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The equivalence graph of the comaximal graph of a group

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Abstract. Let G be a finite group. The comaximal graph of G, denoted by $\Gamma_m(G)$, is a graph whose vertices are the proper subgroups of G that are not contained in the Frattini subgroup of G and join two distinct vertices H and K, whenever $G = \langle H, K\rangle$. In this paper, we define an equivalence relation $\sim$ on $V(\Gamma_m(G))$ by taking $H \sim K$ if and only if their open neighborhoods are the same. We introduce a new graph determined by equivalence classes of $V(\Gamma_m(G))$, denoted $\Gamma_E(G)$, as follows. The vertices are $V(\Gamma_E(G)) = \{[H] | H \in V(\Gamma_m(G))\}$ and two equivalence classes $[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if H and K are adjacent in $\Gamma_m(G)$. We will state some basic graph theoretic properties of $\Gamma_E(G)$ and study the relations between some properties of graph $\Gamma_m(G)$ and $\Gamma_E(G)$, such as the chromatic number, clique number, girth and diameter. Moreover, we classify the groups for which $\Gamma_E(G)$ is complete, regular or planar. Among other results, we show that if the number of maximal subgroups of the group G is less or equal than 4, then $\Gamma_m(G)$ and $\Gamma_E(G)$ are perfect graphs.

1. Introduction

The study of algebraic structures using the properties of graphs has been an exciting research topic, leading to many fascinating results and questions. Associating a graph to a group or a ring and using information on one of the two objects to solve a problem for the other is an interesting research topic, for instance, see [?, ?, ?]. For example, in [?] Sharma and Bhatwadekar defined a graph on a non-zero commutative ring with identity R, $\Gamma(R)$, with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In [?] the authors introduced and studied the comaximal graph of a finite bounded lattice, denoted by $\Gamma(R)$. They investigated some graph-theoretic properties of $\Gamma(R)$. It is shown that for two finite semi-local rings R and S, if R is reduced, then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.

Let G be a group and $L(G)$ be the set of all subgroups of G. We can associate a graph to G in many different ways (see, for example, [1, 2, 3, 14]). Here we consider the following way: Let $\Phi(G)$ be the Frattini subgroup of G. Associate a graph $\Gamma_m(G)$ to G, the comaximal graph of G, as follows: The vertex set is all proper subgroups of G that are not contained in $\Phi(G)$ and two distinct vertices H

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and K joined by an edge if and only if $G = \langle H, K\rangle$. Note that if $G \cong C_{p^n}$, a cyclic group of order p^n, then $\Phi(G) \cong C_{p^{n-1}}$ and so $\Gamma_m(G)$ is a null graph. Recently, this graph was investigated by H. Ahmadi and B. Taeri in [?, ?, ?], in which it is referred to as *the graph related to the join of subgroups*.

For a simple graph Γ, two vertices H, K are equivalent if and only if their open neighborhoods are the same, i.e., $H \sim K$ if and only if $N(H) = N(K)$ where $N(H) = \{L \in V(\Gamma) \mid H \text{ and } L \text{ are adjacent in } \Gamma\}$. It is clear that $\sim$ is an equivalence relation on $V(\Gamma)$ and we denote the class of H by $[H]$. The graph of equivalence classes of Γ, denoted by Γ_E, is the simple graph whose vertex set is $V(\Gamma_E) = \{[H] | H \in V(\Gamma)\}$ and two distinct equivalence classes $[H]$ and $[K]$ are adjacent in Γ_E, denoted $[H] - [K]$, if H and K are adjacent in Γ. The remarkable thing is that Γ_E can be considered as a subgraph of Γ, and it can inherit many properties of Γ. In particular, in many cases, some graph theoretic properties of Γ and Γ_E are the same, such as the chromatic number, clique number and diameter. For example, in [?] the authors considered the graph of equivalence classes of the non-commuting graph of a group G and investigated some graph-theoretic properties of this graph.

In this paper, we will introduce the graph of equivalence classes of $\Gamma_m(G)$ and we will state some of basic graph theoretical properties of $\Gamma_E(G)$, for instance determining diameter, girth, dominating set, planarity of the graph and we give some relation between the graph properties of $\Gamma_m(G)$ and $\Gamma_E(G)$. We will classify all solvable groups G for which $\Gamma_E(G)$ is a complete graph. Furthermore, we show that for a non-nilpotent group G, $\Gamma_E(G)$ is planner if and only if $|G| = 2^n 3^m$ and $G/\Phi(G) \cong S_3$. In Section 3, some results on groups whose equivalence graph of comaximal graphs are complete are given. In Section 4, we will state some results on planarity of $\Gamma_E(G)$. Finally, in Section 5 we will study on the perfection of $\Gamma_E(G)$ and we will show that if $|\mathrm{Max}(G)| \leqslant 4$, then $\Gamma_E(G)$ is a perfect graph and conclude if $|\mathrm{Max}(G)| \leqslant 4$, then $\Gamma_m(G)$ is a perfect graph, too, where $\mathrm{Max}(G)$ is the set of all maximal subgroups of the group G.

