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Continuous homomorphisms, the left-gyroaddition action and topological quotient gyrogroups

Watchareepan Atiponrat and Rasimate Maungchang

Abstract. Recently, many properties of gyrogroups have been discovered. In this work, we investigate some properties of topological gyrogroups, specifically, the continuity of some homomorphisms, the canonical decomposition, and the continuity of the left-gyroaddition action.

1. Introduction

A gyrogroup is a generalization of a group of which the associative law is replaced by a more generalized version called, the left gyroassociative law and an additional property called, the left loop property, see Section 2 for more details and examples. Its structures were discovered by A. A. Ungar from the study of the Einstein velocity addition, see [13] and the references therein. Since then, many properties of gyrogroups have been discovered by active researchers in the field, see [3], [4], [7], [8], [9], [11], [12], [14]. A large portion of its algebraic properties was studied by T. Suksumran, for example, the isomorphism theorems, Cayley's Theorem, Lagrange's Theorem, gyrogroup actions, etc., see [7], [8], [11]. He is now extending his study to metric aspect of the gyrogroups, see [10].

From the topological aspect, W. Atiponrat, R. Maungchang, and T. Suksumran have been studying the separation axioms of the topological gyrogroups, see [1], [2], [15]. In this work, we continue the study of topological gyrogroups, in particular, we investigate the continuity of some homomorphisms, the canonical decomposition, and the continuity of the left-gyroaddition action.

2. Definitions and background

In this section, we include basic definitions, examples, and theorems involving the topological gyrogroups. Readers are recommended to see [1], [8], [11], and [14] for further details and examples.

Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be groupoids. A function $f : G_1 \to G_2$ is a called a *homomorphism* if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any $x, y \in G_1$. A bijective homomorphism is called an *isomorphism*. An isomorphism of a groupoid $(G, \oplus)$

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to itself is called a *groupoid automorphism* and we denote the set of all groupoid automorphisms of a groupoid $(G, \oplus)$ by $\mathrm{Aut}(G, \oplus)$.

Definition 2.1 (Definition 2.7 of [14])**.** Let $(G, \oplus)$ be a nonempty groupoid. We say that $(G, \oplus)$ or just G (when it is clear from the context) is a *gyrogroup* if the following hold:

1. There is a unique identity element $0_G \in G$ such that
$$0_G \oplus x = x = x \oplus 0_G \qquad \text{for all } x \in G;$$
2. For each $x \in G$, there exists a unique *inverse* element $\ominus x \in G$ such that
$$\ominus x \oplus x = 0_G = x \oplus (\ominus x);$$
3. For any $x, y \in G$, there exists $\mathrm{gyr}[x, y] \in \mathrm{Aut}(G, \oplus)$ such that
$$x \oplus (y \oplus z) = (x \oplus y) \oplus \mathrm{gyr}[x, y](z)$$
for all $z \in G$; (left gyroassociative law)
4. For any $x, y \in G$, $\mathrm{gyr}[x \oplus y, y] = \mathrm{gyr}[x, y]$. (left loop property)

We give an example of a gyrogroup which is not a group. It is called a *Möbius gyrogroup*.

Example 2.2. Let $\mathbb{D}$ be the complex open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Define a *Möbius addition* $\oplus_M : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ by
$$a \oplus_M b = \frac{a + b}{1 + \bar{a}b},$$
for all $a, b \in \mathbb{D}$. This map is well defined and its image lies in $\mathbb{D}$, see Theorem 5.5.2 of [5] for the proof. It is obvious that 0 is the identity and $-a$ is the inverse of a, for any $a \in \mathbb{D}$. $(\mathbb{D}, \oplus_M)$ is not a group because the associative property does not hold. For example, if $a = 1/2, b = i/2$, and $c = -1/2$, then $a \oplus_M (b \oplus_M c) = (10+15i)/26$ but $(a \oplus_M b) \oplus_M c = (8 + 15i)/34$. However, $(\mathbb{D}, \oplus_M)$ is a gyrogroup with
$$\mathrm{gyr}[a, b](c) = \frac{1 + a\bar{b}}{1 + \bar{a}b}c \qquad \text{for any } a, b, c \in \mathbb{D},$$
as proved in section 3.4 of [14].

Adding a topology to a gyrogroup motivates the following definition.

Definition 2.3 (Definition 1 of [1])**.** A triple $(G, \mathcal{T}, \oplus)$ is called a *topological gyrogroup* if and only if

1. $(G, \mathcal{T})$ is a topological space;

2. $(G, \oplus)$ is a gyrogroup; and

3. The binary operation $\oplus : G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology, and the operation of taking the inverse, i.e., $\ominus(\,\cdot\,) \colon G \to G,\ \ x \mapsto \ominus x$, is continuous.

