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Translatable quadratical quasigroups

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Abstract. The concept of a k-translatable groupoid is introduced. Those k-translatable quadratical quasigroups induced by the additive group of integers modulo m, where $k < 40$, are listed for $m \leqslant 1200$. The fine structure of quadratical quasigroups is explored in detail and the Cayley tables of quadratical quasigroups of orders 5, 9, 13 and 17 are produced. All but those of order 9 are k-translatable, for some k. Quadratical quasigroups induced by the additive group of integers modulo m are proved to be k-translatable, for some k. Open questions and thoughts about future research in this area are given.

1. Introduction

Geometrical motivations for the study of quadratical quasigroups have been given in [9, 10, 11, 12]. In particular Volenec [9, 10] defined a product $*$ on $\mathbb{C}$, the complex numbers, that defines a quadratical quasigroup. The product $x * y$ of two distinct elements is the third vertex of a positively oriented, isosceles right triangle in the complex plane, at which the right angle occurs.

The main aim of this paper is to give insight into the fine algebraic structure of quadratical quasigroups, in order to set the stage for, and to stimulate, further development of the general theory that is still in its relative infancy. This is the second of a series of four papers that advance this theory. We concern ourselves here mainly with the fine algebraic structure, rather than with the geometrical representations, of quadratical quasigroups. However, as noted by Volenec, each algebraic identity valid in the quadratical quasigroup $(\mathbb{C}, *)$ can be interpreted as a geometrical theorem and the theory of quadratical quasigroups gives a better insight into the mutual relations of such theorems ([9], page 108).

Volenec [9] proved that quadratical quasigroups have a number of properties, such as idempotency, mediality and cancellativity. These properties were applied by the authors in [3] to prove that quadratical quasigroups form a variety $\mathcal{Q}$. The spectrum of $\mathcal{Q}$ was proved to be contained in the set of all integers equal to 1 plus a multiple of 4. Quadratical quasigroups are uniquely determined by certain abelian groups and their automorphisms [1]. Necessary and sufficient conditions under which $\mathbb{Z}_m$, the additive group of integers modulo m, induces quadratical quasigroups are given in [3].

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This paper builds on the authors' work in [3], as well as the prior work of Polonijo [7], Volenec [9] and Dudek [1]. In Sections 3, 4, 5, 6 and 7 the notion of a *four-cycle*, which was introduced in [3], is used to explore in detail the fine structure of quadratical quasigroups. The concept of a four-cycle is applied in Sections 4 and 6 to produce Cayley tables for quadratical quasigroups of orders 5, 9, 13 and 17. These tables can be reproduced by model builders, but we would not achieve our aim of stimulating thought about the fine algebraic structure in that manner.

In Section 8, all of these quadratical quasigroups except those of order 9 are proved to be k-translatable, for some k. We prove that, up to isomorphism, there is only one quadratical quasigroup of order 9 and that it is self-dual. Quadratical quasigroups of order 25 and 29 are found. The one of order 25 is 18-translatable, its dual is 7-translatable, the quadratical quasigroup of order 29 is 12-translatable and its dual is 17-translatable.

Sections 8 and 9 of this paper explore other ways of constructing k-translatable quasigroups. We introduce the central concept of a k-translatable groupoid in Section 8 and use it to characterize quadratical quasigroups. In Section 9 necessary and sufficient conditions are found for a quasigroup induced by $\mathbb{Z}_m$ to be k-translatable. We prove that a quadratical quasigroup induced by $\mathbb{Z}_m$ is always k-translatable, for some k. The existence of k-translatable quadratical quasigroups induced by some $\mathbb{Z}_m$ is established for each integer k, where $1 < k < 11$. Values of m for which a quadratical quasigroup induced by $\mathbb{Z}_m$ is $(m-k)$-translatable are determined for each integer k, where $1 < k < 11$.