2. Definitions and basic results

For a simple graph Γ, we denote the sets of the vertices and the edges of Γ by $V(\Gamma)$ and $E(\Gamma)$, respectively. A graph Γ is said to be connected if there exists a path between any two distinct vertices. The distance between two distinct vertices H and K, denoted by $d(H, K)$, is the length of the shortest path connecting H and K, if such a path exists; otherwise, we set $d(H, K) := \infty$. The degree of H, denoted by $\deg(H)$, is the number of edges incident with H. The graph Γ is regular if and only if for any two distinct vertices of graph have a same degree. Moreover, the diameter of a connected graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is $\sup\{d(H, K) : H, K \in V(\Gamma)\}$. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n for the complete graph with n vertices. For a positive integer r, an r-partite graph is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The girth of Γ, denoted by girth(Γ), is the length of the shortest cycle in Γ, if Γ contains a cycle; otherwise, we set girth$(\Gamma) := \infty$. A subset X of $V(\Gamma)$ is called a clique if the induced subgraph on X is a complete graph. The maximum size of a clique in a graph Γ is called the clique number of Γ and denoted by $\omega(\Gamma)$. The chromatic number of a graph Γ, denoted by $\chi(\Gamma)$, is the minimal number of colors which can be assigned to the vertices of Γ in such a way that every two adjacent vertices have different colors. A subset X of the vertices of Γ is called an independent set if the induced subgraph on X has no edges. The maximum size of an independent set in a graph Γ is called the independence number of Γ and denoted by $\alpha(\Gamma)$. A subset D of $V(\Gamma)$ is a dominating set of Γ if every vertex in $V(\Gamma)\setminus D$ is adjacent to some vertex in D. The domination number $\lambda(\Gamma)$ of Γ is the minimum cardinality of a dominating set. The complement of a graph Γ, denoted by $\overline{\Gamma}$, is the graph with the same vertex set such that two distinct vertices H and K are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.

Let $\Gamma_m(G)$ be the comaximal graph of a group G and

$$N(H) = \{L \in V(\Gamma_m(G)) \mid H \text{ and } L \text{ are adjacent in } \Gamma_m(G)\}$$

be the open neighborhood of the vertex H in $\Gamma_m(G)$. Two vertices H and K are equivalent in $\Gamma_m(G)$ if and only if their open neighborhoods are the same, i.e., $H \sim K$ if and only if $N(H) = N(K)$. One can see that $\sim$ is an equivalence relation on $V(\Gamma_m(G))$ and we denote the class of H by $[H]$.

Definition 2.1. Let G be a group and $\Gamma_m(G)$ be its comaximal graph. The *graph of equivalence classes of* $\Gamma_m(G)$, denoted by $\Gamma_E(G)$, is the graph whose vertex set is $V(\Gamma_E(G)) = \{[H] : H \in V(\Gamma_m(G))\}$, and two distinct equivalence classes $[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if H and K are adjacent in $\Gamma_m(G)$.

Proposition 2.2. *Let C_n be a cyclic group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$, where $\alpha_i \in \mathbb{N}$ and $m \geqslant 2$. Then $\Gamma_E(C_n) \cong \Gamma_E(C_{p_1\dots p_m})$.*

Proof. Assume that $C_n = \langle a\rangle$. It is easy to check that

$$N(\langle a^{p_{i_1}^{\beta_1}p_{i_2}^{\beta_2}\dots p_{i_k}^{\beta_k}}\rangle) = N(\langle a^{p_{i_1}p_{i_2}\dots p_{i_k}}\rangle)$$

where $\{i_1, i_2, \dots i_k\} \subset \{1, 2, \dots, m\}$ and $1 \leqslant \beta_i \leqslant \alpha_i$. Therefore $[\langle a^{p_{i_1}^{\beta_1}p_{i_2}^{\beta_2}\dots p_{i_k}^{\beta_k}}\rangle] = [\langle a^{p_{i_1}p_{i_2}\dots p_{i_k}}\rangle]$ and so the result follows. $\square$

Let $\pi(G)$ be the set of all prime divisors of $|G|$. By Proposition 2.2 we have the following result.

Proposition 2.3. *Let C_n and C_m be two cyclic groups of order n, m. If $\pi(C_n) = \pi(C_m) = \{p_1, \dots, p_k\}$, then $\Gamma_E(C_n) \cong \Gamma_E(C_m) \cong \Gamma_E(C_{p_1\dots p_k})$.*

Let H be a proper subgroup of G. Set $M(H) = \{M \in \mathrm{Max}(G) | H \subseteq M\}$.

Lemma 2.4. *Let H and K be proper subgroups of G. Then*

(*i*) *$[H]$ and $[K]$ are adjacent in $\Gamma_E(G)$ if and only if $M(H) \cap M(K) = \emptyset$.*

(*ii*) *$[H] = [K]$ if and only if $M(H) = M(K)$.*

In particular, if H is only contained in a single maximal subgroup M, then $[H] = [M]$.