Sometimes we will just say that G is a topological gyrogroup if the binary operation and the topology are clear from the context.

From the previous example, if we consider $\mathbb{D}$ as a subspace of $\mathbb{C}$ endowed with the standard topology, then it can be shown that $\oplus_M$ and $\ominus_M$ are continuous. So $\mathbb{D}$ is a topological gyrogroup.

The following are some basic algebraic and topological properties of gyrogroups and topological gyrogroups which will be needed later in our work.

Proposition 2.4 (Proposition 6 of [11]). *Suppose $(G, \oplus)$ is a gyrogroup and $A \subseteq G$. Then the following are equivalent:*

1. $\mathrm{gyr}[x, y](A) \subseteq A$ *for all* $x, y \in G$.

2. $\mathrm{gyr}[x, y](A) = A$ *for all* $x, y \in G$.

Lemma 2.5 (Proposition 32 of [7]). *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be gyrogroups, and let $f : G_1 \to G_2$ be a homomorphism. Then the following are true:*

1. $f(0_{G_1}) = 0_{G_2}$.

2. *For any* $x \in G_1$, $f(\ominus_1 x) = \ominus_2 f(x)$.

Following the notation in [14], for any pair of elements x, y in a gyrogroup $(G, \oplus)$, we let $x \boxplus y$ denote $x \oplus \mathrm{gyr}[x, \ominus y](y)$, and let $x \boxminus y$ denote $x \oplus \mathrm{gyr}[x, y](\ominus y)$.

Theorem 2.6 (Theorem 2.10, 2.22 and 2.35 of [14]). *Let $(G, \oplus)$ be a gyrogroup. For any $x, y, z \in G$, the following are true:*

1. $(\ominus x) \oplus (x \oplus y) = y$. (left cancellation law)

2. $(x \boxminus y) \oplus y = x$. (right cancellation law)

3. $\mathrm{gyr}[x, y](z) = \ominus(x \oplus y) \oplus (x \oplus (y \oplus z))$. (gyrator identity)

4. $(x \oplus y) \oplus z = x \oplus (y \oplus \mathrm{gyr}[y, x](z))$. (right gyroassociative law)

Akin to the case of topological groups, topological gyrogroups admit the following properties.

Proposition 2.7 (Proposition 3 of [1]). *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup. Then, for each $a \in G$, the maps $x \mapsto x \oplus a$ and $x \mapsto a \oplus x$, where $x \in G$, are homeomorphisms.*

Proposition 2.8 (Corollary 5 of [1]). *Suppose that $(G, \mathcal{T}, \oplus)$ is a topological gyrogroup, $x \in G$, and $A, B \subseteq G$. Then the following are true:*

1. *A is open if and only if $\ominus A$, $x \oplus A$ and $A \oplus x$ are open.*

2. *If A is open, then $A \oplus B$ and $B \oplus A$ are open.*

Next we introduce subgyrogroups and necessary concepts. This also leads us to the definition of quotient gyrogroups and the left-gyroaddition action.

Definition 2.9 (Section 4 of [11]). Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then H is called a *subgyrogroup* of G and denoted by $H \leqslant G$ if $(H, \oplus_{|H \times H})$ is a gyrogroup and $\mathrm{gyr}[a, b]|_H \in \mathrm{Aut}(H, \oplus_{|H\times H})$ for all $a, b \in H$.

A subgyrogroup H is called an *L-subgyrogroup* and denoted by $H \leqslant_L G$ if

$$\mathrm{gyr}[a, h](H) = H,$$

for all $a \in G, h \in H$.

It is easy to see that $\{0\}$ is trivially an L-subgyrogroup. For a nontrivial example, see Example 18 of [11].

Proposition 2.10 (Proposition 14 of [11]). *Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then $H \leqslant G$ if and only if $\ominus h \in H$ and $h \oplus k \in H$ for all $h, k \in H$.*

Lemma 2.11. *Let H be a subgyrogroup of a gyrogroup $(G, \oplus)$. Then $h \oplus H = H$ for each $h \in H$.*

Proof. Let $h \in H$. By Proposition 2.10, $h \oplus H \subseteq H$. On the other hand, if $k \in H$, then $k = h \oplus (\ominus h \oplus k)$ by the left cancellation law. Again, by Proposition 2.10, $\ominus h \oplus k \in H$ so $k = h \oplus (\ominus h \oplus k) \in h \oplus H$ which implies $H \subseteq h \oplus H$. As a result, $h \oplus H = H$. $\square$

When H is a subgyrogroup of a gyrogroup $(G, \oplus)$, we use the notation G/H to stand for the set of all left cosets of H, i.e. $G/H = \{x \oplus H : x \in G\}$. The notion of L-subgyrogroups enables us to work with the set of all left cosets easily.

