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On Belousov-Moufang quasigroups

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Abstract. We recall a few little-studied quasigroups defined by V.D. Belousov. We describe the linear form of these quasigroups and characterize their parastrophes. We then examine for which values of k these quasigroups are k-translatable.

1. In the theory of quasigroups, an important role is played by loops, i.e. quasigroups with a neutral element. In many cases, for a given class of quasigroups satisfying certain axioms, there is no explicit axiom for the existence of a neutral element but, as a consequence of those axioms, a neutral element exists. However, there are classes of quasigroups where if a quasigroup in that class has a neutral element then the quasigroup is a group or has only one element.

V. D. Belousov dealt with such classes of quasigroups, several of them described in his monograph [1].

In this short post we will recall some of these types of quasigroups and describe their linear form. We will also describe their parastrophes (conjugates) and show when such quasigroups are translatable.

2. A *Moufang quasigroup* is a quasigroup $(Q,\cdot)$ satisfying any one of these identities

$$(x\cdot yz)x = xy\cdot zx, \qquad xz\cdot yx = x(zy\cdot x),$$
$$(xy\cdot z)y = x(y\cdot zy), \qquad (yz\cdot y)x = y(z\cdot yx).$$

Galkin announced in 1988 (cf. [9]) that a quasigroup satisfying one of the first two identities is a loop. In 1996 Kunen proved (cf. [11]) that a quasigroup satisfying one of the above identities is a loop. The same result was obtained in [14] by Shcherbacov and Izbash. By results of Bol and Bruck (cf. [2], p. 115) these four identities are equivalent in loops. Consequently, these identities are equivalent in quasigroups. So, Moufang quasigroups are in fact Moufang loops.

Since all the examples of Moufang quasigroups known to Belousov were loops but quasigroups isotopic to Moufang quasigroups may not have neutral elements, he proposed a different, more general definition of Moufang quasigroups. His Moufang quasigroups (sometimes called $\mathcal{M}$-quasigroups) have neutral elements only in the case when they are Moufang loops. Moreover every loop isotopic to Belousov's Moufang quasigroups is a Moufang loop. The first results about such

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quasigroups appeared in the work of Belousov and Florja in 1966 (cf. [3]), the next in the book [1].

According to this new definition a quasigroup $(Q,\cdot)$ is called a Moufang quasigroup if it satisfies the identity

$$x(y\cdot xz)=(x\cdot yf_x)x\cdot z, \tag{1}$$

where $f_x\cdot x=x$.

This identity is equivalent (cf. [3]) to the identity

$$(xy\cdot z)y=x(y(e_yz\cdot y)),$$

where $y\cdot e_y=y$.

Any Moufang quasigroup (in the classical sense) is a Moufang quasigroup in the sense of Belousov but not conversely. To emphasize the difference, Moufang quasigroups in the sense of Belousov will be called the *Belousov-Moufang quasigroups* (shortly *BM-quasigroups*).

Example 0.1. *Let $(Q,+)$ be an arbitrary abelian group. Then $(Q,\cdot)$, where $x\cdot y=-x+y$, is an example of a quasigroup satisfying* (1). *In this quasigroup $f_x=0$ for all $x\in Q$. So, 0 is a left neutral element of $(Q,\cdot)$ but $(Q,\cdot)$ is not a loop.*

Example 0.2. Also quasigroups defined by the following three tables

·	1	2	3
1	2	1	3
2	1	3	2
3	3	2	1

·	1	2	3	4
1	2	1	3	4
2	1	2	4	3
3	3	4	1	2
4	4	3	2	1

·	1	2	3	4	5
1	2	1	4	3	5
2	1	3	2	5	4
3	4	2	5	1	3
4	3	5	1	4	2
5	5	4	3	2	1

are Belousov-Moufang qusigroups without left neutral elements. So, they are not Moufang quasigroups in the classical sense.

The following two facts are obvious.

Fact 1. *A Belousov-Moufang quasigroup is a Moufang quasigroup if and only if it has a neutral element.*

A Belousov-Moufang quasigroup that is not a loop will be called *proper*.

