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On the connectivity of the proper intersection power graph of a finite group

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Abstract. Let G be a group with an identity element e. The intersection power graph of G, denoted by $P_I(G)$, is the graph with vertex set G and two distinct vertices x and y are adjacent if there exists an element $z \in G \setminus \{e\}$ such that $x^m = z = y^n$, for some $m, n \in \mathbb{N}$ and e is adjacent to all other vertices. The proper intersection power graph of G, denoted by $P_I^*(G)$, is obtained by removing the identity element from the vertex set of $P_I(G)$. In this paper, we study the connectivity of $P_I^*(G)$.

1. Introduction

The investigation of graphs related to groups is an important topic in algebraic combinatorics. Moreover, there are many ways of building a graph from a group. This paper is devoted to the study of the intersection power graph, which was introduced by Bera in [3]. Of course, it must be mentioned that he was motivated by definitions of the power graph and the enhanced power graph. Hence, at first, we will review the definition of them which are the main reason for defining the intersection power graph.

In [6], Kelarev and Quinn introduced the directed power graph of a group, then Chakrabarty et al. introduced the undirected power graph of a group in [5]. Assume that G is a group. The power graph of G, showed by $P(G)$, is a graph with vertex set equal to G where two distinct vertices x and y are adjacent if either $x \in \langle y \rangle$ or $y \in \langle x \rangle$. Let us refer to [1], for a survey of all recent results on the power graphs associated with groups and semigroups.

Another graph associated with the group G, which its vertex set is equal to G, is the commuting graph, that is denoted by $C(G)$. In this graph two vertices x and y are adjacent if they commute. It is clear that $P(G)$ is a subgraph of $C(G)$.

Then the enhanced power graph was introduced in [2], in order to find that for an arbitrary group how much its power graph is closed to its commuting graph. It is denoted by $P_E(G)$ and its vertex set is G where x is adjacent to y in $P_E(G)$ if there exists an element z in G such that $x, y \in \langle z \rangle$. For more information see [4].

The intersection power graph is a new representation of groups by graphs. It was introduced by Bera in [3]. The intersection power graph is denoted by $P_I(G)$.

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Its vertex set is G where x is adjacent to y in $P_I(G)$ if there exists an element z in $G \setminus \{e\}$ such that $x^n = z = y^m$, for some $m, n \in \mathbb{N}$ and e is adjacent to all other vertices in $P_I(G)$. In other words, x is adjacent to y in $P_I(G)$ if $\langle x \rangle \cap \langle y \rangle \neq \{e\}$. It is easy to see that for a group G, the power graph $P(G)$ is a subgraph of the intersection power graph $P_I(G)$. We encourage the interested readers to study [3] where Bera has beautifully explained his reason for defining this new graph there.

We denote by $P_I^*(G)$ the proper intersection power graph which is obtained by removing the identity element from its vertex set.

In [2], the authors asked which groups do have the property that the power graph is connected when the identity is removed?
In this paper, we try to answer to this question for the intersection power graph. We will show that if the prime graph of a finite group G is disconnected, then $P_I^*(G)$ is disconnected. Furthermore, we will prove several statements about the connectivity of the proper intersection power graph.

Throughout this paper, G is a finite group and $G^* = G \setminus \{e\}$. We use $x \sim y$ to denote two distinct vertices x and y of a graph are joined.

2. Main results

Let G be a group and M_i for $i \in \{1, \dots t\}$ be all maximal cyclic subgroups of G. We use $\Gamma_M(G)$ to denote a graph which its vertex set is the set of all maximal cyclic subgroups of G and two distinct vertices M_i and M_j are adjacent if their intersection is not trivial.

Let $P_I^*(G)$ be connected and let M and M' be two maximal cyclic subgroups of G. Consider $M = \langle m \rangle$ and $M' = \langle m' \rangle$, where $m, m' \in G$. We know that there exists a path between m and m' in $P_I^*(G)$, since $P_I^*(G)$ is connected. Hence,

$$m = x_1 \sim x_2 \sim \dots \sim x_n = m',$$

where $x_i \in G$. Assume that M_i is a maximal cyclic subgroup of G such that $x_i \in M_i$, for every $1 \leqslant i \leqslant n$. Since $x_i \sim x_{i+1}$ in $P_I^*(G)$, so $M_i \cap M_{i+1} \neq \{e\}$, which implies that $M_i \sim M_{i+1}$ in $\Gamma_M(G)$, for every $1 \leqslant i \leqslant n-1$. It follows that $\Gamma_M(G)$ is connected.