Proposition 2.12 (Proposition 19 of [11]). *Let H be an L-subgyrogroup of a gyrogroup $(G, \oplus)$. Then, for any $a, b \in G$, $a \oplus H = b \oplus H$ if and only if $\ominus a \oplus b \in H$.*

Proposition 2.13 (Proposition 20 of [11]). *Let H be an L-subgyrogroup of a gyrogroup $(G, \oplus)$. Then the set $G/H = \{x \oplus H : x \in G\}$ forms a partition of G.*

Being a subgyrogroup and an L-subgyrogroup are preserved by homomorphisms in the following sense.

Proposition 2.14 (Proposition 24 of [11]). *Let $f : G \to H$ be a homomorphism between gyrogroups, and let $K \leqslant G$. Then $f(K) \leqslant H$. Moreover, if $K \leqslant_L G$ and f is surjective, then $f(K) \leqslant_L H$.*

Proposition 2.15 (Proposition 25 of [11]). *Let $f : G \to H$ be a homomorphism between gyrogroups, and let $K \leqslant H$. Then $f^{-1}(K) \leqslant G$. Moreover, if $K \leqslant_L H$, then $f^{-1}(K) \leqslant_L G$. In particular,* $\ker f \leqslant_L G$.

Upcoming, trying to obtain a nice object like normal subgroups, we define normal subgyrogroups which allow a familiar binary operation on the set of all left cosets.

Definition 2.16 (Section 5 of [11]). Let H be a nonempty subset of a gyrogroup $(G, \oplus)$. Then H is called a *normal subgyrogroup* of G and denoted by $H \trianglelefteq G$ if $H = \ker f$ for some homomorphism $f : G \to K$ where K is a gyrogroup.

Lemma 2.17 (the paragraph after Proposition 25 of [11]). *Let $(G, \oplus)$ be a gyrogroup. If $H \trianglelefteq G$, then $\mathrm{gyr}[x, y](H) = H$ for all $x, y \in G$. In particular, H is an L-subgyrogroup of G.*

Theorem 2.18 (Theorem 27 of [11]). *Let $(G, \oplus)$ be a gyrogroup, and let $H \trianglelefteq G$. Then the function $\bigoplus : G/H \times G/H \to G/H$ defined by $(x \oplus H, y \oplus H) \mapsto (x \oplus y) \oplus H$ is a binary operation. Furthermore, $(G/H, \bigoplus)$ becomes a gyrogroup such that H is the identity element and $\ominus x \oplus H$ is the inverse of $x \oplus H$ for each $x \oplus H \in G/H$.*

Definition 2.19 (Section 5 of [11]). Let $(G, \oplus)$ be a gyrogroup, and let $H \trianglelefteq G$. The gyrogroup $(G/H, \bigoplus)$ in Theorem 2.18 is called the *quotient gyrogroup*, and the function $q : G \to G/H$ such that $x \mapsto x \oplus H$ is called a *canonical projection*.

Theorem 2.20 (Theorem 28 of [11] (The first isomorphism theorem)). *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be gyrogroups, and let $f : G \to H$ be a homomorphism. Then the map $g \oplus \ker f \mapsto f(g)$ gives rise to an isomorphism between $G/\ker f$ and $f(G)$.*

We end this section with the definition of the left-gyroaddition action.

Definition 2.21 (Definition 3.1 of [8]). Let $(G, \oplus)$ be a gyrogroup, and let X be a set. A function $\cdot : G \times X \to X$, written $\cdot((a, x)) = a \cdot x$, is a (gyrogroup) *action* of G on X if

1. $0_G \cdot x = x$ for all $x \in X$, and
2. $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$.

Theorem 2.22 (Theorem 4.5 of [8]). *Let H be a subgyrogroup of $(G, \oplus)$. Then the function $\cdot : G \times G/H \to G/H$ such that for all $g \in G, x \oplus H \in G/H$,*

$$g \cdot (x \oplus H) = (g \oplus x) \oplus H$$

defines a gyrogroup action of G on G/H if and only if

$$\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$$

for all $a, b \in G, x \oplus H \in G/H$.

Definition 2.23 (Definition 4.4 of [8]). Following the language of Theorem 2.22, the function $\cdot : G \times G/H \to G/H$ is called the *left-gyroaddition action* if it is a gyrogroup action.