Proof. (i). Assume that H and K are adjacent in $\Gamma_m(G)$. If M is a maximal subgroup of G that contains both of them, then $\langle H, K\rangle \neq G$, a contradiction. Conversely, assume that the intersection of $M(H)$ and $M(K)$ is the empty set and $[H]$ and $[K]$ are not adjacent in $\Gamma_E(G)$. Then $\langle H, K\rangle$ is a proper subgroup of G and so H and K lie in a maximal subgroup of G which is a contradiction.

(ii). Let $[H] = [K]$ and M be a maximal subgroup of G such that $M \in N(H) = N(K)$. Then M is adjacent to both of H and K, which implies that for any maximal subgroup N of G, $H \subseteq N$ if and only if $K \subseteq N$. Therefore $M(H) = M(K)$. Conversely, assume that $M(H) = M(K)$ and $[H] \neq [K]$. Then $H \nsim K$ and so $N(H) \neq N(K)$. Therefore there is a vertex L in $\Gamma_m(G)$ such that $G = \langle L, H\rangle$ and $\langle L, K\rangle$ lies in a maximal subgroup of G, which is a contradiction. □

Remark 2.5. Let G be a group and $\mathrm{Max}(G) = \{M_1, \ldots, M_n\}$. For $I_n = \{1, \ldots, n\}$ we put

$$V_{i_1 i_2 \ldots i_r} = \{H \in V(\Gamma_m(G)) | M(H) = \{M_{i_1}, M_{i_2}, \ldots, M_{i_r}\}\}$$

where $i_1, i_2, \ldots, i_r \in I_n$ and $r \leqslant n-1$. By Lemma 2.4 we have $H, H' \in V_{i_1 i_2 \ldots i_r}$ if and only if $[H] = [H']$. Now if $V_{i_1 i_2 \ldots i_r} \neq \emptyset$, we may denote the vertex $V_{i_1 i_2 \ldots i_r}$ in $\Gamma_E(G)$ by $[v_{i_1 i_2 \ldots i_r}]$. Furthermore, for $1 \leqslant i \leqslant n$ we denote the class of V_i by $[M_i]$. Then we have

$$V(\Gamma_E(G)) = \{[M_i] : 1 \leqslant i \leqslant n\} \cup_{r=2}^{n-1} \{[v_{i_1 i_2 \cdots i_r}] : 1 \leqslant i_1, \cdots, i_r \leqslant n, V_{i_1 i_2 \ldots i_r} \neq \emptyset\}.$$

Furthermore, It is clear that $[v_{i_1 i_2 \ldots i_r}]$ and $[v_{j_1 j_2 \ldots j_s}]$ are adjacent in $\Gamma_E(G)$ if and only if $\{i_1, i_2, \ldots, i_r\} \cap \{j_1, j_2, \ldots, j_s\} = \emptyset$ where $1 \leqslant r, s \leqslant n-1$.

Proposition 2.6. *Assume that G is a finite group. Then*

(*i*) $\omega(\Gamma_E(G)) = \omega(\Gamma_m(G)) = \chi(\Gamma_m(G)) = \chi(\Gamma_E(G)) = |\mathrm{Max}(G)|$.

(*ii*) $\mathrm{diam}(\Gamma_E(G)) = \mathrm{diam}(\Gamma_m(G)) \leqslant$ *slant*3. *In particular, $\Gamma_E(G)$ is connected.*

(*iii*) *If* $|\mathrm{Max}(G)| \geqslant 3$, *then* $\mathrm{girth}(\Gamma_E(G)) = 3$.

(*iv*) $\alpha(\Gamma_E(G)) \leqslant \alpha(\Gamma_m(G))$.

Proof. (i). Let $|\mathrm{Max}(G)| = n$. We claim that $\{[M_1], \dots, [M_n]\}$ is a maximum clique in $\Gamma_E(G)$. Let $\{[H_1], \dots, [H_r]\}$ be a clique in graph $\Gamma_E(G)$. Since $[H_i]$ and $[H_j]$ are adjacent, by Lemma 2.4, $M(H_i) \cap M(H_j) = \emptyset$, thus every subgroup H_i is contained in a maximal subgroup of G and so $r \leqslant n$. By the same way we have $\{M_1, \dots, M_n\}$ is a maximum clique in $\Gamma_m(G)$. Therefore $\omega(\Gamma_E(G)) = \omega(\Gamma_m(G)) = |\mathrm{Max}(G)|$. Moreover, it is clear that for any graph Γ, $\omega(\Gamma) \leqslant \chi(\Gamma)$. Now assume that $\omega(\Gamma_m(G)) = t$ and $\mathrm{Max}(G) = \{M_1, \cdots, M_t\}$. Then for $1 \leqslant i \leqslant t$, $S_i = L(M_i) \setminus L(\Phi(G))$ is an independent set and $V(\Gamma_m(G)) = \cup_{i=1}^{t} S_i$, where $L(X)$ is the set of all subgroups of a group X. Hence $\chi(\Gamma) \leqslant \omega(\Gamma)$ and the proof is complete.