As a consequence of Example 1 we obtain

Fact 2. *For every $n>2$ there is a proper Belousov-Moufang quasigroup with n elements.*

3. We say that a quasigroup $(Q,\cdot)$ is *linear over a group* $(Q,+)$ if there are automorphisms α,β of $(Q,+)$ and an element $a\in Z(Q,+)$, where $Z(Q,+)$ is the center of $(Q,+)$, such that

$$x\cdot y=\alpha x+\beta y+a. \tag{2}$$

A quasigroup linear over an abelian group is called a T-*quasigroup* (cf. [13]).

Theorem 0.3. *A Belousov-Moufang quasigroup $(Q, \cdot)$ linear over a group $(Q, +)$ has a left neutral element and can be presented in the form*

$$x \cdot y = \alpha x + y, \tag{3}$$

where $\alpha \in Aut(Q, +)$ and $\alpha^2 = \varepsilon$.

Proof. Applying (2) to (1) we obtain

$$\left.\begin{array}{l} \alpha x + \beta\alpha y + \beta^2\alpha x + \beta^3 z + \beta^2 a + \beta a + a = \\ \alpha^3 x + \alpha^2\beta\alpha y + \alpha^2\beta^2 f_x + \alpha^2 a + \alpha\beta x + \alpha a + \beta z + a, \end{array}\right\} \tag{4}$$

which for $x = y = z = 0$ gives

$$\beta^2 a + \beta a + a = \alpha^2\beta^2 f_x + \alpha^2 a + \alpha a + a. \tag{5}$$

Obviously $\beta^2 a, \beta a, a \in Z(Q, +)$. Thus, putting $x = y = 0$ in (4) and applying (5), we get $\beta^2 = \varepsilon$. Similarly, (4) for $x = z = 0$ gives $\beta\alpha = \alpha^2\beta\alpha$. Hence, $\alpha^2 = \varepsilon$. Now, putting $y = z = 0$ in (4), applying (5) and $\beta^2 = \alpha^2 = \varepsilon$ we obtain $\alpha = \alpha\beta$, so $\beta = \varepsilon$. Therefore $x \cdot y = \alpha x + y + a$. Consequently, $x = f_x \cdot x = \alpha f_x + x + a$ implies $\alpha f_x + a = 0$. But from (5) it follows $a = f_x + \alpha a$. So, $\alpha a = \alpha f_x + a = 0$, i.e. $a = 0 = f_x$. Hence $x \cdot y = \alpha x + y$ and $0 \cdot z = z$ for every $z \in Q$. $\square$

Corollary 0.4. *For each $n > 2$ there are at least two Belousov-Moufang quasigroups induced by the group $\mathbb{Z}_n$.*

Corollary 0.5. *A Belousov-Moufang quasigroup linear over a group has one idempotent. This idempotent is its left neutral element.*

Corollary 0.6. *If a Belousov-Moufang quasigroup linear over a group is unipotent then the group is abelian.*

For $n = 8$ there are four such quasigroups.

As a consequence of Theorem 0.3 and results proved in [4] and [10] we obtain

Corollary 0.7. *Belousov-Moufang quasigroups linear over an abelian group are medial and paramedial.*

4. Another interesting class of quasigroups is the class of left Bol quasigroups. Florja, in his PhD dissertation written under the supervision of Belousov, defined the *left Bol quasigroups* (called by him – Bol quasigroups) as quasigroups satisfying the identity

$$x(y \cdot xz) = R_{e_x}^{-1}(x \cdot yx) \cdot z, \tag{6}$$

where $x \cdot e_x = x$, and proved (cf. [8]) that each quasigroup isotopic to a left Bol quasigroup is a Bol loop.

Example 0.8. Examples of left Bol quasigroups are as follows:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	5	1	2	3	6	4
3	6	5	1	2	4	3
4	4	6	5	1	3	2
5	2	3	4	6	1	5
6	3	4	6	5	2	1

$\cdot$	1	2	3	4	5	6
1	1	5	6	4	2	3
2	2	1	5	6	3	4
3	3	2	1	5	4	6
4	4	3	2	1	6	5
5	5	6	4	3	1	2
6	6	4	3	2	5	1

Both these quasigroups are isotopic to the group $\mathbb{Z}_6$, but they are not linear over $\mathbb{Z}_6$. The first has left neutral element, the second has a right neutral element.