In Section 9 lists are given for k-translatable $(k < 40)$ quadratical quasigroups of orders $m < 1200$, induced by $\mathbb{Z}_m$ and k-translatable quadratical quasigroups induced by $\mathbb{Z}_m$ for $m < 500$.

In a future publication, the two different approaches to the construction of quadratical quasigroups are united. It will be proved that a quadratical quasigroup is translatable if and only if it is induced by some $\mathbb{Z}_{4n+1}$. Finally, open questions and possible future directions for research are discussed in Section 9.

2. Preliminaries

Volenec [9] defined a *quadratical groupoid* as a right solvable groupoid satisfying the following condition:

$$xy \cdot x = zx \cdot yz. \qquad (A)$$

He proved that such groupoids are quasigroups and satisfy the identities listed below.

Theorem 2.1. *A quadratical groupoid satisfies the following identities:*

$$x = x^2 \quad (idempotency), \tag{1}$$
$$x \cdot yx = xy \cdot x \quad (elasticity), \tag{2}$$
$$x \cdot yx = xy \cdot x = yx \cdot y \quad (strong\ elasticity), \tag{3}$$
$$yx \cdot xy = x \quad (bookend), \tag{4}$$
$$x \cdot yz = xy \cdot xz \quad (left\ distributivity), \tag{5}$$
$$xy \cdot z = xz \cdot yz \quad (right\ distributivity), \tag{6}$$
$$xy \cdot zw = xz \cdot yw \quad (mediality), \tag{7}$$
$$x(y \cdot yx) = (xy \cdot x)y, \tag{8}$$
$$(xy \cdot y)x = y(x \cdot yx), \tag{9}$$
$$xy = zw \longleftrightarrow yz = wx \quad (alterability). \tag{10}$$

These identities can be used to characterize quadratical quasigroups. Namely, the following theorem is proved in [3].

Theorem 2.2. *The class of all quadratical quasigroups form a variety uniquely defined by*
- *(A), (3), (4), (7), or*
- *(1), (4), (7), or*
- *(2), (4), (7), or*
- *(4), (5), (10).*

Quadratical quasigroups are uniquely characterized by commutative groups and their automorphisms. This characterization (proved in [1]) is presented below.

Theorem 2.3. *A groupoid $(G, \cdot)$ is a quadratical quasigroup if and only if there exists a commutative group $(G, +)$ in which for every $a \in G$ the equation $z + z = a$ has a unique solution $z \in G$ and φ, ψ are automorphisms of $(G, +)$ such that*

$$xy = \varphi(x) + \psi(y),$$

$$\varphi(x) + \psi(x) = x,$$

$$2\varphi\psi(x) = x$$

for all $x, y \in G$.

In this case we say that the quadratical quasigroup is *induced by* $(G, +)$.

We also will need the following two results proved in [3].

Theorem 2.4. *A finite quadratical groupoid has order $m = 4t + 1$.*

So, later it will be assumed that $m = 4t + 1$ for some natural t.

Theorem 2.5. *A quadratical groupoid induced by the additive group $\mathbb{Z}_m$ has the form*

$$x \cdot y = ax + (1-a)y\,,$$

where $a \in \mathbb{Z}_m$ *and*

$$2a^2 - 2a + 1 = 0. \tag{11}$$

3. Products in quadratical quasigroups

Let Q be a quadratical quasigroup and $a, b \in Q$ be two different elements. Suppose that $C = \{x_1, x_2, \ldots, x_n\} \subseteq Q$ consists of n distinct elements, such that $aba = x_1x_2 = x_2x_3 = x_3x_4 = \ldots = x_{n-1}x_n = x_nx_1$. Then C will be called an (*ordered*) *n-cycle based on aba*. Note that $x_1 \neq aba$, or else $x_1 = x_2 = \ldots = x_n = aba$. Note also that if $C = \{x_1, x_2, x_3, \ldots, x_n\} \subseteq Q$ is an n-cycle based on aba, then so is $C_i = \{x_i, x_{(i+1) mod\, n}, x_{(i+2) mod\, n}, \ldots, x_{(i+n-1) mod\, n}\}$.