3. Continuous homomorphisms

In this section, we prove the continuity of some homomorphisms and the canonical decomposition of topological gyrogroups.

Proposition 3.24. *Let $(G_1, \mathcal{T}_1, \oplus_1)$ and $(G_2, \mathcal{T}_2, \oplus_2)$ be topological gyrogroups. Let $f : G_1 \to G_2$ be a homomorphism. Then f is continuous if and only if it is continuous at 0_{G_1}.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $x \in G_1$. If U is a neighborhood of $f(x)$, then $\ominus_2 f(x) \oplus_2 U$ is a neighborhood of 0_{G_2} by Proposition 2.8. So there is a neighborhood W of 0_{G_1} such that $f(W) \subseteq \ominus_2 f(x) \oplus_2 U$. As a result, $x \oplus_1 W$ is a neighborhood of x such that $f(x \oplus_1 W) = \{f(x \oplus_1 w) : w \in W\} = \{f(x) \oplus_2 f(w) : w \in W\} = f(x) \oplus_2 f(W) \subseteq f(x) \oplus_2 (\ominus_2 f(x) \oplus_2 U) = \{f(x) \oplus_2 (\ominus_2 f(x) \oplus_2 u) : u \in U\} = U$ by the left cancellation law (see Theorem 1). Hence f is continuous at x. Since x is arbitrary, f is continuous. □

Lemma 3.25. *Let H be a subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that* $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ *for all $a, b \in G, x \oplus H \in G/H$ [or let $H \trianglelefteq G$]. Suppose G/H is equipped with the quotient topology induced by q. Then the canonical projection $q : G \to G/H$ is a continuous open map.*

Proof. Since G/H is endowed with the quotient topology induced by q, q is continuous. Next, let $U \subseteq G$ be an open set. Then $q(U) = \{u \oplus H : u \in U\}$. We will show that $q^{-1}(q(U)) = U \oplus H$. If $a \in q^{-1}(q(U))$, then $q(a) = a \oplus H = u \oplus H$ for some $u \in U$. As a result, $\ominus u \oplus a \in H$ by Proposition 2.12. Thus $\ominus u \oplus a = h$ for some $h \in H$ so $a = u \oplus h \in U \oplus H$ by the left cancellation law. On the other hand, if $x \in U \oplus H$, then $x = v \oplus k$ for some $v \in U$, $k \in H$. We obtain that $q(x) = x \oplus H = (v \oplus k) \oplus H = v \oplus (k \oplus \mathrm{gyr}[k, v](H)) = v \oplus (k \oplus H) = v \oplus H \in q(U)$; the fourth and fifth equalities come from our assumption together with Proposition 2.4 [or come from Lemma 2.17 for the case $H \trianglelefteq G$] and Lemma 2.11. So $x \in q^{-1}(q(U))$ and we can conclude that $q^{-1}(q(U)) = U \oplus H$ which is an open set by Proposition 2.8. Hence q is an open map. □

Theorem 3.26. (Canonical decomposition) *Let $(G_1, \mathcal{T}_1, \oplus_1)$ and $(G_2, \mathcal{T}_2, \oplus_2)$ be topological gyrogroups. Let $f : G_1 \to G_2$ be a continuous homomorphism. Then the following are true:*

$$\begin{array}{ccc} G_1 & \xrightarrow{f} & G_2 \\ q \big\downarrow & & \big\downarrow i \\ G_1/\ker f & \xrightarrow[\widetilde{f}]{} & f(G_1) \end{array}$$

(1) *The above diagram commutes where* $q : G_1 \to G_1/\ker f$ *is the canonical projection,* $\widetilde{f} : G_1/\ker f \to f(G_1)$ *is a function defined by* $g \oplus_1 \ker f \mapsto f(g)$ *for all* $g \in G_1$*, and* $i : f(G_1) \to G_2$ *is the inclusion map.*

(2) $i : f(G_1) \to G_2$ *is an injective continuous homomorphism, and* $\widetilde{f}$ *is a continuous isomorphism.*

(3) f *is an open map if and only if* $f(G_1)$ *is open in* G_2 *and* $\widetilde{f}$ *is an open map.*

(4) $\widetilde{f}$ *is an open map if and only if* $f(U)$ *is open in* $f(G_1)$ *for all open subset* U *of* G_1.