Conversely, let $\Gamma_M(G)$ be connected and $g, g' \in G^*$, where $G^* = G \setminus \{e\}$. There exist maximal cyclic subgroups M and M' of G such that $g \in M$ and $g' \in M'$. Since $\Gamma_M(G)$ is connected, so there is a path between M and M' in $\Gamma_M(G)$. Hence

$$M = M_1 \sim M_2 \sim \dots \sim M_n = M'.$$

Let $M_i = \langle m_i \rangle$, for every $1 \leqslant i \leqslant n$. We have $M_i \sim M_{i+1}$. As a result, $M_i \cap M_{i+1} \neq \{e\}$, for every $1 \leqslant i \leqslant n-1$. Assume $x_i \in M_i \cap M_{i+1}$. Then $m_i^s = x_i = m_{i+1}^t$, for some $s, t \in \mathbb{N}$ and for every $1 \leqslant i \leqslant n-1$. Consequently, $m_i \sim m_{i+1}$ in $P_I^*(G)$ and so $m_1 \sim m_2 \sim \dots \sim m_n$ is a path in $P_I^*(G)$. Moreover, we have $g = m_1^l$, for some $l \in \mathbb{N}$, which implies that $g \sim m_1$ in $P_I^*(G)$. Similarly, we have $g' \sim m_n$ in $P_I^*(G)$. It follows that $P_I^*(G)$ is connected. Therefore, we have proved the following result.

Lemma 2.1. *Let G be a finite group. Then $P_I^*(G)$ is connected if and only if $\Gamma_M(G)$ is connected.*

The set of all element orders of G is denoted by $\omega(G)$, which is closed and partially ordered by the divisibility. As a result, it is uniquely determined by the set $\mu(G)$, the subset of its maximal elements.

We use $\Gamma_\mu(G)$ to denote a graph whose vertex set is $\mu(G)$ where $s, t \in \mu(G)$ are adjacent if $gcd(s,t) \neq 1$.

Let $\Gamma_M(G)$ be connected. Moreover, assume that m and m' are two arbitrary elements of $\mu(G)$. There exist maximal cyclic subgroups M and M' such that $|M| = m$ and $|M'| = m'$. Since $\Gamma_M(G)$ is connected, so we can consider the following path from M to M' in $\Gamma_M(G)$:

$$M = M_0 \sim M_1 \sim \ldots \sim M_{n-1} \sim M_n = M'.$$

Since $M_i \sim M_{i+1}$, we deduce that $gcd(|M_i|, |M_{i+1}|) \neq 1$, for every $0 \leqslant i \leqslant n-1$. We know that M_i is a maximal cyclic subgroup of G, which implies that $|M_i| \in \mu(G)$, for every $0 \leqslant i \leqslant n$. Consequently, by definition of $\Gamma_\mu(G)$ we can consider the following path between m and m' in $\Gamma_\mu(G)$:

$$m = |M_0| \sim |M_1| \sim \ldots \sim |M_{n-1}| \sim |M_n| = m',$$

which implies that $\Gamma_\mu(G)$ is connected. Therefore, we have proved the following result.

Lemma 2.2. *Let G be a finite group. If $\Gamma_M(G)$ is connected, then $\Gamma_\mu(G)$ is connected.*

By Lemmas 2.1 and 2.2, we have:

Corollary 2.3. *Let G be a finite group. If $\Gamma_\mu(G)$ is disconnected then $P_I^*(G)$ is disconnected.*

Let G be a finite group. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. The prime graph is a graph associated to a group G. Its vertex set is equal to $\pi(G)$ where two distinct prime numbers p and q of $\pi(G)$ are adjacent if and only if G has an element of order pq. This graph is denoted by $\Gamma(G)$.

Let $\Gamma_\mu(G)$ be connected. Moreover, suppose that p and p' are two prime numbers of $\pi(G)$. It follows that there are m and m' in $\mu(G)$ such that $p \mid m$ and $p' \mid m'$. We know that there exists a path between m and m' in $\Gamma_\mu(G)$. As a consequence, $m = m_0 \sim m_1 \sim \ldots \sim m_n = m'$, where $m_i \in \mu(G)$, for every $0 \leqslant i \leqslant n$. Since $m_i \sim m_{i+1}$, so there is a prime number r_i such that $r_i \mid gcd(m_i, m_{i+1})$, for every $0 \leqslant i \leqslant n-1$. Consequently, $r_i r_{i+1} \mid m_{i+1}$ and hence $r_i \sim r_{i+1}$ in $\Gamma(G)$, for every $0 \leqslant i \leqslant n-1$. Furthermore, we have $p, r_0 \mid m_0$ and $p', r_n \mid m_n$ and so we get $p \sim r_0$ and $p' \sim r_n$ in $\Gamma(G)$. Therefore, $p \sim r_0 \sim r_1 \sim \ldots \sim r_n \sim p'$ is a path in $\Gamma(G)$. It follows that $\Gamma(G)$ is connected.