In [3] is proved that in a quadratical quasigroup all n-cycles have the length $n = 4$. Moreover, if $a, b \in Q$ and $a \neq b$, then each element $x_1 \neq aba$ of Q is a member of a 4-cycle based on aba. Two 4-cycles based on aba, where $a \neq b$, are equal or disjoint. Note that in any 4-cycle $C = \{x_1, x_2, x_3, x_4\}$, $x_4 = x_1x_3$. Hence, $C = \{x, yx, y, xy\}$, where $x = x_1$ and $y = x_3$.

Definition 3.1. Let Q be a quadratical quasigroup with $\{a, b\} \subseteq Q$ and $a \neq b$. Then $\{a, b, ab, ba, aba\}$ contains five distinct elements. We will use the notation $[1,1] = a$, $[1,2] = ab$, $[1,3] = ba$ and $[1,4] = b$. We omit the commas and square brackets in the notation, when this causes no confusion, and write $11 = a$, $12 = ab$, $13 = ba$ and $14 = b$. For $n \geqslant 2$, by induction we define $n1 = (n-1)1 \cdot (n-1)2$, $n2 = (n-1)2 \cdot (n-1)4$, $n3 = (n-1)3 \cdot (n-1)1$, $n4 = (n-1)4 \cdot (n-1)3$ and $Hn = \{n1, n2, n3, n4\}$. On the occasions when we need to highlight that the element fk, $f \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, 3, 4\}$, is in the dual quadratical quasigroup Q^* we will denote it by fk^*. Similarly, $Hn^* = \{n1^*, n2^*, n3^*, n4^*\}$. Note that the values of both fk and fk^* depend on the choice of the elements a and b.

Example 3.2. $H2 = \{a \cdot ab, ab \cdot b, ba \cdot a, b \cdot ba\}$,
$H3 = \{(a \cdot ab)(ab \cdot b), (ab \cdot b)(b \cdot ba), (ba \cdot a)(a \cdot ab), (b \cdot ba)(ba \cdot a)\}$,
$H4 = \{(31 \cdot 32)(32 \cdot 34), (32 \cdot 34)(34 \cdot 33), (33 \cdot 31)(31 \cdot 32), (34 \cdot 33)(33 \cdot 31)\}$, where $31 = (a \cdot ab)(ab \cdot b)$, $32 = (ab \cdot b)(b \cdot ba)$, $33 = (ba \cdot a)(a \cdot ab)$ and $34 = (b \cdot ba)(ba \cdot a)$.

Example 3.3. $11^* = a$, $12^* = a * b$, $13^* = b * a$, $14^* = b$ and, for $n \geqslant 2$, by induction we define $n1^* = (n-1)1^* * (n-1)2^*$, $n2^* = (n-1)2^* * (n-1)4^*$, $n3^* = (n-1)3^* * (n-1)1^*$ and $n4^* = (n-1)4^* * (n-1)3^*$.

Example 3.4. $H2^* = \{a*(a*b), (a*b)*b, (b*a)*a, b*(b*a)\} = \{ba \cdot a, b \cdot ba, a \cdot ab, ab \cdot b\}$ and $52^* = 42^* \cdot 44^* = (32^* \cdot 34^*)(34^* \cdot 33^*) = (((ab*b)*(b*ba))*((b*ba)*(ba*a)))*(((b*ba)*(ba*a))*(ba*a)*(a*ab))$, where $a*ab = a*(a*b)$, $ab*b = (a*b)*b$, $ba*a = (b*a)*a$ and $b*ba = b*(b*a)$.

Note that the expression ab, when working in the dual groupoid $Q^* = (Q, *)$, equals $a * b$, which equals $b \cdot a$ in the original groupoid itself. This notation will cause no problems, as we will either calculate values only using the dot product or the star product, or when we are calculating using both products, as in Theorem 5.1, the distinction will be obvious.

