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On the properties of zero-divisor graphs of posets

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Abstract. We determine the cut vertices in the zero-divisor graphs of posets and study the posets with end-regular zero-divisor graph. Also, we investigate the zero-divisor graph of the product of two posets. In particular, we determine all posets with planar and outerplanar zero-divisor graphs.

1. Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [13], [16], [17], [20], [21], [24] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have been also actively investigated. For example, power graphs and divisibility graphs have been considered in [14], [15], zero-divisor graphs have been studied in [3], [4], [5], [8], [9], and cozero-divisor graphs and annihilating-ideal graphs have been considered in [1] and [2], respectively.

Recently, the zero-divisor graph of a poset was defined and studied in [11], [12], [19] and [23]. In this paper, we deal with the zero-divisor graphs of posets based on terminology of [19]. In [19], Lu and Wu defined the zero-divisor graph for an arbitrary partially ordered set $(P, \leqslant)$ (poset, briefly) with a least element 0, as an undirected graph whose vertices consists of all nonzero zero-divisors of P, and two distinct vertices x and y are adjacent if and only if $\{x, y\}^\ell = \{0\}$, where for a subset S of P, $\{S\}^\ell$ denotes the set of lower bounds of S. In this paper, we denote this graph by $\Gamma(P)$. In Section 2, we study the cut vertices in $\Gamma(P)$. Also, we investigate some basic properties of $\Gamma(P_1 \times P_2)$, where P_1 and P_2 are two finite posets. In Section 3, we study the planarity of $\Gamma(P_1 \times P_2)$. In Section 4, we investigate the outerplanarity in the zero-divisor graphs of posets. In the last section, we study the posets with end-regular zero-divisor graphs.

Now we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [6] and partially ordered sets in [7]. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. In a graph G, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise,

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we set $\mathrm{d}(a,b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a,b) : a$ and b are distinct vertices of $G\}$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. The *valency* of a vertex a is the number of the edges of the graph G incident with a. A *clique* of a graph is a maximal complete subgraph of it and the number of vertices in a largest clique of G is called *clique number* of G and is denoted by $\omega(G)$. In the graph theory, a *unicycle graph* is a graph that has exactly one cycle. The graph with no vertices and no edges is the *null graph*.

In a partially ordered set $(P, \leqslant)$ with a least element 0, an element a is called an *atom* if $a \neq 0$, and, for an element x in P, the relation $0 \leqslant x \leqslant a$ implies either $x = 0$ or $x = a$. Also, for $a, b \in P$, we say that $a < b$, whenever $a \leqslant b$ and $a \neq b$. Assume that S is a subset of P. Then an element x in P is a *lower bound* of S if $x \leqslant s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S by S^u, that is,

$$S^\ell := \{x \in P \mid x \leqslant s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leqslant x, \text{ for all } s \in S\}.$$

We say that a non-empty subset I of P is an *ideal* of P if, for arbitrary elements x and y in P, the relations $x \in I$ and $y \leqslant x$ imply that $y \in I$. Also the ideal I is *prime* if $x, y \in P$ with $\{x,y\}^\ell \subseteq I$, then $x \in I$ or $y \in I$. A *maximal* ideal of P is a proper ideal of P which is maximal among all ideals of P.

2. Cut vertices in the zero-divisor graph of a poset

Throughout the paper, P is a finite poset and $A(P) = \{a_1, a_2, ..., a_n\}$ is the set of all atoms of P. Also, we denote the set of zero-divisors of the poset P by $Z(P)$, that is,

$$Z(P) = \{x \in P \mid \{x,y\}^\ell = 0, \text{ for some } y \in P\}.$$

Clearly, if $|A(P)| = 1$, then $\Gamma(P)$ is a null graph. Therefore, we suppose that $|A(P)| \geqslant 2$.

A vertex a of a graph G is called a *cut vertex* if the removal of a and any edges incident on a creates a graph with more connected components than G.

Theorem 2.1. *If a is a cut vertex in $\Gamma(P)$, then $\{0, a\}$ is an ideal of P.*

Proof. One can easily see that $\{0, a\}$ is an ideal of P if and only if a is an atom of P. Hence it is sufficient to show that $a = a_i$, for some $i = 1, 2, \ldots, n$. Assume that a is not an atom. Since a is a cut vertex, $\Gamma(P) \backslash \{a\}$ has at least two components X and Y. We claim that $A(P) \subseteq X$ or $A(P) \subseteq Y$. Otherwise there are atoms

a_i and a_j, where $1 \leqslant i \neq j \leqslant n$, such that $a_i \in X$ and $a_j \in Y$. Now we have that a_i is adjacent to a_j, which is impossible. Without loss of generality, we may assume that $A(P) \subseteq X$. Then, for all $y \in Y$, we have $y \in \{a_i\}^u$, for $i = 1, 2, \ldots, n$. Thus $y \in \cap_{i=1}^n \{a_i\}^u$. This implies that $y \notin Z(P)$, which is impossible. Therefore $a \in A(P)$, and so $\{0, a\}$ is an ideal of P. $\square$

The following example shows that the converse of Theorem 2.1 is not true in general.