(ii). Assume that $[H]$ and $[K]$ are two distinct vertices in $\Gamma_E(G)$. If $H \cap K \nsubseteq \Phi(G)$, then there is a maximal subgroup M of G such that $G = \langle M, H \rangle = \langle M, K \rangle$ and so $d([H], [K]) \leqslant 2$. Now assume that $H \cap K \subseteq \Phi(G)$. Then there are maximal subgroups M_1 and M_2 of G such that

$$G = \langle M_1, H \rangle = \langle M_2, K \rangle = \langle M_1, M_2 \rangle$$

and so $d([H], [K]) \leqslant 3$. Therefore $\mathrm{diam}(\Gamma_E(G)) \leqslant slant3$. By the same way one may have $\mathrm{diam}(\Gamma_m(G)) \leqslant slant3$, as required.

(iii). Suppose that a group G contains at least three maximal subgroups M_1, M_2 and M_3. Then $\{M_1, M_2, M_3\}$ and $\{[M_1], [M_2], [M_3]\}$ are triangles in $\Gamma_m(G)$ and $\Gamma_E(G)$ respectively and so $\mathrm{girth}(\Gamma_m(G)) = \mathrm{girth}(\Gamma_E(G)) = 3$.

(iv). It is clear that if $\{H_1, \dots, H_r\}$ is an independent set in the graph $\Gamma_m(G)$, then $\{[H_1], \dots, [H_r]\}$ is an independent set in $\Gamma_E(G)$. Thus $\alpha(\Gamma_E(G)) \leqslant \alpha(\Gamma_m(G))$. □

3. On the completeness of $\Gamma_E(G)$

Let G be a finite group. In [14], the authors have introduced the concept of *maximal graph*, denoted by $\Gamma M(G)$, as the graph whose vertices are the maximal subgroups of G and join two distinct vertices M_1 and M_2, whenever $M_1 \cap M_2 \neq 1$. If the intersection of every pair of distinct maximal subgroups of G is trivial, then the graph $\Gamma M(G)$ has no edges. Now we may recall the following theorem.

Theorem 3.1. [14, Proposition 1.3] *Let G be a finite group. The intersection of every pair of distinct maximal subgroups of G is trivial if and only if G is solvable and one of the following holds:*

(i) $G \cong C_{p^n}$ *(p is prime).*

(ii) $G \cong C_{pq}$ *(p, q different primes).*

(iii) $G \cong C_p \times C_p$ *(p is prime).*

(iv) $G = P \rtimes Q$, *where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P.*

In the following theorem, we characterize all groups whose graph of equivalence classes of comaximal graph of G are complete.

Theorem 3.2. *The equivalence graph of the comaximal graph of G is complete if and only if G is solvable and one of the following holds.*

- (i) $G \cong C_{p^n}$ *(p is prime).*
- (ii) $G \cong C_{p^r q^s}$ *(p, q different primes).*
- (iii) *G is a p-group, where $G/\Phi(G) \cong C_p \times C_p$ (p a prime). In particular, if G is an abelian p-group then $G \cong C_{p^r} \times C_{p^s}$ and $\Gamma_E(G) \cong K_{p+1}$.*
- (iv) *$G/\Phi(G) \cong P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p is prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P. Moreover, in this case, G is not nilpotent.*

Proof. Let $\Gamma_E(G)$ be a complete graph and $\mathrm{Max}(G) = \{M_1, \dots, M_k\}$. Since M_i and $M_i \cap M_j$ are not joined by an edge, then $[M_i \cap M_j]$ is not one of the vertices of $\Gamma_E(G)$. Hence $M_i \cap M_j = \Phi(G)$ and so $V(\Gamma_E(G)) = \{[M_1], \dots, [M_k]\}$. Moreover, the intersection of every pair of distinct maximal subgroups of $G/\Phi(G)$ is trivial. Now by Theorem 3.1 we have the following cases:

(i). If $G/\Phi(G) \cong C_{p^n}$, then $n = 1$ and $G \cong C_{p^m}$, for some integer m. Thus in this case $\Gamma_E(G)$ is an empty graph.

(ii). If $G/\Phi(G) \cong C_{pq}$, then G is a cyclic group with two maximal subgroups. Therefore $G \cong C_{p^r q^s}$.