Conversely, let $\Gamma(G)$ be connected and $m, m' \in \mu(G)$. Let p and p' be two prime numbers such that $p \mid m$ and $p' \mid m'$. Since $\Gamma(G)$ is connected, so we assume that $p = r_0 \sim r_1 \sim \ldots \sim r_n = p'$ is a path in $\Gamma(G)$. We know that $r_i \sim r_{i+1}$ in $\Gamma(G)$. As a result, there exists $u_i \in \mu(G)$ such that $r_i r_{i+1}$ divides u_i, for every $0 \leqslant i \leqslant n-1$. It follows that $r_i \mid gcd(u_{i-1}, u_i)$, which implies that $u_{i-1} \sim u_i$, for every $1 \leqslant i \leqslant n-1$ in $\Gamma_\mu(G)$. Moreover, we know that $p \mid gcd(m, u_0)$ and $p' \mid gcd(m', u_{n-1})$ and so $m \sim u_1$ and $m' \sim u_{n-1}$ in $\Gamma_\mu(G)$. Therefore, $m \sim u_0 \sim u_2 \sim \ldots \sim u_{n-1} \sim m'$ is a path in $\Gamma_\mu(G)$. It follows that $\Gamma_\mu(G)$ is connected. Therefore, we have the following result.

Lemma 2.4. *Let G be a finite group. Then $\Gamma_\mu(G)$ is connected if and only if $\Gamma(G)$ is connected.*

Now by Corollary 2.3 and Lemma 2.4, we obtain the following result.

Theorem 2.5. *Let G be a finite group. If $\Gamma(G)$ is disconnected, then $P_I^*(G)$ is disconnected.*

In [7], all groups which have disconnect prime graphs, have been introduced. Hence Theorem 2.5 is so important and it motivates the following theorem.

Theorem 2.6. *If G is isomorphic to one of the following groups, then $P_I^*(G)$ is disconnected.*

1) *A finite simple group whose prime graph is disconnected.*
2) *The symmetric groups S_p, S_{p+1}, S_{p+2}.*
3) *The dihedral group D_{2n}, where n is an odd number.*

In an arbitrary graph, a vertex is named a dominating vertex if it is adjacent to every other vertex of the graph. For instance, if G is a group, then the identity element e is a dominating vertex of the intersection power graph of G.

Let $\langle x \rangle$ be the unique minimal subgroup of G. If $g \in G^*$, then $x \sim g$ in $P_I^*(G)$, since $\langle x \rangle \subseteq \langle g \rangle$. It follows that $P_I^*(G)$ is connected. Moreover, x is a dominating vertex of $P_I^*(G)$. Therefore, we get the following result.

Theorem 2.7. *Let G be a finite group. Then $P_I^*(G)$ is connected if G has a unique minimal subgroup.*

Let $x, y \in G^*$ such that $xy = yx$ and $(o(x), o(y)) = 1$. Since x and y commute, so $(xy)^{o(y)} = x^{o(y)}$. It follows that $\langle x^{o(y)} \rangle \subseteq \langle xy \rangle$. On the other hand, since $(o(x), o(y)) = 1$, we conclude that $\langle x \rangle = \langle x^{o(y)} \rangle$, which implies that $x \in \langle xy \rangle$. According to the definition of power graph, $x \sim xy$ in $P(G)$. We know that $P(G)$ is a subgraph of $P_I(G)$ and hence $x \sim xy$ in $P_I^*(G)$. Similarly, we can show that $y \sim xy$ in $P_I^*(G)$. Therefore, $x \sim xy \sim y$ is a path in $P_I^*(G)$. Hence we have:

Lemma 2.8. *Let $x, y \in G^*$ such that $xy = yx$ and $(o(x), o(y)) = 1$. Then there exists a path from x to y in $P_I^*(G)$.*

Let G be a finite group. Suppose that x is the unique element of order 2 in G. As a result, $x \in Z(G)$. Let y be an arbitrary element of G^*. If $x \in \langle y \rangle$, then $x \sim y$ in $P^*(G)$ and so $x \sim y$ in $P_I^*(G)$. Otherwise, $o(y)$ is odd and so x and y are on a path in $P_I^*(G)$, by Lemma 2.8. Consequently, we get that.

Lemma 2.9. *Let G be a finite group. If G has a unique element of order 2, then $P_I^*(G)$ is connected.*

Example 2.10. Consider the generalized quaternion group

$$Q_{2^n} = \langle a, b | a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, a^b = a^{-1} \rangle.$$

It has just one element of order 2. Therefore, its proper intersection power graph is connected.