Example 2.2. Suppose that P is a poset in Figure 1. Then, it is easy to see that a_1 is an atom, but it is not a cut vertex in $\Gamma(P)$.

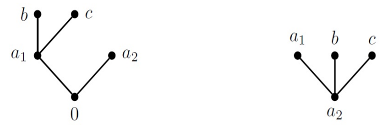

Figure 1. P and $\Gamma(P)$

Notation. Let $i_1, i_2, \ldots, i_n$ be integers with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. The notation $U^P_{i_1 i_2 \ldots i_k}$ stands for the following set:

$$\{x \in P; \quad x \in \cap_{s=1}^k \{a_{i_s}\}^u \setminus \cup_{j \neq i_1, i_2, \ldots, i_k} \{a_j\}^u\}$$

Note that no two distinct elements in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\Gamma(P)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover $P \setminus \{0\} = \cup_{k=1,\ 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n}^n U_{i_1 i_2 \cdots i_k}$. Also, if there is no ambiguity, we denote $U^P_{i_1 i_2 \ldots i_k}$ by $U_{i_1 i_2 \ldots i_k}$. Also by $1 \cdots \hat{i} \cdots n$ we means that $1 \cdots i-1\ i+1 \cdots n$.

In the next theorem, we provide some conditions under which the converse of Theorem 2.1 holds.

Theorem 2.3. *Let $|P| \geqslant 4$. Then there exists i with $1 \leqslant i \leqslant n$ such that a_i is a cut vertex in $\Gamma(P)$, if $|U_i| = 1$ and $U_{1 \cdots \hat{i} \cdots n} \neq \emptyset$, for some $1 \leqslant i \leqslant n$.*

Proof. It is enough to show that there exist vertices b and c in P such that a_i is in every path from b to c in $\Gamma(P)$. Since $U_{1 \cdots \hat{i} \cdots n} \neq \emptyset$, there is an element b in $U_{1 \cdots \hat{i} \cdots n}$. Now, for some $j \neq i$, consider $c \in U_j$. Thus a_i is in every path from b to c in $\Gamma(P)$, and so it is a cut vertex in $\Gamma(P)$. $\square$

Proposition 2.4. *Let a be a cut vertex in $\Gamma(P)$ and X be connected component of $\Gamma(P) \setminus \{a\}$. Also suppose that X is complete with at least two vertices. Then $V(X) \cup \{0\}$ is an ideal of P.*

Proof. Since a is a cut vertex in $\Gamma(P)$, by Theorem 2.1, a is an atom of P. Suppose that $a = a_1$. Now, we have the following cases:

Case 1. $A(P)\backslash\{a\} \subseteq X$. If X contains an element b such that b is not an atom, then since X is complete, we have that $b \in U_1$. Now, let $Y \neq X$ be another connected components of $\Gamma(P)\backslash\{a\}$ and let $c \in Y$. Clearly, $c \in U_{23\ldots n}$. Thus b and c are adjacent which is impossible. So we have that $X = A(P)\backslash\{a\}$, and thus $V(X) \cup \{0\}$ is an ideal of P.

Case 2. $A(P)\backslash\{a\} \nsubseteq X$. It is easy to see that in this situation X does not contain any atom. Now, let x and y be distinct elements in X. Then we have $x, y \in U_{23\ldots n}$, and so x is not adjacent to y, which is impossible. Therefore this case does not happen. □

The next example shows that the converse of Proposition 2.4 is not true in general.

Example 2.5. Suppose that P is a poset of Figure 2. Clearly a_1 is the cut vertex in $\Gamma(P)$. Let $V(X) = \{a_2, a_3, c\}$. Then, by Figure 2, it is easy to see that $V(X) \cup \{0\}$ is an ideal of P, but X is not a complete subgraph of $\Gamma(P)$.

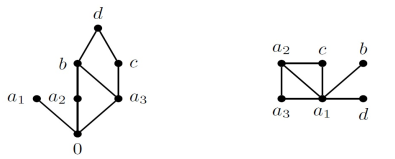

Figure 2. P and $\Gamma(P)$

Definition 2.6. Suppose that x is a vertex in $\Gamma(P)$. Set

$$Z_x := \{y \in P \mid \{x, y\}^\ell = \{0\}\}.$$

We say that Z_x is properly maximal if $Z_x \subseteq Z_b$, for some $b \in P\backslash\{0, x\}$, then we have $Z_x = Z_b$.