(iii). If $G/\Phi(G) \cong C_p \times C_p$, then G is nilpotent. Therefore

$$G \cong S(p_1) \times \ldots \times S(p_k),$$

where $S(p_i)$ is the Sylow p_i-subgroup of G and $\pi(G) = \{p_1, \dots, p_k\}$ is the set of all prime divisors of $|G|$. Assume that $k \geqslant 2$. We know $\Phi(G) \cong \Phi(S(p_1)) \times \ldots \times \Phi(S(p_k))$ and $\Phi(S(p_i)) \neq 1$. Therefore

$$C_p \times C_p \cong \frac{G}{\Phi(G)} \cong \frac{S(p_1)}{\Phi(S(p_1))} \times \ldots \times \frac{S(p_k)}{\Phi(S(p_k))},$$

which contradicts $\pi(G) = \pi(G/\Phi(G))$. Hence $k = 1$ and so G is a p-group, where $G/\Phi(G) \cong C_p \times C_p$. In particular, if G is an abelian p-group, $G/\Phi(G) \cong C_p \times C_p$ follows that $G \cong C_{p^r} \times C_{p^s}$ and so $\Gamma_E(G) \cong K_{p+1}$.

(iv). If $G/\Phi(G) = P \rtimes Q$, Since Q is a non-normal maximal subgroup of G, then G is non-nilpotent.

Conversely, If $G \cong C_{p^n}$ or $C_{p^r q^s}$, then it is clear that $\Gamma_E(G)$ is complete. Now assume that G is a p-group of order p^n, where $G/\Phi(G) \cong C_p \times C_p$. Then $|\Phi(G)| = p^n$ and for all M_i and M_j in $\mathrm{Max}(G)$, $|M_i \cap M_j| = |\Phi(G)|$. Therefore $M_i \cap M_j = \Phi(G)$ for all M_i and M_j in $\mathrm{Max}(G)$ and so $V(\Gamma_E(G)) = \{[M_1], \dots, [M_k]\}$. Thus $\Gamma_E(G)$ is a complete graph.

For the last case there is a bijection between $\mathrm{Max}(G)$ and $\mathrm{Max}(G/\Phi(G))$ and we may assume that $G/\Phi(G) \cong P'/\Phi(G) \rtimes Q'/\Phi(G)$, where $P = P'/\Phi(G)$ and $Q = Q'/\Phi(G)$. Then $V(\Gamma_E(G)) = \{[P'], [Q'], [Q'^g] |\ g \in G\}$ and so $\Gamma_E(G)$ is a complete graph. □

Proposition 3.3. $\lambda(\Gamma_E(G)) = 1$ *if and only if* $\Gamma_E(G)$ *is a complete graph.*

Proof. Let $D = \{[H]\}$ be a dominating set. It is easy to show that H is only contained in a single maximal subgroup M and so $[H] = [M]$ by Lemma 2.4. On the other hand, one can see that $M \cap N = \Phi(G)$ for all $N \in \mathrm{Max}(G) \setminus \{M\}$. Therefore $M/\Phi(G) \cap N/\Phi(G) = \{\Phi(G)\}$ and so the maximal graph of $G/\Phi(G)$, $\Gamma M(G/\Phi(G))$, is nonconnected. Thanks to Theorem 1.2 in [14], $G/\Phi(G)$ is isomorphic to one of the groups $C_p \times C_p$, C_{pq} or $P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p, and Q acts irreducibly and fixed point freely on P. Now the result follows by Theorem 3.2. □

Proposition 3.4. $\Gamma_E(G)$ *is a regular graph if and only if* $\Gamma_E(G)$ *is a complete graph.*

Proof. Let $\Gamma_E(G)$ be a regular graph and let, for a contradiction, there is maximal subgroups M_i and M_j of G such that $\Phi(G) \subsetneq M_i \cap M_j$. Then $[M_i \cap M_j]$ is one of the vertices of $\Gamma_E(G)$. But $\deg([M_i \cap M_j]) < \deg([M_i])$, which contradicts the regularity of $\Gamma_E(G)$. Therefore $\Phi(G) = M_i \cap M_j$ and so $V(\Gamma_E(G)) = \mathrm{Max}(G)$ and the result follows. □

Proposition 3.5. *If* G *is a finite* p*-group which has a maximal cyclic subgroup, then* $\Gamma_E(G)$ *is a complete graph.*

Proof. Thanks to Theorem 5.3.4 in [?], G is one of the following groups:

(i) C_{p^n}

(ii) $C_{p^n} \times C_{p^{n-1}}$

(iii) $D_{2^n} =< x, y | x^{2^{n-1}} = y^2 = (xy)^2 = 1 >,\ n \geqslant 3.$

$iv)$ $Q_{2^n} =< x, y | x^{2^{n-1}} = 1, y^2 = x^{2^{n-2}}, x^y = x^{-1} >,\ n \geqslant 3.$

(v) $SD_{2^n} =< x, a | x^2 = 1 = a^{2^{n-1}}, a^x = a^{2^{n-2}-1} >,\ n \geqslant 3.$

(vi) $M_n(p) =< x, a | x^p = 1 = a^{p^{n-1}}, a^x = a^{1+p^{n-2}} >,\ n \geqslant 3.$

Now by using the parts (i) and (iii) of Theorem 3.2, $\Gamma_E(G)$ is a complete graph. □

