb{D}_{\delta}$, for all $\delta \in S'_x$.*

Corollary 5.5. *Suppose X is an s-separated finitely supported Cb-set with 2-equivariant support map. Let $x \in X$ with $|\mathrm{supp}\, x| > 1$. Then,*

(1) *For all $d \in \mathrm{supp}\, x$, we have $(0/d)x = (1/d)x$.*
(2) *For all $\delta_1, \delta_2 \in S'_x$, we have $\delta_1\delta_2 x = \delta_2\delta_1 x$.*
(3) *For all $d \neq d' \in \mathrm{supp}\, x$, we have $Cb(0/d)x = Cb(0/d')x$.*
(4) *If X is a non-singleton cyclic, then X has a unique zero element.*

Proof. (1) Since the support map of X is 2-equivariant, and $|\mathrm{supp}\, x| > 1$, by Lemma 5.4(ii), we get that $\mathrm{supp}\,(1/d)x = \mathrm{supp}\,(0/d)x = \mathrm{supp}\, x \setminus \{d\} \neq \emptyset$. Now, by Definition 5.2, we have $(0/d)x = (1/d)x$.

(2) By (1), we have

$$\begin{aligned}&(0/d)(1/d)x = (0/d)(0/d)x = (0/d)x \\ &= (1/d)x = (1/d)(1/d)x = (1/d)(0/d)x.\end{aligned}$$

Now, applying Remark 2.3, we get that $\delta_1\delta_2 x = \delta_2\delta_1 x$.

(3) Let $d, d' \in \mathrm{supp}\, x$. Then, since $\mathrm{supp}\,(0/d)x = \mathrm{supp}\, x \setminus \{d\} \neq \emptyset$ and $\mathrm{supp}\,(0/d')x = \mathrm{supp}\, x \setminus \{d'\} \neq \emptyset$, we get that $|\mathrm{supp}\,(0/d)x| = |\mathrm{supp}\,(0/d')x|$. Therefore, applying Lemma 5.4(i), $Cb(0/d)x = Cb(0/d')x$.

(4) Suppose $X = Cbx$, for some non-zero element $x \in X$. If $\theta_1 \neq \theta_2 \in Z(Cbx)$, then there exist $\delta_1, \delta_2 \in S'_x$ with $\theta_1 = \delta_1 x$ and $\theta_2 = \delta_2 x$. Now, by (2),

$$\theta_1 = \delta_2\theta_1 = \delta_2\delta_1 x = \delta_1\delta_2 x = \delta_1\theta_2 = \theta_2,$$

which is a contradiction. □

In the following lemma, for an s-separated finitely supported Cb-set, by Corollary 5.5, we show that the sub Cb-set $\mathrm{Perm_f}(\mathbb{D})S'_x x$ of a cyclic Cb-set Cbx is cyclic.

Lemma 5.6. *Suppose X is an s-separated finitely supported Cb-set with 2-equivariant support map. Let $x \in X$ with $|\mathrm{supp}\, x| > 1$. Then, there exists $d \in \mathrm{supp}\, x$ with $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb(0/d)x$.*

Proof. Let $\mathrm{supp}\, x = \{d_1, \cdots, d_k\}$ with $k > 1$. Then, applying Theorem 2.14, we get that $\mathrm{Perm_f}(\mathbb{D})S'_x x = \bigcup_{i=1}^{k} Cb(b_i/d_i)x$ is a sub Cb-set of Cbx where $b_i \in 2$. Now, by Corollary 5.5(1,3), we get that $\mathrm{Perm_f}(\mathbb{D})S'_x x$ is cyclic. Therefore, there exists $d \in \mathrm{supp}\, x$ such that $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb(0/d)x$. □

Remark 5.7. Suppose X is an s-separated finitely supported Cb-set and $x \neq x'$ are two non-zero elements of X. Let the support map of X be 2-equivariant. Then,

Case (1): If $|\mathrm{supp}\, x| = |\mathrm{supp}\, x'|$, then by Lemma 5.4(i), $Cbx = Cbx'$.