Theorem 2.7. *If a is a cut vertex in $\Gamma(P)$, then Z_a is properly maximal.*

Proof. Assume on the contrary that $Z_a \subsetneqq Z_b$, for some vertices b in $\Gamma(P)$ with $b \neq a$. Then clearly all vertices adjacent to a are also adjacent to b. This is a contradiction with the fact that a is a cut vertex. □

Let $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ be two posets with the least elements. Then the cartesian product $P_1 \times P_2$ is also a poset with the following relation. For two distinct elements $(x, y), (x', y') \in P_1 \times P_2$ we say that $(x, y) \leqslant (x', y')$ if and only if $x \leqslant_1 x'$ and $y \leqslant_2 y'$. Clearly $(P_1 \times P_2, \leqslant)$ has the minimum element $(0, 0)$. Suppose that P_1 and P_2 are two finite posets such that $A(P_1) = \{a_1, a_2, \ldots, a_n\}$ and $A(P_2) = \{b_1, b_2, \ldots, b_m\}$. In the following we study some properties of the zero-divisor graph $\Gamma(P_1 \times P_2)$.

Lemma 2.8. *In the poset $P_1 \times P_2$, we have $A(P_1 \times P_2) = (A(P_1) \times \{0\}) \cup (\{0\} \times A(P_2))$, and so $|A(P_1 \times P_2)| = |A(P_1)| + |A(P_2)|$.*

Proof. Suppose that (a,b) belongs to the set $A(P_1 \times P_2)$. If $a,b \neq 0$, then we have $(0,0) < (a,0) < (a,b)$ which is impossible. Then we have $a = 0$ or $b = 0$. Without loss of generally, we may assume that $b = 0$. If $a \notin A(P_1)$, then there exists an atom $a_i \in A(P_1)$, for some $1 \leqslant i \leqslant n$, such that $a_i < a$. Hence we have that $(0,0) < (a_i,0) < (a,0)$ which is impossible. Thus $a \in A(P_1)$, and so the result holds. □

We can extend the concept of $P_1 \times P_2$ for a product of finite number of posets.

Corollary 2.9. *Let $P = P_1 \times P_2 \times \cdots \times P_n$, where $(P_i, \leqslant_i)$'s are partially ordered sets for $i = 1, 2, \ldots, n$. Then $A(P)$ consists of elements $(a_1, a_2, \ldots, a_n)$ such that there exists $1 \leqslant j \leqslant n$ with $a_j \in A(P_j)$, and, for all i with $1 \leqslant i \neq j \leqslant n$, $a_i = 0$.*

Proposition 2.10. *Let $P = P_1 \times P_2 \times \cdots \times P_n$ be a poset such that $P \neq P_1 \times P_2$, with $|P_1| = |P_2| = 2$. If $a = (0, 0, \ldots, u_i, 0, \ldots, 0) \in Z(P)$ is a cut vertex with nonzero component u_i such that $u_i \notin Z(P_i)$, then $|P_i| = 2$.*

Proof. Assume on the contrary that P_i has at least three elements and so there exists v_i in $P_i \backslash \{0, u_i\}$. It is easy to see that $Z_a \subseteq Z_{(0,0,\ldots,v_i,0,\ldots,0)}$. Since a is a cut vertex, by Theorem 2.7, we have that $Z_a = Z_{(0,0,\ldots,v_i,0,\ldots,0)}$, which implies that $a = (0, 0, \ldots, v_i, 0, \ldots, 0)$. Hence $u_i = v_i$, which is a contradiction. □

3. Planarity of $\Gamma(P_1 \times P_2)$

Recall that a graph is said to be *planar* if it can be drown in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

Theorem 3.1. *If $\Gamma(P_1 \times P_2)$ is planar, then $|A(P_1)| + |A(P_2)| \leqslant 4$.*

Proof. Suppose on the contrary that $|A(P_1)| + |A(P_2)| \geqslant 5$. Since the induced subgraph of $\Gamma(P_1 \times P_2)$ on the vertex-set $A(P_1 \times P_2)$ is a complete graph, one can find a subgraph of $\Gamma(P_1 \times P_2)$ isomorphic to K_5, and so, by Kuratowski's Theorem, $\Gamma(P_1 \times P_2)$ is not planar. Hence we have $|A(P_1)| + |A(P_2)| \leqslant 4$. □

By Theorem 3.1, we must study the cases that $|A(P_1)| + |A(P_2)|$ is equal to 2, 3 and 4. In the following proposition, we state the necessary and sufficient condition for planarity of $\Gamma(P_1 \times P_2)$, when $|A(P_1)| + |A(P_2)| = 2$.