Proof. To see (1), we first show that $\widetilde{f}$ is well defined. If $a, b \in G$ are so that $a \oplus_1 \ker f = b \oplus_1 \ker f$, then $\ominus_1 b \oplus_1 a \in \ker f$ by Proposition 2.12. Thus $f(\ominus_1 b) \oplus_2 f(a) = f(\ominus_1 b \oplus_1 a) = 0_{G_2}$ so $\ominus_2 f(\ominus_1 b) = f(a)$ by the left cancellation law. Hence $f(b) = f(a)$ by Lemma 2.5. Next, the diagram commutes because for any $a \in G_1$, $f(a) = i(f(a)) = i(\widetilde{f}(a \oplus_1 \ker f)) = i(\widetilde{f}(q(a)))$.

To prove (2), i is injective and continuous because it is a restriction of the identity map. Moreover, it is a homomorphism since $f(G_1)$ is a gyrogroup by Proposition 2.14. On the other hand, $\widetilde{f}$ is an isomorphism by the first isomorphism theorem. Next, we show that $\widetilde{f}$ is continuous. Let U be an open subset of $f(G_1)$. Then there is an open subset W of G_2 such that $U = W \cap f(G_1)$. Since f is continuous, $f^{-1}(W)$ is open in G_1. Then $q(f^{-1}(W))$ is an open subset of $G_1/\ker f$ by Lemma 3.25. Now observe that

$$\begin{aligned}\widetilde{f}^{-1}(U) &= \widetilde{f}^{-1}(W \cap f(G_1)) = \widetilde{f}^{-1}(i^{-1}(W \cap f(G_1))) = \widetilde{f}^{-1}(i^{-1}(f(f^{-1}(W)))) \\ &= \widetilde{f}^{-1}(i^{-1}((i \circ \widetilde{f} \circ q)(f^{-1}(W)))) = q(f^{-1}(W)).\end{aligned}$$

So $\widetilde{f}^{-1}(U)$ is open in $G_1/\ker f$, and hence $\widetilde{f}$ is continuous.

Now we prove (3). ($\Rightarrow$): Suppose that f is an open map. Then $f(G_1)$ is open in G_2. To see that $\widetilde{f}$ is an open map, let U be an open subset of $G_1/\ker f$. Since q is continuous, $q^{-1}(U)$ is open. Moreover, $f(q^{-1}(U))$ is open because f is an open map. Then $\widetilde{f}(U) = (i^{-1} \circ f \circ q^{-1})(U)$ is open because i, q are continuous.
($\Leftarrow$): Let $f(G_1)$ be open in G_2, and let $\widetilde{f}$ be an open map. We will show that f is an open map. Let U be an open subset of G_1. Then $(\widetilde{f} \circ q)(U)$ is open in $f(G_1)$ because q and $\widetilde{f}$ are open maps. Since $f(G_1)$ is open in G_2, $(\widetilde{f} \circ q)(U)$ is open in G_2. Notice that $f(U) = (i \circ \widetilde{f} \circ q)(U) = i((\widetilde{f} \circ q)(U)) = (\widetilde{f} \circ q)(U)$. Hence $f(U)$ is open in G_2 which implies that f is an open map.

Finally, we prove (4). ($\Rightarrow$): Assume that $\widetilde{f}$ is an open map. Let U be an open subset of G_1. Then $(\widetilde{f} \circ q)(U)$ is open in $f(G_1)$ because q and $\widetilde{f}$ are open maps. Observe that $f(U) = i((\widetilde{f} \circ q)(U)) = (\widetilde{f} \circ q)(U)$. So $f(U)$ is open in $f(G_1)$.
($\Leftarrow$): Suppose that $f(U)$ is open in $f(G_1)$ for all open subset U of G_1. To see that $\widetilde{f}$ is an open map, let W be an open subset of $G_1/\ker f$. Then $(i^{-1} \circ f \circ q^{-1})(W) =$

$(f \circ q^{-1})(W) = f(q^{-1}(W))$ is open in $f(G_1)$ by the assumption and the fact that q is continuous. Since $\widetilde{f}(W) = (i^{-1} \circ f \circ q^{-1})(W)$, $\widetilde{f}$ is an open map. □

4. Action and topological quotient gyrogroups

In our last section, we consider the set of all left cosets of an L-subgyrogroup H in a topological gyrogroup $(G, \mathcal{T}, \oplus)$. According to Proposition 2.13, we can assign the quotient topology induced by canonical projection to G/H and study the continuity of the left-gyroaddition action $\cdot : G \times G/H \to G/H$ where $G \times G/H$ is endowed with the product topology. In addition, if $H \trianglelefteq G$, then $(G/H, \bigoplus)$ is a gyrogroup so we can examine the continuity of $\bigoplus$.