Proposition 3.6. $\Gamma_E(G) \cong K_4$ *if and only if one of the following holds.*

(*i*) *G is a 3-group and $G/\Phi(G) \cong C_3 \times C_3$. In particular, if G is an abelian 3-group then $G \cong C_{3^r} \times C_{3^s}$, $r, s \geqslant 1$.*

(*ii*) *$G/\Phi(G) \cong S_3$.*

Proof. Assume that $\Gamma_E(G) \cong K_4$. Since $\Gamma_E(G)$ is complete graph, then $|V(\Gamma_E(G))| = |\mathrm{Max}(G)| = 4$. Then we have the following cases:
(*i*). By part (*iii*) of Theorem 3.2, G is a 3-group and $G/\Phi(G) \cong C_3 \times C_3$. In particular, if G is an abelian 3-group then $G \cong C_{3^r} \times C_{3^s}, r, s \geqslant 1$.
(*ii*). By part (*iv*) of Theorem 3.2, assume that $G/\Phi(G) \cong P \rtimes Q$, where P is an elementary abelian p-group of order p^n (p a prime), $|Q| = q$, where q is a prime different from p. One can see that the number of Sylow q-subgroups and Sylow p-subgroup of $G/\Phi(G)$ are $q + 1 = 3$ and 1 respectively. Therefore $G/\Phi(G) \cong C_3 \rtimes C_2 \cong S_3$.
□

Proposition 3.7. *$\Gamma_E(G) \cong K_5$ if and only of $G/\Phi(G) \cong A_4$.*

Proof. Assume that $\Gamma_E(G) \cong K_5$. Since $\Gamma_E(G)$ is complete graph, then $|V(\Gamma_E(G))| = |\mathrm{Max}(G)| = 5$ and so by the last part of Theorem 3.2 the number of Sylow q-subgroups and of $G/\Phi(G)$ are $q+1 = 4$ and so $G/\Phi(G) \cong (C_2 \times C_2) \rtimes C_3 \cong A_4$. □

4. On the planarity of $\Gamma_E(G)$

In this section, we will investigate the planarity of the equivalence graph $\Gamma_E(G)$. First we recall the following well-known theorem of Kuratowski.

Theorem 4.1. [13, Theorem 4.4.6] *A graph is planar if and only if it has no subdivisions of K_5 or $K_{3,3}$.*

In the following theorem, we characterize all cyclic groups whose equivalence graph are planar.

Theorem 4.2. *Let C_n be a cyclic group of order n. $\Gamma_E(C_n)$ is planar if and only if $|\pi(C_n)| = 2$ or 3.*

Proof. Since $|\mathrm{Max(C_n)}| = |\pi(C_n)|$, then $|\mathrm{Max(C_n)}| \leqslant 4$, otherwise $\Gamma_E(G)$ will have a subgraph isomorphic to K_5 which is a contradiction. First we assume that $|\mathrm{Max(C_n)}| = 4$. According to Proposition 2.3 if $\pi(C_n) = \{p_1, \ldots, p_4\}$, we have $\Gamma_E(C_n) = \Gamma_E(C_m) = \Gamma_E(C_{p_1 \ldots p_4})$. Hence the induced subgraph on vertices

$$\{< a^{p_1} >, < a^{p_2} >, < a^{p_3} >, < a^{p_4} >, < a^{p_1 p_3} >, < a^{p_2 p_4} >\}$$

contains a subgraph isomorphic to $K_{3,3}$ and so $\Gamma_E(C_n)$ is not planar. Now, one can check that if $|\pi(C_n)| = 2$ or 3, then $\Gamma_E(C_n)$ is planar. □

Theorem 4.3. *Assume that G is a p-group of order p^n where p is a prime and $n \geqslant 2$. Then $\Gamma_E(G)$ is planar if and only if $G/\Phi(G) \cong C_2 \times C_2$ or $C_3 \times C_3$. In particular, if G is an abelian non-cyclic p-group of order p^n and $n \geqslant 2$, then $\Gamma_E(G)$ is planar if and only if $G \cong C_{3^r} \times C_{3^s}$ or $G \cong C_{2^r} \times C_{2^s}$, where $r, s \geqslant 1$.*

Proof. Let G be a p-group of order p^n and $\Gamma_E(G)$ be planar. Then $G/\Phi(G) \cong C_p \times \cdots \times C_p$ with rank r, $|\mathrm{Max}(G)| = (p^r - 1)/(p-1)$ and $|\mathrm{Max}(G)| \leqslant 4$. Hence we must have $p = 2$ or $p = 3$ and $r = 2$ and so by Theorem 3.2 $\Gamma_E(G) \cong K_3$ or K_4, which they are planar.