Theorem 0.9. *A left Bol quasigroup $(Q,\cdot)$ linear over a group $(Q,+)$ has the form*

$$x\cdot y=\beta^{-2}x+\beta y,$$

where $\beta\in Aut(Q,+)$.

Proof. Applying (2) to (6) we obtain

$$\left.\begin{array}{l}\alpha^2x+\alpha\beta\alpha y+\alpha\beta^2\alpha x+\alpha\beta^3z+\alpha\beta^2a+\alpha a+\beta e_x+a=\\ \alpha^2x+\alpha\beta\alpha y+\alpha\beta^2x+\alpha\beta a+\alpha a+\beta z+a,\end{array}\right\}\tag{7}$$

which for $x=y=z=0$ implies

$$\alpha\beta^2a+\beta e_0=\alpha\beta a.\tag{8}$$

Now, from (7) putting $x=y=0$ and using (8), we obtain $\alpha\beta^2=\varepsilon$. Thus, $a+\beta e_0=\alpha\beta a$. Hence, $0=0\cdot e_0=\alpha 0+\beta e_0+a=\alpha\beta a$, so $a=0$ and $\alpha=\beta^{-2}$. □

Corollary 0.10. *A left Bol quasigroup linear over an abelian group is medial.*

Corollary 0.11. *A Belousov-Moufang quasigroup linear over a group is a left Bol quasigroup if and only if it is a group.*

In general, a left Bol quasigroup is not a Belousov-Moufang quasigroups but, as proved Florja (cf. [8]) a Belousov-Moufang quasigroup is always a left Bol quasigroup. Below we give a short proof of this fact.

Proposition 0.12. *A Belousov-Moufang quasigroup is a left Bol quasigroup.*

Proof. Let $(Q,\cdot)$ be a Belousov-Moufang quasigroup. Let's choose $v=(x\cdot yf_x)x$. Then $v\cdot e_x=(x\cdot yf_x)x\cdot e_x\overset{(1)}{=}x(y\cdot xe_x)=x\cdot yx$. This implies $v=R_{e_x}^{-1}(x\cdot yx)$. Hence $(x\cdot yf_x)x=R_{e_x}^{-1}(x\cdot yx)$. Consequently $x(y\cdot xz)\overset{(1)}{=}(x\cdot yf_x)x\cdot z=R_{e_x}^{-1}(x\cdot yx)\cdot z$. □

5. A *right Bol quasigroup* is a quasigroup $(Q,\cdot)$ satisfying the identity

$$(yx\cdot z)x = yL_{f_x}^{-1}(xz\cdot x), \tag{9}$$

where $f_x\cdot x = x$.

Theorem 0.13. *A quasigroup $(Q,\cdot)$ linear over a group $(Q,+)$ is a right Bol quasigroup if and only if*

$$x\cdot y = \alpha x + \beta y + a$$

for some fixed $a\in Q$ and $\alpha,\beta\in Aut(Q,+)$ such that $\alpha^2=\varepsilon$.

Proof. Then

$$(yx\cdot z)x = \alpha y + \beta L_{f_x}^{-1}(xz\cdot x) + a$$

whence

$$\begin{aligned} L_{f_x}^{-1}(xz\cdot x) &= -\beta^{-1}\alpha y + \beta^{-1}(yx\cdot z)x - \beta^{-1}a\\ (xz\cdot x) &= f_x\cdot(-\beta^{-1}\alpha y + \beta^{-1}(yx\cdot z)x - \beta^{-1}a)\\ (xz\cdot x) &= \alpha f_x - \alpha y + (yx\cdot z)x\\ \alpha^2 x &= \alpha f_x - \alpha y + \alpha^3 y + \alpha^2\beta x + \alpha^2 a \end{aligned}$$

This for $x=y=0$ gives $0=\alpha f_0+\alpha^2 a$, so for $x=0$ we get $\alpha y=\alpha^3 y$. Thus $\alpha^2=\varepsilon$. □

Corollary 0.14. *A Belousov-Moufang quasigroup linear over a group is a right Bol quasigroup.*

Proposition 0.15. *An idempotent right Bol quasigroup $(Q,\cdot)$ linear over a group is medial and has the form*

$$x\cdot y = \alpha x + (\varepsilon-\alpha)y,$$

where $(Q,+)$ is an abelian group, $\alpha\in Aut(Q,+)$ and $\alpha^2=\varepsilon$.