From now on, let $\mathfrak{T}$ denote the quotient topology induced by the canonical projection $q : G \to G/H$. In addition, we will assume that G/H is endowed with $\mathfrak{T}$ in our proof when the topology is needed to be specify. We begin this section by providing some basic facts of G/H in the following proposition which the proof in topological group version can be adopted.

Proposition 4.1. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \leqslant G$ be such that $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$. Then the following are equivalent:*

1. *$(G/H, \mathfrak{T})$ is T_2.*
2. *$(G/H, \mathfrak{T})$ is T_1.*
3. *H is a closed subset of G.*

Proof. $(1 \Rightarrow 2)$: Trivial.

$(2 \Rightarrow 3)$: Observe that $H = q^{-1}(\{H\})$ because of Lemma 2.11 and Proposition 2.12. Since q is continuous and $\{H\}$ is closed because $(G/H, \mathfrak{T})$ is T_1, we gain the result.

$(3 \Rightarrow 1)$: We will show that the set $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ is closed in $G/H \times G/H$ together with the product topology. Notice that $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} \subseteq \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\}$ by Proposition 2.12. On the other hand, $\{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\} \subseteq \{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ by the fact that $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$, Proposition 2.4 and, again, Proposition 2.12. So $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} = \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\}$. Next, observe that $G/H \times G/H - \{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\} = G/H \times G/H - \{(x \oplus H, y \oplus H) : \ominus x \oplus y \in H\} = \{(x \oplus H, y \oplus H) : \ominus x \oplus y \notin H\} = (q \times q) \circ (\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$, where $q \times q$ is the product of two open quotient maps and $Id : G \to G$ is the identity function. Since $\oplus$ is continuous and H is closed, $\oplus^{-1}(G - H)$ is open. Moreover, $\ominus(\cdot) \times Id : G \times G \to G \times G$ is a homeomorphism so $(\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$ is open. Finally, it is a well-known fact in topology that the product of two open maps is an open map. Hence $(q \times q) \circ (\ominus(\cdot) \times Id) \circ (\oplus^{-1})(G - H)$ is open. This implies that $\{(x \oplus H, y \oplus H) : x \oplus H = y \oplus H\}$ is closed. □

Lemma 4.2. *Let H be a subgyrogroup of a gyrogroup $(G, \oplus)$, $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$. Then, for all $a \in G$ and $x \oplus H, y \oplus H \in G/H$, $(a \oplus x) \oplus H = (a \oplus y) \oplus H$ if and only if $x \oplus H = y \oplus H$.*

Proof. ($\Leftarrow$): Use Theorem 2.22.

($\Rightarrow$): Suppose $(a \oplus x) \oplus H = (a \oplus y) \oplus H$. We will show that $\ominus y \oplus x \in H$ which implies $x \oplus H = y \oplus H$. Let $(a \oplus x) \oplus h_1 \in (a \oplus x) \oplus H$. By assumption, $\mathrm{gyr}[a,b](H) \subseteq H$, for all $a, b \in G$. So $\mathrm{gyr}[a,b](H) = H$, for all $a, b \in G$, by Proposition 2.4. Then, for some $h_2, h_3, h_4, h_5 \in H$,

$$\begin{aligned}
(a \oplus x) \oplus h_1 &= (a \oplus y) \oplus h_2,\\
a \oplus (x \oplus h_3) &= a \oplus (y \oplus h_4),\\
x \oplus h_3 &= y \oplus h_4,\\
\ominus y \oplus (x \oplus h_3) &= h_4,\\
(\ominus y \oplus x) \oplus h_5 &= h_4,\\
\ominus y \oplus x &= h_4 \boxminus h_5.
\end{aligned}$$

Moreover, $h_4 \boxminus h_5 = h_4 \oplus \mathrm{gyr}[h_4, h_5](\ominus h_5) \in H$. Hence $\ominus y \oplus x \in H$. □

Theorem 4.3. *Let H be a subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$. Then the left-gyroaddition action $\cdot : G \times G/H \to G/H$ is transitive. Furthermore, for each $a \in G$, the function $f_a : G/H \to G/H$ defined by $f_a(x \oplus H) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for all $x \oplus H \in G/H$ is a homeomorphism.*

Proof. To begin with, we show that the action is transitive. Let $x \oplus H, y \oplus H \in G/H$. Then $(y \boxminus x) \cdot (x \oplus H) = ((y \boxminus x) \oplus x) \oplus H = y \oplus H$, by the right cancellation law.