Proposition 3.2. *Suppose that $|A(P_1)| + |A(P_2)| = 2$ such that $|A(P_1)| = 1 = |A(P_2)|$. Then $\Gamma(P_1 \times P_2)$ is planar if and only if $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$.*

Proof. Since $|A(P_1)| + |A(P_2)| = 2$, we have that $\Gamma(P_1 \times P_2)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P_1 \times P_2)$ is planar if and only if $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$. □

Now, suppose that P_1 and P_2 are posets such that $|A(P_1)| + |A(P_2)| = 3$. Let $|A(P_1)| = 1$ and $|A(P_2)| = 2$. If $|P_1|, |P_2| \geqslant 4$, then we can find a copy of $K_{3,3}$ in the graph $\Gamma(P_1 \times P_2)$. Thus, by Kuratowski's Theorem, $\Gamma(P_1 \times P_2)$ is not planar. Therefore, if $\Gamma(P_1 \times P_2)$ is planar, then $|P_1| \leqslant 3$ or $|P_2| \leqslant 3$. Now, we have the following cases:

Case 1. Suppose that $|P_1| = 2$ and $|U_i^{P_2}| \geqslant 2$, for all $1 \leqslant i \leqslant 2$. In this situation we can find a subdivision of K_5 as in Figure 3, where $y_i \in U_i^{P_2} \backslash \{b_i\}$, for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

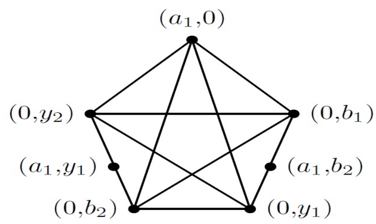

Figure 3.

If $|U_i^{P_2}| = 1$ and $|U_j^{P_2}| \geqslant 3$, for some $1 \leqslant i \neq j \leqslant 2$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (0, b_1), (a_1, b_1)\} \cup \{(0, b_2), (0, y_2), (0, y_2')\}$, where $y_i, y_i' \in U_i^{P_2} \backslash \{b_i\}$, for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| = 1$ and $|U_j^{P_2}| \leqslant 2$, for all $1 \leqslant i \neq j \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 4, and so $\Gamma(P_1 \times P_2)$ is planar.

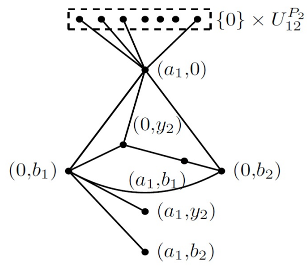

Figure 4.

Case 2. Suppose that $|P_1| = 3$ and $|U_i^{P_2}| \geqslant 3$, for some $1 \leqslant i \leqslant 2$. In this situation one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (0, b_2), (x, b_2)\} \cup \{(0, b_1), (0, y_1), (0, y_1')\}$, where $x \in P_1 \backslash \{0, a_1\}$ and $y_i, y_i' \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| \leqslant 2$, for all $1 \leqslant i \leqslant 2$, then one of the following situations happen:

(i) If $|U_i^{P_2}| = 2$, for all $1 \leqslant i \leqslant 2$, then we can find a subdivision of K_5 as in Figure 3, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for all $1 \leqslant i \leqslant 2$, and so $\Gamma(P_1 \times P_2)$ is not planar.

(ii) If $|U_i^{P_2}| = 2$, $|U_j^{P_2}| = 1$, for all $1 \leqslant i \neq j \leqslant 2$ and $U_{12}^{P_2} \neq \emptyset$, then we can find a subdivision of K_5 as in Figure 5, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 2$ and $c_{12} \in U_{12}^{P_2}$. So $\Gamma(P_1 \times P_2)$ is not planar.

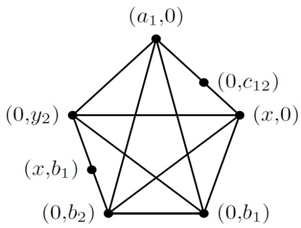

Figure 5.

If $|U_i^{P_2}| = 2$, $|U_j^{P_2}| = 1$, for all $1 \leqslant i \neq j \leqslant 2$ and $U_{12}^{P_2} = \emptyset$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 6, and so $\Gamma(P_1 \times P_2)$ is planar.

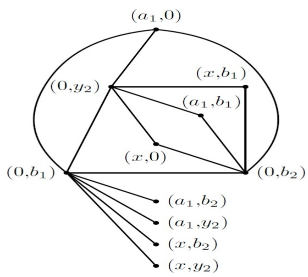

Figure 6.

(iii) If $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 7, and so $\Gamma(P_1 \times P_2)$ is planar.

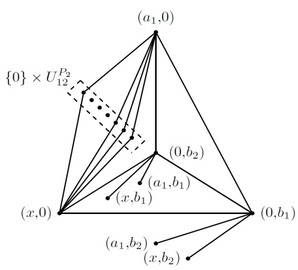

Figure 7.

Case 3. Suppose that $|P_2| = 3$. In this situation $\Gamma(P_1 \times P_2)$ is pictured in Figure 8, and hence $\Gamma(P_1 \times P_2)$ is planar.

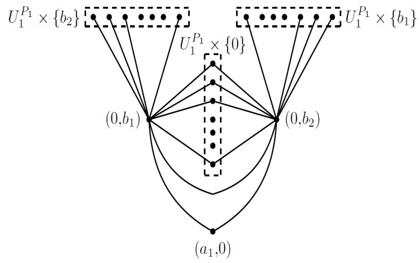

Figure 8.

Thus we have the following theorem.