Theorem 2.11. *Let G be a finite group. If $Z(G)$ is not a p-group, then $P_I^*(G)$ is connected.*

Proof. Assume that G is a finite group and $Z(G)$ is not a p-group. Suppose that $z_1, z_2 \in Z(G)$, such that $o(z_1) = p$ and $o(z_2) = q$, where p and q are distinct prime numbers. Moreover, let x and y be two arbitrary elements of G^*. We consider the following two cases:
1. Let $(o(x), o(y)) = 1$. In this case we have:

- Assume p and q do not divide $o(x)$ and $o(y)$. Then, by Lemma 2.8,

$$x \sim xz \sim z \sim yz \sim y \quad \text{for} \quad z \in \{z_1, z_2\}.$$

- If p, q and $o(x)$ are coprime but $p^\alpha q^\beta \mid o(y)$, where $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and both α and β are not zero in addition $p^{\alpha+1} \nmid o(y)$ and $q^{\beta+1} \nmid o(y)$. Without loss of generality, let $\alpha \neq 0$. Now by Lemma 2.8,

$$x \sim xz_2 \sim z_2 \sim y^{o(y)/p^\alpha} z_2 \sim y^{o(y)/p^\alpha} \sim y.$$

- If $p^\alpha \mid o(x)$ and $p^{\alpha+1} \nmid o(x)$ also $q^\beta \mid o(y)$ and $q^{\beta+1} \nmid o(y)$, where $\alpha, \beta \in \mathbb{N}$. Consequently, by Lemma 2.8,

$$x \sim x^{o(x)/p^\alpha} \sim x^{o(x)/p^\alpha} z_2 \sim z_2 \sim z_1 z_2 \sim z_1 \sim y^{o(y)/q^\beta} z_1 \sim y^{o(y)/q^\beta} \sim y.$$

2. Let $r = (o(x), o(y)) \neq 1$. In this case we have:

- If p and q do not divide r, then by Lemma 2.8 we have:

$$x \sim x^{o(x)/r} \sim x^{o(x)/r} z \sim z \sim y^{o(y)/r} z \sim y^{o(y)/r} \sim y \quad \text{for} \quad z \in \{z_1, z_2\}.$$

- If $p^\alpha q^\beta \mid r$, where $\alpha, \beta \in \mathbb{N} \cup \{0\}$ and both α and β are not zero in addition $p^{\alpha+1} \nmid r$ and $q^{\beta+1} \nmid r$. Hence, by Lemma 2.8,

$$x \sim x^{o(x)/p^\alpha} \sim x^{o(x)/p^\alpha} z_2 \sim z_2 \sim z_1 z_2 \sim z_1 \sim y^{o(y)/q^\beta} z_1 \sim y^{o(y)/q^\beta} \sim y.$$

Therefore, there exists a path between x and y and so $P_I^*(G)$ is connected. $\square$

Example 2.12. *The proper intersection power graph of every abelian group which is not a p-group is connected.*

Theorem 2.13. *Assume that G is a finite nilpotent group of order n, where n is not a prime power. Then $P_I^*(G)$ is connected.*

Proof. Suppose that G is a finite nilpotent group of order n. Consider $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where $k > 1$ and for every $1 \leqslant i \leqslant k$, p_i is a prime and $n_i \in \mathbb{N} \cup \{0\}$. Let x and y be two arbitrary elements of G^*. Suppose that $o(x) = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ and $o(y) = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$, where $\alpha_i, \beta_i \in \mathbb{N} \cup \{0\}$, for every $1 \leqslant i \leqslant k$. We consider the following two cases:
1. Let $(o(x), o(y)) = 1$. Since G is nilpotent, so x and y commute. Now by Lemma 2.8, there exists a path between x and y.
2. Let $(o(x), o(y)) = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_k^{\gamma_k} \neq 1$. We consider the following two cases:

- Let $o(x)$ and $o(y)$ be powers of a single prime number. For instance, $o(x) = p_1^{\alpha_1}$ and $o(y) = p_1^{\beta_1}$. Since n is not a prime power, so there exists $z \in G$ such that $o(z) = p_i$, where $i \neq 1$. By Lemma 2.8, we have:
$$x \sim xz \sim z \sim yz \sim y.$$
- Assume that there exist $i, j \in \{1, 2, \dots k\}$ such that $\gamma_i \neq 0 \neq \gamma_j$. Now by Lemma 2.8,
$$x \sim x^{o(x)/p_i^{\alpha_i}} \sim x^{o(x)/p_i^{\alpha_i}} y^{o(y)/p_j^{\beta_j}} \sim y^{o(y)/p_j^{\beta_j}} \sim y.$$

Consequently, we conclude that there is a path between x and y. □

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