Next, we prove the last sentence of the theorem. Let $a \in G$. We first show that the function $f_a : G/H \to G/H$ defined by $f_a(x \oplus H) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for each $x \oplus H \in G/H$ is a continuous bijection. Lemma 4.2 shows that f_a is injective. Moreover, for any $x \oplus H \in G/H$, $f_a((\ominus a \oplus x) \oplus H) = (a \oplus (\ominus a \oplus x)) \oplus H = x \oplus H$. So f_a is bijective. To see the continuity of f_a, let $L_a : G \to G$ be such that $L_a(x) = a \oplus x$ for all $x \in G$. Then L_a is a homeomorphism by Proposition 2.7. Observe that $q \circ L_a = f_a \circ q$ where $q : G \to G/H$ is the canonical projection. So, for each open set $U \subseteq G/H$, we have $f_a^{-1}(U) = q(L_a^{-1}(q^{-1}(U)))$ which is open by Lemma 3.25. We conclude that f_a is a continuous bijection. It is not hard to check that $f_a^{-1} = f_{\ominus a}$ which is a continuous bijection by similar proof. Thus f_a is a homeomorphism. □

In some special occasion, the continuity of the left-gyroaddition action is established.

Theorem 4.4. *Suppose that H is a compact subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that $\mathrm{gyr}[a,b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$.*

Then the left-gyroaddition action of G on G/H is transitive. Moreover, it is continuous when $G \times G/H$ is endowed with the product topology.

Proof. The action is transitive by Theorem 4.3. Next, we show that the map $\cdot : G \times G/H \to G/H$ defined by $\cdot((a, x \oplus H)) = a \cdot (x \oplus H) = (a \oplus x) \oplus H$ for all $a \in G, x \oplus H \in G/H$ is continuous when the topology on $G \times G/H$ is the product topology. Suppose $(a, x \oplus H) \in G \times G/H$. Let $U \subseteq G/H$ be an open set containing $\cdot((a, x \oplus H)) = (a \oplus x) \oplus H$. Observe that $a \oplus (x \oplus H) = (a \oplus x) \oplus \mathrm{gyr}[a, x](H) = (a \oplus x) \oplus H$ by our assumption and Proposition 2.4. Moreover, $q((a \oplus x) \oplus H) = q(\{(a \oplus x) \oplus h : h \in H\}) = \{((a \oplus x) \oplus h) \oplus H : h \in H\} = \{(a \oplus x) \oplus (h \oplus \mathrm{gyr}[h, a \oplus x](H)) : h \in H\} = \{(a \oplus x) \oplus (h \oplus H) : h \in H\} = \{(a \oplus x) \oplus H : h \in H\} \subseteq U$; the fourth and fifth equalities come from our assumption together with Proposition 2.4 and Lemma 2.11. So $a \oplus (x \oplus H) = (a \oplus x) \oplus H \subseteq q^{-1}(U)$ which is an open set because q is continuous. Thus, for each $h \in H$, there are open sets U_h, V_h of G such that $a \in U_h$, $x \oplus h \in V_h$, and $U_h \oplus V_h \subseteq q^{-1}(U)$ because $\oplus$ is continuous. It is clear that $x \oplus H \subseteq \bigcup_{h \in H} V_h$. Since H is compact, $x \oplus H$ is compact by Proposition 2.7. Hence $x \oplus H \subseteq V_{h_1} \cup ... \cup V_{h_l}$ for some $h_1, ..., h_l \in H, l \in \mathbb{N}$. Let $\widetilde{U} = U_{h_1} \cap ... \cap U_{h_l}$ and $\widetilde{V} = V_{h_1} \cup ... \cup V_{h_l}$. Then $\widetilde{U} \oplus \widetilde{V} \subseteq q^{-1}(U)$, $a \in \widetilde{U}$ and $x \oplus H \subseteq \widetilde{V}$ where $\widetilde{U}, \widetilde{V}$ are open in G. Notice that $x \in x \oplus H \subseteq \widetilde{V}$ which implies $x \oplus H = q(x) \in q(\widetilde{V})$. Moreover, $q(\widetilde{V})$ is open by Lemma 3.25. Hence $\widetilde{U} \times q(\widetilde{V})$ is a neighborhood of $(a, x \oplus H)$ such that

$$\begin{aligned}
\cdot(\widetilde{U} \times q(\widetilde{V})) &= \{u \cdot q(v) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{u \cdot (v \oplus H) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{(u \oplus v) \oplus H : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= \{q(u \oplus v) : u \in \widetilde{U} \text{ and } v \in \widetilde{V}\} \\
&= q(\widetilde{U} \oplus \widetilde{V}) \subseteq q(q^{-1}(U)) = U.
\end{aligned}$$

We conclude that the action is continuous. □

Next, we will explore the continuity of $\bigoplus$ when $H \trianglelefteq G$. Let us start with the following theorem.