Theorem 3.3. *Suppose that $|A(P_1)| + |A(P_2)| = 3$ such that $|A(P_1)| = 1$ and $|A(P_2)| = 2$. Then $\Gamma(P_1 \times P_2)$ is planar if and only if one of the following conditions hold.*

(i) $|P_1| = 2$, $|U_i^{P_2}| = 1$ *and* $|U_j^{P_2}| \leqslant 2$, *for all* $1 \leqslant i \neq j \leqslant 2$.

(ii) $|P_1| = 3$ *and* $|U_i^{P_2}| = 1$, *for all* $1 \leqslant i \leqslant 2$.

(iii) $|P_1| = 3$, $|U_i^{P_2}| = 2$ *and* $|U_j^{P_2}| = 1$, *for some* $1 \leqslant i \neq j \leqslant 2$ *and* $U_{12}^{P_2} = \emptyset$.

(iv) $|P_2| = 3$.

Finally, in order to complete the study of planarity of $\Gamma(P_1 \times P_2)$, we assume that $|A(P_1)| + |A(P_2)| = 4$. Now, we have the following cases:

Case 1. Suppose that $|A(P_1)| = 1$ and $|A(P_2)| = 3$. In this situation if $\Gamma(P_1 \times P_2)$ is planar, then $|P_1| \leqslant 3$. Note that if $\Gamma(P_1 \times P_2)$ is planar and $|P_1| \geqslant 4$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1, 0), (x, 0), (x', 0)\} \cup \{(0, b_1), (0, b_2), (0, b_3)\}$, where $x, x' \in P_1 \backslash \{0, a_1\}$. Thus $\Gamma(P_1 \times P_2)$ is not planar. Therefore $|P_1| \leqslant 3$.

Now, we investigate the planarity of $\Gamma(P_1 \times P_2)$ whenever, $|P_1| \leqslant 3$. To this end, we consider the following situations:

(i) Suppose that $|P_1| = 2$. If $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 3$, then we can find a subdivision of K_5 as in Figure 9, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 3$.

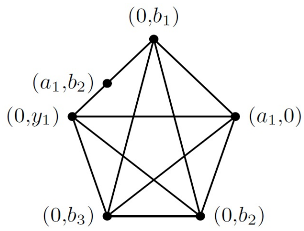

Figure 9.

If $|U_{ij}^{P_2}| \geqslant 1$, for some $1 \leqslant i \neq j \leqslant 3$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1,0),(0,b_3),(a_1,b_3)\} \cup \{(0,b_1),(0,b_2),(0,c_{12})\}$, where $c_{ij} \in U_{ij}^{P_2}$ for some $1 \leqslant i \neq j \leqslant 3$. So $\Gamma(P_1 \times P_2)$ is not planar.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 3$ and $U_{ij}^{P_2} = \emptyset$, for all $1 \leqslant i \neq j \leqslant 3$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 10, and so $\Gamma(P_1 \times P_2)$ is planar.

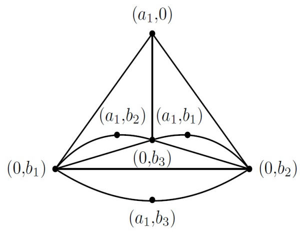

Figure 10.

(ii) Assume that $|P_1| = 3$. If $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 3$, then we can find a subdivision of K_5 as in Figure 9, where $y_i \in U_i^{P_2} \backslash \{b_i\}$ for some $1 \leqslant i \leqslant 3$.

If $|U_{ij}^{P_2}| \geqslant 1$, for some $1 \leqslant i \neq j \leqslant 3$, then one can find a copy of $K_{3,3}$ with vertex-set $\{(a_1,0),(0,b_3),(a_1,b_3)\} \cup \{(0,b_1),(0,b_2),(0,c_{12})\}$, where $c_{ij} \in U_{ij}^{P_2}$, for some $1 \leqslant i \neq j \leqslant 3$. So $\Gamma(P_1 \times P_2)$ is not planar.

If $U_{123}^{P_2} \neq \emptyset$, then we can find a subdivition of K_5 as in Figure 11, where $c_{123} \in U_{123}^{P_2}$. So $\Gamma(P_1 \times P_2)$ is not planar.

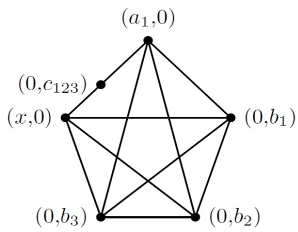

Figure 11.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 3$ and $U_{i\ldots j}^{P_2} = \emptyset$, for all $1 \leqslant i \neq j \leqslant 3$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 12, and so $\Gamma(P_1 \times P_2)$ is planar.

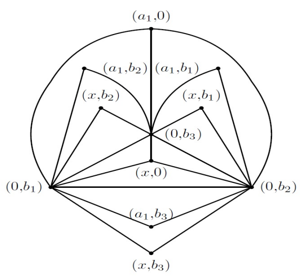

Figure 12.