Case (2): If $|\mathrm{supp}\, x| < |\mathrm{supp}\, x'|$, then $Cbx \subsetneq Cbx'$ and if $|\mathrm{supp}\, x'| < |\mathrm{supp}\, x|$, then $Cbx' \subsetneq Cbx$. To prove this, let $|\mathrm{supp}\, x| = k$ and $|\mathrm{supp}\, x'| = l$. Assuming $k < l$, we show that $Cbx \subsetneq Cbx'$. The other part is proved similarly. Take $\mathrm{supp}\, x' = \{d_1, \cdots, d_k, d_{k+1}, \cdots, d_l\}$. Since the support map of X is 2-equivariant, we get that

$$\begin{array}{rcl} \mathrm{supp}\,(0/d_l)\cdots(0/d_{k+1})x' & = & \mathrm{supp}\, x' \setminus \{d_l, \cdots, d_{k+1}\} \\ & = & \{d_1, \cdots, d_k\}. \end{array}$$

Thus, $|\mathrm{supp}\,(0/d_l)\cdots(0/d_{k+1})x'| = k$. Now, applying Lemma 5.4(i), we get that $Cb(0/d_l)\cdots(0/d_{k+1})x' = Cbx$ and so $x \in Cbx'$.

Theorem 5.8. *Suppose X is an s-separated finitely supported Cb-set. Let the support map of X be 2-equivariant. Then, X is decomposable if and only if X is discrete or $X = Y \cup Z$ is a disjoint union of a non-singleton indecomposable sub Cb-set of Y and a discrete sub Cb-set Z.*

Proof. To prove the non-trivial part, suppose X is non-discrete. Take $X = \coprod_\alpha X_\alpha$ to be a decomposition of X into indecomposable sub Cb-sets. We show that all the non-zero elements of X belong to exactly one component of X. On the contrary, let $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$ be two non-zero elements. Then, $|\mathrm{supp}\, x_\alpha| \leq |\mathrm{supp}\, x_\beta|$ or $|\mathrm{supp}\, x_\beta| \leq |\mathrm{supp}\, x_\alpha|$. Now, applying Remark 5.7, $Cbx_\alpha \subseteq Cbx_\beta \subseteq X_\beta$ or $Cbx_\beta \subseteq Cbx_\alpha \subseteq X_\alpha$ which is a contradiction. Thus, there exists a unique α_0 with $X \setminus Z(X) \subseteq X_{\alpha_0}$ which means that X can be written as a disjoint union of a non-singleton indecomposable sub Cb-set and a discrete sub Cb-set. □

Now, we are ready to characterize retractable s-separated finitely supported Cb-sets with 2-equivariant support maps.

In the following lemma, we characterize retractable s-separated cyclic finitely supported Cb-sets with 2-equivariant support maps.

Lemma 5.9. *Suppose X is an s-separated cyclic finitely supported Cb-set. Let the support map of X be 2-equivariant. Then, X is retractable if and only if X is simple.*

Proof. If X is singleton, then it is clear that X is retractable and simple. Suppose $X = Cbx$ is cyclic with a non-zero element x of X. Also, let X be retractable. Then, by Corollary 4.4, X has a unique zero element θ. Notice that, by Remark 4.6(1), X has a θ-simple sub Cb-set, say $Cbx_0 = \mathrm{Perm_f}(\mathbb{D})x_0 \cup \{\theta\}$. Thus, applying Remark 5.3, Cbx_0 is simple. Since $x_0 \in Cbx$, by Theorem 2.4, we get that $x_0 = \pi x$ or $x_0 = \pi\delta_0 x$. If $x_0 = \pi x$, then $Cbx = Cbx_0$, and so, X is simple. Suppose $x_0 = \pi\delta_0 x$. In this case, we also show that $\delta_0 \in S_x$, and so, $Cbx_0 = Cbx$. On the contrary, let $\delta_0 \in S'_x$. Then, by Lemma 2.9(i), $\delta_0 x \neq x$, and so, Cbx_0 is a proper sub Cb-set of Cbx. Since X is retractable, there exists a retraction $\varphi : Cbx \to Cb\delta_0 x$.