Proof. The idempotency of $(Q,\cdot)$ implies $x=\alpha x+\beta x+a$. Thus, $\alpha x = x+\alpha\beta x+\alpha a$. This for $x=0$ gives $\alpha a=0$, so $a=0$. Consequently, $x=\alpha x+\beta x$. Hence $\alpha\beta=\beta\alpha$ and $\alpha x = x-\beta x$. Since $\alpha\in Aut(Q,+)$ the last implies the commutativity of $(Q,+)$. This means (cf. [10]) that $(Q,\cdot)$ is medial. □

6. A quasigroup $(Q,\cdot)$ satisfying the identity

$$x(xy\cdot z) = y(zx\cdot x)$$

is called an *i-quasigroup* (cf. [5]).

Theorem 0.16. *Any i-quasigroup $(Q,\cdot)$ linear over a group $(Q,+)$ can be presented in the form*

$$x\cdot y = \alpha x + \alpha^2 y + a, \tag{10}$$

where $\alpha\in Aut(Q,+)$, $\alpha^4=\varepsilon$ and $\alpha x + x\in Z(Q,+)$ for every $x\in Q$.

Proof. Applying (2) to (10) we obtain

$$\alpha x+\beta\alpha^2x+\beta\alpha\beta y+\beta^2z=\alpha y+\beta\alpha^2z+\beta\alpha\beta x+\beta^2x, \tag{11}$$

which for $x=y=0$ implies $\beta=\alpha^2$.

Similarly, (11) for $x=z=0$ gives $\beta^2=\alpha^4=\varepsilon$. Now, (11) for $y=0$ implies $\alpha x+x\in Z(Q,+)$ for every $x\in Q$. □

Corollary 0.17. *Any i-loop $(Q,\cdot)$ linear over a group $(Q,+)$ has the form $x\cdot y=x+y+a$.*

Proof. In any i-loop $x\cdot xy=y\cdot xx$. This implies $\alpha=\varepsilon$. □

As a consequence of the above results we obtain

Corollary 0.18. *A medial Belousov-Moufang quasigroup linear over group is an i-quasigroup.*

Proposition 0.19. *An idempotent i-quasigroup linear over a group is medial.*

Proof. Let $(Q,\cdot)$ be linear over a group $(Q,+)$. Since $(Q,\cdot)$ is an idempotent i-quasigroup we have $x=\alpha x+\alpha^2x+a$, which implies $a=0$. So, $x=\alpha x+\alpha^2x$. Consequently, $\alpha^{-1}x=x+\alpha x$. Now from $\alpha^{-1}(x+y)=\alpha^{-1}x+\alpha^{-1}y$ we obtain $y+\alpha x=\alpha x+y$ for all $x,y\in(Q,+)$. Thus $(Q,+)$ is an abelian group. □

Proposition 0.20. *A unipotent i-quasigroup linear over a group is medial and paramedial, and has one left neutral element.*

Proof. A quasigroup $(Q,\cdot)$ is unipotent if there is $b\in Q$ such that $x\cdot x=b$ for all $x\in Q$. Then $b=0\cdot 0=\alpha 0+\alpha^2 0+a=a$. So, $a=x\cdot x=\alpha x+\alpha^2x+a$ implies $0=x+\alpha x$ for all $x\in Q$. Hence $\alpha x=-x$. Consequently, $(Q,+)$ is an abelian group and $x\cdot y=-x+y+a$. □

7. Each quasigroup $(Q,\cdot)$ determines five new quasigroups $(Q,\circ_i)$ with the operations $\circ_i$ defined as follows:

$$\begin{aligned} x\circ_1 y=z &\longleftrightarrow x\cdot z=y,\\ x\circ_2 y=z &\longleftrightarrow z\cdot y=x,\\ x\circ_3 y=z &\longleftrightarrow z\cdot x=y,\\ x\circ_4 y=z &\longleftrightarrow y\cdot z=x,\\ x\circ_5 y=z &\longleftrightarrow y\cdot x=z.\end{aligned}$$

Such defined (not necessarily different) quasigroups are called *parastrophes* or *conjugates* of $(Q,\cdot)$.