Case 2. Assume that $|A(P_1)| = 2 = |A(P_2)|$. In this situation we can find a subdivision of $K_{3,3}$ as in Figure 13, and so $\Gamma(P_1 \times P_2)$ is not planar.

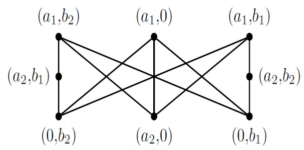

Figure 13.

Hence we have the following theorem.

Theorem 3.4. *Suppose that $|A(P_1)| + |A(P_2)| = 4$ such that $|A(P_1)| = 1$ and $|A(P_2)| = 3$. Then $\Gamma(P_1 \times P_2)$ is planar if and only if one of the following conditions hold.*

(*i*) $|P_1| = 2$ *and* $|U_i^{P_2}| = 1$ *for all* $1 \leqslant i \leqslant 3$ *and* $U_{ij}^{P_2} = \emptyset$ *for all* $1 \leqslant i \neq j \leqslant 3$.

(*ii*) $|P_1| = 3$, $|U_i^{P_2}| = 1$ *for all* $1 \leqslant i \leqslant 3$ *and* $U_{i\ldots j}^{P_2} = \emptyset$ *for all* $1 \leqslant i \neq j \leqslant 3$.

4. Outerplanarity of $\Gamma(P)$ and $\Gamma(P_1 \times P_2)$

A directed graph is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

In the following, we characterize all posets P such that $\Gamma(P)$ is outerplanar.

Lemma 4.1. *If $\Gamma(P)$ is outerplanar, then $|A(P)| \leqslant 3$.*

Proof. Assume to the contrary that $|A(P)| \geqslant 4$. Since the induced subgraph of $\Gamma(P)$ on vertex-set $A(P)$ is a complete subgraph, one can find a copy of K_4 in $\Gamma(P)$, and so $\Gamma(P)$ is not outerplanar. Hence we have $|A(P)| \leqslant 3$. □

By Lemma 4.1, we must study the cases that $|A(P)|$ is equal to 2 and 3. In the following proposition, we state the necessary and sufficient condition for outerplanarity of $\Gamma(P)$, when $|A(P)| = 2$.

Proposition 4.2. *Suppose that $|A(P)| = 2$. Then $\Gamma(P)$ is outerplanar if and only if $|U_i| = 1$, for some $1 \leqslant i \leqslant 2$, or $|U_i| \leqslant 2$, for all $1 \leqslant i \leqslant 2$.*

Proof. Since $|A(P)| = 2$, we have that $\Gamma(P)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P)$ is outerplanar if and only if $|U_i| = 1$, for some $1 \leqslant i \leqslant 2$, or $|U_i| \leqslant 2$, for all $1 \leqslant i \leqslant 2$. □

In the sequel of this section, we investigate the outerplanarity of $\Gamma(P)$, when $|A(P)| = 3$. If $|\cup_{i=1}^3 U_i| \geqslant 5$, then we can find a copy of $K_{2,3}$ in the structure of $\Gamma(P)$, and so $\Gamma(P)$ is not outerplanar. Therefore, if $\Gamma(P)$ is outerplanar, then $|\cup_{i=1}^3 U_i| \leqslant 4$. Now, we have the following cases:

Case 1. Suppose that $|\cup_{i=1}^3 U_i| = 3$. In this situation $\Gamma(P)$ is a unicyclic graph which is in pictured in Figure 14, and so it is outerplanar.

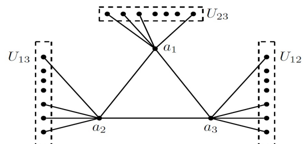

Figure 14.

Case 2. Suppose that $|\cup_{i=1}^3 U_i| = 4$. Suppose that $|U_i| = 2$. If $|U_{jk}| \geqslant 1$, for some $1 \leqslant i \neq j \neq k \leqslant 3$, then we can find a copy of $K_{2,3}$ with vertex-set $\{a_1, a_1'\} \cup \{a_2, a_3, c_{23}\}$, where $a_i' \in U_i \backslash \{a_i\}$ and $c_{jk} \in U_{jk}$, for some $1 \leqslant i \neq j \neq k \leqslant 3$, and so $\Gamma(P)$ is not outerplanar.

Now, if $U_{jk} = \emptyset$, for all $1 \leqslant i \neq j \neq k \leqslant 3$, then $\Gamma(P)$ is isomorphic to the graph which is pictured in Figure 15, and so $\Gamma(P)$ is outerplanar.

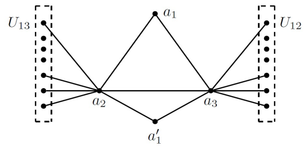

Figure 15.

Theorem 4.3. *Suppose that $|A(P)| = 3$. Then $\Gamma(P)$ is outerplanar if and only if one of the following conditions holds:*

(*i*) $|\cup_{i=1}^3 U_i| = 3$.