First, we show that $\varphi(x) = \delta_0 x$. Since $\varphi(x) \in Cb\delta_0 x$, by Remark 2.13, we have $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})S'_{\delta_0 x}\delta_0 x$ or $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})\delta_0 x$. If $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})S'_{\delta_0 x}\delta_0 x$, then $\varphi(x) = \pi'\delta'\delta_0 x$ where $\delta' \in S'_{\delta_0 x}$ and $\pi' \in \mathrm{Perm_f}(\mathbb{D})$. Since φ is a retraction and $\delta_0 x \in Cb\delta_0 x$, we get that $\delta_0 x = \varphi(\delta_0 x) = \delta_0\varphi(x) = \delta_0\pi'\delta'\delta_0 x$.

Now, applying Lemma 2.9, we get that $|\mathrm{supp}\,\delta_0 x| = |\mathrm{supp}\,\delta_0\pi'\delta'\delta_0 x| < |\mathrm{supp}\,\delta_0 x|$, which is impossible. Therefore, $\varphi(x) \in \mathrm{Perm_f}(\mathbb{D})\delta_0 x$, and so there exists $\pi' \in \mathrm{Perm_f}(\mathbb{D})$ with $\varphi(x) = \pi'\delta_0 x$. Also, since φ is a retraction and $\delta_0 x \in Cb\delta_0 x$, we get that

$$\delta_0 x = \varphi(\delta_0 x) = \delta_0\varphi(x) = \delta_0\pi'\delta_0 x = \pi'\delta'_0\delta_0 x.$$

where the last equality is true by Remark 2.3(2). Now, $\delta'_0 \in S_{\delta_0 x}$, since otherwise, if $\delta'_0 \in S'_{\delta_0 x}$, then by Lemma 2.9

$$|\mathrm{supp}\,\delta_0 x| = |\mathrm{supp}\,\pi'\delta'_0\delta_0 x| = |\mathrm{supp}\,\delta'_0\delta_0 x| < |\mathrm{supp}\,\delta_0 x|,$$

which is impossible. Thus, $\delta'_0 \in S_{\delta_0 x}$ and so $\delta_0 x = \pi'\delta'_0\delta_0 x = \pi'\delta_0 x$. Therefore, $\varphi(x) = \delta_0 x$.

Now, take $d \in (\mathrm{supp}\,x) \setminus \mathrm{supp}\,\delta_0 x$, and $d' \in \mathrm{supp}\,\delta_0 x$. Then, since X is s-separated, we have $(d\ d')x = x$. Also, since φ is a retraction, we get that

$$(d\ d')\delta_0 x = (d\ d')\varphi(x) = \varphi((d\ d')x) = \varphi(x) = \delta_0 x.$$

Thus,

$$d = (d\ d')d' \in (d\ d')\mathrm{supp}\,\delta_0 x = \mathrm{supp}\,(d\ d')\delta_0 x = \mathrm{supp}\,\delta_0 x,$$

which is impossible.

The other part follows by Corollary 4.7. □

Theorem 5.10. *Suppose X is an s-separated finitely supported Cb-set. Let the support map of X be 2-equivariant. Then, X is retractable if and only if X is discrete or simple or X is a disjoint union of a simple sub Cb-set and a discrete sub Cb-set.*

Proof. Discrete Cb-sets are retractable. Also, by Corollary 4.7, simple finitely supported Cb-sets are retractable.