Parastrophes of each quasigroup can be divided into separate classes containing isotopic parastrophes. The number of such classes is always 1, 2, 3 or 6 (cf. [6]). In some cases (described in [12]) parastrophes of a given quasigroup Q are pairwise equal. Examples of quasigroups for which all parastrophes are equal to the output quasigroup are the Belousov-Moufang quasigroups defined in Example 2. However, this is quite a rare case.

In Example 0.8 are given two left Bol quasigroup isotopic to the group $\mathbb{Z}_6$. All their parastrophes are also isotopic to this group. Moreover for the first quasigroup: $(Q,\cdot) = (Q,\circ_2)$, $(Q,\circ_1) = (Q,\circ_4)$, $(Q,\circ_3) = (Q,\circ_5)$; for the second: $(Q,\cdot) = (Q,\circ_1)$, $(Q,\circ_2) = (Q,\circ_3)$, $(Q,\circ_4) = (Q,\circ_5)$.

It is not difficult to see that the parastrophes $(Q,\circ_1)$ and $(Q,\circ_2)$ of a quasigroup $(Q,\cdot)$ linear over a group $(Q,+)$ are isotopic to $(Q,\cdot)$, also in the case when a is not a central element of $(Q,+)$. This together with the results obtained in [6] gives

Proposition 0.21. *All parastrophes of a quasigroup $(Q,\cdot)$ linear over a group $(Q,+)$ are isotopic to $(Q,\cdot)$ and to $(Q,+)$.*

In general, parastrophes do not retain the basic properties of the initial quasigroup: parastrophes of loops may not be loops, parastrophes of commutative quasigroups may not be commutative.

However, in the case of quasigroups linear over a group:

– *the parastrophe $(Q,\circ_1)$ of a BM-quasigroup is a BM-quasigroup,*

– *the parastrophe $(Q,\circ_1)$ of a right Bol quasigroup is a right Bol quasigroup,*

– *the parastrophe $(Q,\circ_2)$ of an i-quasigroup is an i-quasigroup.*

8. A finite quasigroup $Q = \{1,2,\ldots,n\}$ is called *k-translatable* (cf. [7]), where $1 \leqslant k < n$, if its Cayley table is obtained by the following rule: If the first row of the Cayley table is $a_1, a_2, \ldots, a_n$, then the q-th row is obtained from the $(q-1)$-st row by taking the last k entries in the $(q-1)$–st row and inserting them as the first k entries of the q-th row and by taking the first $n-k$ entries of the $(q-1)$-st row and inserting them as the last $n-k$ entries of the q-th row, where $q \in \{2,3,\ldots,n\}$.

It is important to note that a quasigroup may be k-translatable for one ordering but not for another (see Example 8.13 in [7]).

We assume that $\mathbb{Z}_n$ has a natural order $1,2,\ldots,n$ and 0 is identified with n.

It is not difficult to see that the additive group $\mathbb{Z}_n$ is $(n-1)$-translatable. It is also a consequence of the following result proved in [7].

Proposition 0.22. *A quasigroup of the form $xy = ax + by + c$ induced by the group $\mathbb{Z}_n$ is k-translatable if and only if $a + kb \equiv 0 (\operatorname{mod} n)$, i.e. if and only if $k = -ab^{-1} (\operatorname{mod} n)$.*

Quasigroups and their parastrophes linear over the group $\mathbb{Z}_n$ considered in this note are k-translatable for the values of k presented in the following table.

type	*form*	$(Q,\cdot)$	$(Q,\circ_1)$	$(Q,\circ_2)$	$(Q,\circ_3)$	$(Q,\circ_4)$	$(Q,\circ_5)$
$BM-$quasigroup	$ax+y$	$n-a$	a	1	1	a^{-1}	$-a^{-1}$
left Bol	$a^{-2}x+ay$	a^{-3}	a^{-2}	a^{-1}	a	a^2	$-a^{-3}$
right Bol	$ax+by+c$	$n-ab^{-1}$	a	b^{-1}	ab	a^{-1}	$-ab$
$i-$quasigroup	$ax+a^2y$	$-a^3$	a	a^2	a^2	a^3	$-a$

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