The proofs of the following propositions are straightforward, using induction on n and the properties of quadratical quasigroups, and are omitted.

Proposition 3.5. *For any positive integer* t, $t1 \cdot t4 = t2$, $t2 \cdot t3 = t4$, $t3 \cdot t2 = t1$ *and* $t4 \cdot t1 = t3$.

Proposition 3.6. *For* $t > 1$, $aba \cdot tk = (t-1)k$ *for any* $k \in \{1, 2, 3, 4\}$.

Proposition 3.7. *For* $t > 1$, $t1 \cdot aba = (t-1)2$, $t2 \cdot aba = (t-1)4$, $t3 \cdot aba = (t-1)1$ *and* $t4 \cdot aba = (t-1)3$.

Proposition 3.8. *For any positive integer* t, Ht *contains* 4 *distinct elements.*

Proposition 3.9. *For any positive integer* t, $Ht \cap \{aba\} = \emptyset$.

Proposition 3.10. *For any positive integer* t, $t1 \cdot t3 = t2 \cdot t1 = t3 \cdot t4 = t4 \cdot t2 = aba$.

Proposition 3.11. $Ht = \{t1, t3, t4, t2\}$ *is a* 4*-cycle based on* aba.

Definition 3.12. We say that a *groupoid* Q *is of the form* Qn, for some positive integer n, if $Q = \{aba\} \bigcup_{t=1}^{n} Ht$ for some $\{a, b\} \subseteq Q$, where each Ht is as in Definition 3.1.

4. Quadratical quasigroups of form Q1 and Q2

We are now in a position to examine more closely the Cayley tables of quadratical quasigroups. This will aid in the construction of the tables for quadratical quasigroups of orders 5, 9, 13 and 17. Dudek [1] gave two examples of quadratical quasigroups of orders 5, 13 and 17 and six examples of quadratical quasigroups of order 9. A close examination of the fine structure will aid us in proving that all these quadratical quasigroups are of the form Qn, for some positive integer n. Each pair of quadratical quasigroups of orders 5, 13 or 17 will be proved to be dual groupoids. The 6 quadratical quasigroups of order 9 will be proved to be of form $Q2$ and self-dual. That is, up to isomorphism, there is only one quadratical quasigroup of order 9.

A method of constructing quadratical quasigroups of the form Qn is as follows. Proposition 3.6 implies that $aba \cdot Ht = H(t-1)$ for all $t \neq 1$. Since quadratical quasigroups are cancellative, we can assume that $aba \cdot H1 = Hn$. If we choose the value of $aba \cdot 11$ in $Hn = \{n1, n2, n3, n4\}$ then, using the properties of quadratical quasigroups, we can attempt to fill in the remaining unknown products in the

Cayley table. If this can be done without contradiction, then, using Theorem 2.2, we can check that the groupoid thus obtained is quadratical, by checking that it is bookend and medial. Completing the Cayley table is this way is not always possible, as shown in the following example.

Example 4.1. Suppose Q is a quadratical quasigroup of the form $Q2$. Then $aba \cdot 11 = aba \cdot a \in H2 = \{21, 22, 23, 24\} = \{a \cdot ab, ab \cdot b, ba \cdot a, b \cdot ba\}$. Now $aba \cdot a = a(ba \cdot a)$ and so $aba \cdot a \notin \{a \cdot ab, ba \cdot a\}$, since cancellativity, idempotency and alterability would imply that $a = b$ (if $aba \cdot a = ba \cdot a$) and $b = a \cdot ab$ (if $aba \cdot a = a \cdot ab$), the latter contradicting to the fact that two 4-cycles based on aba are equal or disjoint (cf. [3]). Hence, $aba \cdot a$ must be in the set $\{ab \cdot b, b \cdot ba\}$. However, if $aba \cdot a = b \cdot ba$, then by (10), $ab = ba \cdot aba = (b \cdot ab)a = aba \cdot a = b \cdot ba$, a contradiction since $H1 \cap H2 = \emptyset$.