(*ii*) $|\cup_{i=1}^3 U_i| = 4$ *and if* $|U_i| = 2$, *for some* $1 \leqslant i \leqslant 3$, *then* $U_{jk} = \emptyset$, *for all* $1 \leqslant i \neq j \neq k \leqslant 3$.

In the following, we characterize all posets P_1 and P_2 such that $\Gamma(P_1 \times P_2)$ is outerplanar. Clearly, if $\Gamma(P_1 \times P_2)$ is outerplanar, then, by Lemmas 2.8 and 4.1, $|A(P_1)| + |A(P_2)| \leqslant 3$. In the next two Theorems, we investigate the cases $|A(P_1)| + |A(P_2)| = 2$ and $|A(P_1)| + |A(P_2)| = 3$.

Theorem 4.4. *Suppose that $|A(P_1)| + |A(P_2)| = 2$ such that $|A(P_1)| = 1 = |A(P_2)|$. Then $\Gamma(P_1 \times P_2)$ is outerplanar if and only if $|P_i| \leqslant 2$ or, $|P_j| \leqslant 3$ with $|P_i| \leqslant 2$, for some $1 \leqslant i \neq j \leqslant 2$.*

Proof. Since $|A(P_1)| + |A(P_2)| = 2$, we have that $\Gamma(P_1 \times P_2)$ is a complete bipartite graph. Now one can easily see that $\Gamma(P_1 \times P_2)$ is an outerplanar graph if and only if $|P_i| \leqslant 2$ or, $|P_j| \leqslant 3$ and $|P_i| \leqslant 2$, for some $1 \leqslant i \neq j \leqslant 2$. $\square$

Now, suppose that P_1 and P_2 are posets such that $|A(P_1)| = 1$ and $|A(P_2)| = 2$. If $|P_i| \geqslant 3$ and $|P_j| \geqslant 4$, for all $1 \leqslant i \neq j \leqslant 2$, then we can find a copy of $K_{2,3}$ in the graph $\Gamma(P_1 \times P_2)$. Thus $\Gamma(P_1 \times P_2)$ is not outerplanar. Therefore, if $\Gamma(P_1 \times P_2)$ is outerplanar, then $|P_1| = 2$, or $|P_2| = 3$ with $|P_1| \leqslant 3$. Now, in the following two cases, we study the outerplanarity of $\Gamma(P_1 \times P_2)$ whenever $|P_1| = 2$, or $|P_1| \leqslant 3$ with $|P_2| = 3$.

Case 1. Suppose that $|P_1| = 2$ and $|U_i^{P_2}| \geqslant 2$, for some $1 \leqslant i \leqslant 2$. In this case we can find a subdivision of K_4 as in Figure 16, where $y_i \in U_i^{P_2} \backslash \{b_i\}$, and so $\Gamma(P_1 \times P_2)$ is not outerplanar.

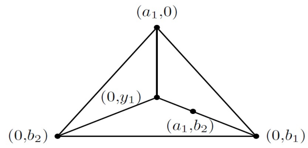

Figure 16.

Now, if $|U_i^{P_2}| = 1$, for all $1 \leqslant i \leqslant 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 17, and so $\Gamma(P_1 \times P_2)$ is outerplanar.

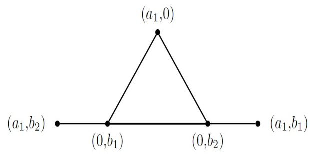

Figure 17.

Case 2. Suppose that $|P_2| = 3$ and $|P_1| \leqslant 3$. If $|P_1| = 3$. Then $\Gamma(P_1 \times P_2)$ is pictured in Figure 18, where $x \in P_1 \backslash \{0, a_1\}$, and so it is outerplanar.

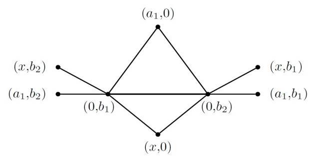

Figure 18.

If $|P_1| = 2$, then $\Gamma(P_1 \times P_2)$ is pictured in Figure 17, and so it is outerplanar.

Theorem 4.5. *Suppose that $|A(P_1)| + |A(P_2)| = 3$ such that $|A(P_1)| = 1$ and $|A(P_2)| = 2$. Then $\Gamma(P_1 \times P_2)$ is outerplanar if and only if one of the following conditions hold.*

(*i*) $|P_1| = 2$ *and* $|U_i^{P_2}| = 1$, *for all* $1 \leqslant i \leqslant 2$.

(*ii*) $|P_2| = 3$ *and* $|P_1| \leqslant 3$.

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge. The number frank(G) is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number rank(G)=$q - n + r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [10, Proposition 2.2], we have rank$(G) \leqslant$ frank (G). A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [10]).

(i) rank(G)= frank(G),

(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 4.2 and Theorem 4.3 we have the following result.