To prove the other part, let X be non-discrete and retractable. Then, by Theorem 5.8, $X = Y \cup Z$ is a disjoint union of a discrete sub Cb-set Z, and an indecomposable sub Cb-set Y. Notice that, by Remark 4.2(1), Y is retractable. So, applying Corollary 4.4, we get that Y has a unique zero element θ. We show that Y is simple. To show this, first, we prove that Y has a unique simple sub Cb-set. By Remark 4.6(1), Y has a θ-simple sub Cb-set. Since Y is s-separated, by Remark 5.3, we get that Y has a simple sub Cb-set. Now, suppose B_1 and B_2 are two simple sub Cb-sets of Y. So, applying Theorem 4.5, $B_1 = Cby_1$ and $B_2 = Cby_2$ are cyclic. Assuming $B_1 = Cby_1$, we show that $|\mathrm{supp}\,y_1| = 1$. Notice that, $(0/d)y_1 = \theta$, for all $d \in \mathrm{supp}\,y_1$. Since Y is s-separated with 2-equivariant support map, we get that $\emptyset = \mathrm{supp}\,(0/d)y_1 = (\mathrm{supp}\,y_1) \setminus \{d\}$. Thus $\mathrm{supp}\,y_1 = \{d\}$, and so, $|\mathrm{supp}\,y_1| = 1$.

Similarly, $|\mathrm{supp}\, y_2| = 1$. Thus, since $|\mathrm{supp}\, y_1| = |\mathrm{supp}\, y_2| = 1$, by Remark 5.7, we get that $B_1 = B_2$. Hence, Y has a unique simple sub Cb-set, say B.

Now, we prove that $Y = B$. Let $y \in Y$. Then, since Y is retractable, by Remark 4.6(1), we get that Cby is retractable. Now, applying Lemma 5.9, Cby is simple. Thus, $Cby = B$, and so, $y \in B$. Therefore, $B \subseteq Y \subseteq B$ which means that $Y = B$ is simple. □

In Theorem 5.11, we give a description of a cyclic s-separated finitely supported Cb-set with 2-equivariant support map.

Theorem 5.11. *If Cbx is an s-separated finitely supported Cb-set with 2-equivariant support map and $\mathrm{supp}\, x = \{d_1, \cdots, d_k\}$, then*

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \bigcup_{i=1}^{k} \mathrm{Perm_f}(\mathbb{D})(0/d_i)\cdots(0/d_1)x,$$

where $d_j \in \mathrm{supp}\,(0/d_{j-1})\cdots(0/d_1)x$, for $j = 2, \cdots, k$.

Proof. Let $\mathrm{supp}\, x = \{d_1, \cdots, d_k\}$. Then, applying Lemma 5.6, there exists some $d_i \in \mathrm{supp}\, x$, say $d_i = d_1$, with $\mathrm{Perm_f}(\mathbb{D})S'_x x = Cb(0/d_1)x$. Now, we show that $\mathrm{Perm_f}(\mathbb{D})S'_{(0/d_1)}(0/d_1)x = Cb(0/d_2)(0/d_1)x$, where $d_2 \in \mathrm{supp}\,(0/d_1)x$. Similar to the proof of Theorem 2.14,

$$\mathrm{Perm_f}(\mathbb{D})S'_{(0/d_1)}(0/d_1)x = \bigcup_j Cb(0/d_j)(0/d_1)x,$$

where for all j, $d_j \in \mathrm{supp}\,(0/d_1)x$.

On the other hand, for all j, we have $\mathrm{supp}\,(0/d_j)(0/d_1)x = \mathrm{supp}\, x \backslash \{d_j, d_1\}$. So, for all $r \neq s$, we get $|\mathrm{supp}\,(0/d_r)(0/d_1)x| = |\mathrm{supp}\,(0/d_s)(0/d_1)x|$. Now, applying Lemma 5.4, $Cb(0/d_r)(0/d_1)x = Cb(0/d_s)(0/d_1)x$. Thus, $\mathrm{Perm_f}(\mathbb{D})S'_{(0/d_1)}(0/d_1)x$ is cyclic. So, there exists $d \in \mathrm{supp}\,(0/d_1)x$, say $d = d_2$ with

$$Cb(0/d_1)x = \mathrm{Perm_f}(\mathbb{D})(0/d_1)x \cup Cb(0/d_2)(0/d_1)x.$$

By continuing this process, we get

$$Cbx = \mathrm{Perm_f}(\mathbb{D})x \cup \bigcup_{i=1}^{k} \mathrm{Perm_f}(\mathbb{D})(0/d_i)\cdots(0/d_1)x.$$

□

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