Assume that G is a group isomorphic to D_{2^n}, Q_{2^n} or SD_{2^n}, $n \geqslant 3$. Then $G/\Phi(G) \cong C_2 \times C_2$. Furthermore, $M_n(p)/\Phi(M_n(p)) \cong C_p \times C_p$ for $p = 2$ or 3. Thanks to Theorem 4.3 we have the following result. □

Corollary 4.4. *Let G be a group isomorphic to one of the group $D_{2^n}, Q_{2^n}, SD_{2^n}$, $n \geqslant 3$ or $M_n(p)$, $p = 2$ or 3. Then $\Gamma_E(G)$ is planar.*

Theorem 4.5. *Let G be a non-nilpotent group. $\Gamma_E(G)$ is planar if and only if $|G| = 2^n 3^m$ and $G/\Phi(G) \cong S_3$, where $n, m \geqslant 1$.*

Proof. Assume that $\Gamma_E(G)$ is planar. Then $|\mathrm{Max}(G)| \leqslant 4$. On the other hand, since G is not nilpotent by Lemma 3, in [9], we have $|\mathrm{Max}(G)| \geqslant 4$. So $\mathrm{Max}(G) = 4$ and by theorem 3 in [9], G is a supersolvable group of order $2^n 3^m$ $n, m \geqslant 1$ and $G/\Phi(G) \cong S_3$ and the result follows. □

5. On the perfection of $\Gamma_E(G)$

In this section, we will study the perfection of the equivalence graph. We show that if $|\mathrm{Max}(G)| \leqslant 4$ then $\Gamma_E(G)$ and $\Gamma_m(G)$ are perfect. First, we recall the following definitions and theorems.

Definition 5.1. A graph Γ is *perfect* whenever $\omega(\Gamma') = \chi(\Gamma')$, for all induced subgraphs Γ' of Γ.

Definition 5.2. A graph is *chordal* (or *triangulated*) if each of its cycles of length at least 4 has a chord, i.e., if it contains no induced cycles other than triangles.

Proposition 5.3. [13, Proposition 5.5.1] *Every chordal graph is perfect. In particular, complete graphs, empty graphs and k-partite graphs are perfect.*

Theorem 5.4. [**?**, Theorem 1.2] *A graph Γ is perfect if and only if neither Γ nor $\overline{\Gamma}$ contains an odd cycle of length at least 5 as an induced subgraph.*

Theorem 5.5. *If $|\mathrm{Max}(G)| \leqslant 3$, then $\Gamma_E(G)$ is chordal.*

Proof. If $|\mathrm{Max}(G)| = 1$, then $\Phi(G)$ is the maximal subgroup of G and so $\Gamma_E(G)$ is empty. Furthermore, if $|\mathrm{Max}(G)| = \{M_1, M_2\}$, then $V(\Gamma_E(G)) = \{[M_1], [M_2]\}$

and so $\Gamma_E(G) \cong K_2$. Hence by Proposition 5.3 they are perfect. Now assume that $\mathrm{Max}(G) = \{M_1, M_2, M_3\}$ and

$$[H_1] - [H_2] - \cdots - [H_n] - [H_1]$$

be a cycle of length n in $\Gamma_E(G)$. Since for all $1 \leqslant i \leqslant 3$, $\deg([M_i]) = 2$ or 3 and by Remark 2.5 $\deg([v_{ij}]) = 1$, then $n \leqslant 3$ and so $\Gamma_E(G)$ is chordal. □

Corollary 5.6. *If* $|\mathrm{Max}(G)| \leqslant 3$*, then* $\Gamma_E(G)$ *is perfect.*

It must be noted that if $|\mathrm{Max}(G)| \geqslant 4$, then there exists a finite group like G such that $\Gamma_E(G)$ is not chordal. For example, assume that $G = \langle a \rangle \cong C_{p_1 \dots p_4}$, where $p_1, \dots, p_4$ are primes, then

$$C_4 : [a^{p_1}] - [a^{p_2}] - [a^{p_1 p_3}] - [a^{p_2 p_4}] - [a^{p_1}]$$

is a cycle of length 4 without a chord.

Theorem 5.7. *If* $|\mathrm{Max}(G)| = 4$ *then* $\Gamma_E(G)$ *is perfect.*

Proof. We use Theorem 5.4 and show that $\Gamma_E(G)$ and $\overline{\Gamma_E(G)}$ do not contain an odd cycle of length at least 5 as an induced subgraph. For $\Gamma_E(G)$, by Remark 2.5 we have

$$V(\Gamma_E(G)) = \{[M_1], [M_2], [M_3], [M_4], [v_{ij}], [v_{ijk}] | i, j, k \in \{1, 2, 3, 4\}\}.$$

In the general case, we may assume that all of $[v_{ij}]$'s and $[v_{ijk}]$'s are not empty. It must be noted that there is not a cycle of length at least 5 which contains $[v_{ijk}]$, because each $[v_{ijk}]$ has degree 1 and cannot be part of a cycle. Therefore, if $n \geqslant 5$ and $C_n : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ is an odd cycle in $V(\Gamma_E(G))$, then for $1 \leqslant i \leqslant n$, $[H_i]$ is equal to either $[M_i]$ or $[v_{ij}]$. Without loss of generality, we may assume that $[H_1] = [M_1]$ or $[H_1] = [v_{12}]$.
If $[H_1] = [M_1]$, there are two choices for $[H_2]$.