Theorem 4.6. *The zero-divisor graph* $\Gamma(P)$ *is a ring graph if and only if it is an outerplanar graph.*

5. End-regularity of zero-divisor graphs of posets

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (resp, an isomorphism) from G to itself is called an *endomorphism* (resp, *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By End(G), we denote the set of all the endomorphisms of G. It is well-known that End(G) is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called regular if every element in S is regular. Also, a graph G is called *end-regular* if End(G) is regular.

Now, we recall the following Lemma from [18].

Lemma 5.1. [18, Lemma 2.1] *Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying* $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, *then G is not end-regular.*

Lemma 5.2. *Suppose that* $|A(P)| \geqslant 3$. *If* $U_{i\ldots j}, U_{i\ldots j\ldots k} \neq \emptyset$, *such that* $|U_{i\ldots j}| > 1$, *for some* $1 \leqslant i < j < k < n$, *or* $U_{i\ldots j}$, $U_{i\ldots j\ldots k}, U_{i\ldots j\ldots k\ldots t} \neq \emptyset$, *for some* $1 \leqslant i < j < k < t < n$, *then* $\Gamma(P)$ *is not end-regular.*

Proof. First suppose that $U_{i\dots j}, U_{i\dots j\dots k} \neq \emptyset$ and $|U_{i\dots j}| > 1$, for some $1 \leqslant i < j < k < n$. Let $a, b \in U_{i\dots j}$ and $c \in U_{i\dots j\dots k}$. Then $N(c) \subsetneqq \mathrm{N}(a)$, since $a_k \in \mathrm{N}(a)\backslash\mathrm{N}(c)$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, and so, by Lemma 5.1, $\Gamma(P)$ is not end-regular. If $U_{i\dots j}$, $U_{i\dots j\dots k}, U_{i\dots j\dots k\dots t} \neq \emptyset$, for some $1 \leqslant i < j < k < t < n$, then consider the elements $a \in U_{i\dots j}$, $b \in U_{i\dots j\dots k}$ and $c \in U_{i\dots j\dots k\dots t}$. Now, we have $\mathrm{N}(c) \subsetneqq \mathrm{N}(b) \subseteq \mathrm{N}(a)$. Hence $\Gamma(P)$ is not end-regular. □

Proposition 5.3. *Suppose that $|A(P)| = 2$. Then $\Gamma(P)$ is end-regular.*

Proof. Clearly $\Gamma(P)$ is a complete bipartite graph. Now, by [22, Theorem 3.4], we have that $\Gamma(P)$ is end-regular. □

Lemma 5.4. *Suppose that $x, y \in Z(P)$. Then $N(x) \subseteq N(y)$ if and only if $Z_x \subseteq Z_y$ and $\{x, y\}^\ell \neq \{0\}$.*

Proof. First assume that $\mathrm{N}(x) \subseteq \mathrm{N}(y)$. Then $Z_x \subseteq Z_y$. Also, suppose to the contrary that $\{x, y\}^\ell = \{0\}$. Then x is adjacent to y. This means that $y \in \mathrm{N}(x) \subseteq \mathrm{N}(y)$, and so $y \in \mathrm{N}(y)$, which is a contradiction.

Conversly, one can easy to see that result holds. □

Proposition 5.5. *Suppose that $P = P_1 \times P_2 \times \cdots \times P_n$. Then we have the following statements.*

(*i*) *If $n \geqslant 3$, then $\Gamma(P_1 \times P_2 \times \cdots \times P_n)$ is not end-regular.*

(*ii*) *If $|A(P_1)| = 1 = |A(P_2)|$, then $\Gamma(P_1 \times P_2)$ is end-regular.*

Proof. (*i*) Suppose that $A(P_1) = \{a_1, a_2, \dots, a_n\}$, $A(P_2) = \{b_1, b_2, \dots, b_m\}$ and $A(P_3) = \{c_1, c_2, \dots, c_t\}$, where $m, n, t \geqslant 1$.

Set $x := (a_i, 0, \dots, 0)$, $y := (a_i, b_j, 0, \dots, 0)$ and $z := (a_i, b_j, c_k, 0, \dots, 0)$, for some $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant t$. Then $\mathrm{N}(z) \subsetneqq \mathrm{N}(y) \subsetneqq \mathrm{N}(x)$. Now, by Lemmas 5.1 and 5.4, $\Gamma(P)$ is not end-regular.

(*ii*) Note that in this case, $\Gamma(P_1 \times P_2)$ is a complete bipartite graph and, by [22, Theorem 3.4], every complete bipartite graph is end-regular. □

Lemma 5.6. *Assume that $\Gamma(P_2)$ has distinct vertices x and y such that $x, y \notin A(P_2)$ and $\mathrm{N}(x) \subseteq \mathrm{N}(y)$. Then $\Gamma(P_1 \times P_2)$ is not end-regular.*

Proof. Suppose that $b \in A(P_2)$. Then it follows from the fact that $\mathrm{N}(0, b) \subsetneqq \mathrm{N}(0, x) \subsetneqq \mathrm{N}(0, y)$. □

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