Theorem 4.5. *Let H be a subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$ such that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in G, x \oplus H \in G/H$ [or let $H \trianglelefteq G$]. Then $\mathfrak{T}$ is a discrete topology if and only if H is an open subset of G.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{T}$ is a discrete topology. We obtain that $\{H\}$ is an open subset of G/H. Since q is continuous, $q^{-1}(\{H\})$ is open. It is not hard to prove that $q^{-1}(\{H\}) = H$ by using Lemma 2.11 and Proposition 2.12. The result follows.

($\Leftarrow$) We will show that for each $x \in G$, the singleton set $\{x \oplus H\}$ is open. Since H is an open subgyrogroup of G, $x \oplus H$ is open in G by Proposition 2.8. Observe that $q(x \oplus H) = \{(x \oplus h) \oplus H : h \in H\} = \{x \oplus (h \oplus \mathrm{gyr}[h, x](H)) : h \in H\} = \{x \oplus (h \oplus H) : h \in H\} = \{x \oplus H\}$; again, the third and fourth equalities come from our assumption together with Proposition 2.4 [or come from Lemma 2.17 for the case $H \trianglelefteq G$] and Lemma 2.11. Since q is an open map, $\{x \oplus H\} = q(x \oplus H)$ is open in G/H. $\square$

When H is a normal subgyrogroup of a topological gyrogroup $(G, \mathcal{T}, \oplus)$, it is possible that $(G/H, \mathfrak{T}, \bigoplus)$ turns into a topological gyrogroup. Fortunately, we can show that this is the case.

Definition 4.6. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \trianglelefteq G$. Then the quotient gyrogroup $(G/H, \bigoplus)$ is called the topological quotient gyrogroup if $(G/H, \mathfrak{T}, \bigoplus)$ is a topological gyrogroup.*

Theorem 4.7. *Let $(G, \mathcal{T}, \oplus)$ be a topological gyrogroup, and let $H \trianglelefteq G$. Then $(G/H, \mathfrak{T}, \bigoplus)$ is a topological quotient gyrogroup.*

Proof. It is a well-known result in topology that the product of two open quotient maps is also a quotient map. So $q \times q : G \times G \to G/H \times G/H$ is a quotient map. To prove that $\bigoplus$ is continuous, it is enough to show that $\bigoplus \circ (q \times q)$ is continuous by Theorem 22.2 of [6]. Notice that $(\bigoplus \circ (q \times q))((x, y)) = q(x) \bigoplus q(y) = (x \oplus H) \bigoplus (y \oplus H) = (x \oplus y) \oplus H = (q \circ \oplus)((x, y))$ for all $x, y \in G$. Since q and $\oplus$ are continuous, we have that $\bigoplus \circ (q \times q)$ is continuous which implies the continuity of $\bigoplus$. Next, for each $x \oplus H \in G/H$, $\ominus x \oplus H$ is its inverse element by Theorem 2.18. As a result, the inverse operation $x \oplus H \mapsto \ominus x \oplus H$ is continuous since it is equal to q composed with $\ominus(\,\cdot\,)$. $\square$

A careful reader might ask for the continuity of the left-gyroaddition action in general settings. On one hand, this problem is still open for us. On the other hand, we provide an easy example of occasion that the action is continuous without employing compactness of the subgroup H.

Remark 4.8. *Consider $(\mathbb{D}, \mathcal{T}, \oplus_M)$ where $\mathcal{T}$ is the discrete topology on $\mathbb{D}$ or the subspace topology of $\mathbb{C}$ endowed with the standard topology. It is clear that $(\mathbb{D}, \mathcal{T}, \oplus_M)$ is a topological gyrogroup which is not compact. Let $H = \mathbb{D}$. Then H is not compact, and H is a normal subgyrogroup of $\mathbb{D}$ such that $\mathrm{gyr}[a, b](x \oplus H) \subseteq x \oplus H$ for all $a, b \in \mathbb{D}, x \oplus H \in \mathbb{D}/H$. Since $\mathbb{D}/H$ is a singleton set, the left-gyroaddition action is continuous when $\mathbb{D} \times \mathbb{D}/H$ is equipped with the product topology.*

Finally, we would like to end our work with the succeeding question.

Question 1. *Is the left-gyroaddition action continuous in general?*

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W. Atiponrat
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand
e-mail: watchareepan.a@cmu.ac.th

R. Maungchang
School of Science, Walailak University, Nakhon Si Thammarat, Thailand
e-mail: mate105@gmail.com