Case 1: $[H_2] = [M_2]$, $[M_3]$ or $[M_4]$. If for example $[H_2] = [M_2]$, then we can choose just $[v_{13}]$ or $[v_{14}]$ for $[H_3]$. If $[H_3] = [v_{13}]$, then $[H_4] = [v_{24}]$ and so $[H_1], [H_4]$ are adjacent. Hence $n = 4$, a contradiction. On the other hand, if $[H_3] = [v_{14}]$, then there is no choice for $[H_4]$, a contradiction too.

Case 2: $[H_2] = [v_{23}]$ or $[v_{24}]$. Then $[H_3] = [v_{14}]$ or $[v_{13}]$ respectively and we have no choice for $[H_4]$ which is a contradiction.

Now assume that $[H_1] = [v_{12}]$. We have two choices for $[H_2]$.

Case 1: $[H_2] = [M_3]$ or $[M_4]$. Let for example $[H_2] = [M_3]$. If $[H_3] = [M_1]$ or $[M_2]$, then $[H_4] = [v_{23}]$ or $[v_{13}]$ respectively and there exists no choice for $[H_5]$, a contradiction. Similarly, if $[H_3] = [v_{14}]$ or $[v_{24}]$, then $[H_4] = [v_{23}]$ or $[v_{13}]$ respectively and there exists no choice for $[H_5]$, a contradiction too.

Case 2: $[H_2] = [v_{34}]$. Then $[H_3] = [M_1]$ or $[M_2]$. If for example $[H_3] = [M_1]$, then $[H_4] = [v_{23}]$ or $[v_{24}]$ and so $[H_5] = [v_{14}]$ or $[v_{13}]$ respectively. Now there exists no choice for $[H_6]$ and so this case does not hold. Consequently, $\Gamma_E(G)$ does not contain an odd cycle of length at least 5 as an induced subgraph.

Now, we prove the same result for $\overline{\Gamma_E(G)}$. First we note that since $[v_{ijk}]$ has degree 1 in $\Gamma_E(G)$, all but one vertex of the complement are neighbors of $[v_{ijk}]$, and so it cannot be contained in a chordless cycle of length at least 3. Let $n \geqslant 5$ and $C_n : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ be an odd cycle in $\overline{\Gamma_E(G)}$. Then for $1 \leqslant i \leqslant n$, $[H_i]$ is equal to either $[M_i]$ or $[v_{ij}]$.

Without loss of generality, we may assume that $[H_1] = [M_1]$ or $[H_1] = [v_{12}]$. First assume that $[H_1] = [M_1]$. Then $[H_2] = [v_{12}], [v_{13}]$ or $[v_{14}]$. If for example $[H_2] = [v_{12}]$, then $[H_3] = [v_{23}], [v_{24}]$ or $[M_2]$. If $[H_3] = [M_2]$, then we have no choice for $[H_4]$. Let $[H_3] = [v_{23}]$ (or $[H_3] = [v_{24}]$), then $[H_4] = [M_3]$ or $[v_{34}]$. If $[H_4] = [M_3]$, then there is no choice for $[H_5]$ and if $[H_4] = [v_{34}]$, then $[H_5] = [M_4]$ and we have no choice for $[H_6]$. Therefore in this case we have a contradiction.

Now assume that $[H_1] = [v_{12}]$. We have the following cases for $[H_2]$:

Case 1: If $[H_2] = [M_1]$ or $[M_2]$, then $[H_3] = [v_{13}]$ or $[v_{23}]$ respectively and so we have a cycle of length at most 3, a contradiction.

Case 2: $[H_2] = [v_{13}]$, $[v_{14}]$, $[v_{23}]$ or $[v_{24}]$. If for example $[H_2] = [v_{13}]$, then $[H_3] = [M_3]$ or $[v_{34}]$ and finally we have the paths $[v_{12}] - [v_{13}] - [M_3]$ or $[v_{12}] - [v_{13}] - [v_{34}] - [M_4]$ respectively, which they are not cycles in $\overline{\Gamma_E(G)}$. Then we get a contradiction in this case too.

Therefore $\overline{\Gamma_E(G)}$ does not contain an odd cycle of length at least 5 and so $\Gamma_E(G)$ is a perfect graph. □

One can easily check that if $C_n : H_1 - H_2 - \cdots - H_n - H_1$ is a cycle of length n in $\Gamma_m(G)$, then $\overline{C_n} : [H_1] - [H_2] - \cdots - [H_n] - [H_1]$ is a cycle of length n in $\Gamma_E(G)$. Then by Corollary 5.6 and Theorem 5.7 we have the following result for $\Gamma_m(G)$.

Corollary 5.8. *If* $|\mathrm{Max}(G)| \leqslant 4$, *then* $\Gamma_m(G)$ *is a perfect graph.*

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