Example 4.1 shows that $aba \cdot a = ab \cdot b$. Using the properties of quadratical quasigroups, the Cayley table of the groupoid of the form $Q2$ can only be completed in one way, as shown below here, in Table 1.

We then need to calculate all the possible products $xy \cdot yx$ and $xy \cdot zw$ in Table 1, to prove that they are equal to y and $xz \cdot yw$ respectively. Then, by Theorem 2.2, $Q2$ would be quadratical. This proves to be the case and we omit the detailed calculations. However, to give a flavour of the calculations we find all products $aba \cdot x$ and $x \cdot aba$ when $x \in H1$ and $aba \cdot a = ab \cdot b$.

Since $(a \cdot aba)\,(aba \cdot a) = (a \cdot aba)\,(ab \cdot b)$, it follows that we have $a \cdot aba = b \cdot ba$, $aba \cdot b = ba \cdot a$, $aba \cdot ab = (aba \cdot a)\,(aba \cdot b) = (ab \cdot b)\,(ba \cdot a) = b \cdot ba$ and, similarly $aba \cdot ba = a \cdot ab$. Then $aba \cdot ab = b \cdot ba$ implies $ba \cdot aba = ab \cdot b$. Also, $aba = (ab \cdot aba)\,(aba \cdot ab) = (ab \cdot aba)\,(b \cdot ba)$ implies $ab \cdot aba = ba \cdot a$. Finally, $b \cdot aba = (ab \cdot aba)\,(ba \cdot aba) = (ba \cdot a)\,(ab \cdot b) = a \cdot ab$.

$Q2$	$11=a$	$12=ab$	$13=ba$	$14=b$	aba	$21=a\cdot ab$	$22=ab\cdot b$	$23=ba\cdot a$	$24=b\cdot ba$
$11=a$	a	$a\cdot ab$	aba	ab	$b\cdot ba$	ba	b	$ab\cdot b$	$ba\cdot a$
$12=ab$	aba	ab	b	$ab\cdot b$	$ba\cdot a$	$b\cdot ba$	a	$a\cdot ab$	ba
$13=ba$	$ba\cdot a$	a	ba	aba	$ab\cdot b$	ab	$b\cdot ba$	b	$a\cdot ab$
$14=b$	ba	aba	$b\cdot ba$	b	$a\cdot ab$	$ab\cdot b$	$ba\cdot a$	a	ab
aba	$ab\cdot b$	$b\cdot ba$	$a\cdot ab$	$ba\cdot a$	aba	a	ab	ba	b
$21=a\cdot ab$	$b\cdot ba$	b	$ba\cdot a$	a	ab	$a\cdot ab$	ba	aba	$ab\cdot b$
$22=ab\cdot b$	$a\cdot ab$	$ba\cdot a$	ab	ba	b	aba	$ab\cdot b$	$b\cdot ba$	a
$23=ba\cdot a$	ab	ba	$ab\cdot b$	$b\cdot ba$	a	b	$a\cdot ab$	$ba\cdot a$	aba
$24=b\cdot ba$	b	$ab\cdot b$	a	$a\cdot ab$	ba	$ba\cdot a$	aba	ab	$b\cdot ba$

Table 1.

Proposition 4.2. *A quadratical quasigroup Q of order 9 is of the form $Q = Q2$.*

Proof. We have $Q = H1 \cup \{aba\} \cup C$, where C is a 4-cycle based on aba and $C \cap H1 = \emptyset$. We proceed to prove that $C = H2$.

Consider the following part of the Cayley table: $(H1 \cup \{aba\}) \cdot H1$.

Q	a	ab	ba	b
a	a		aba	ab
ab	aba	ab	b	
ba		a	ba	aba
b	ba	aba		b
aba				

From the table, clearly, if $ba \cdot a \in H1 \cup \{aba\}$, then $ba \cdot a \in \{ab, b\}$.

Assume that $ba \cdot a = ab$. Then we have $a = b \cdot ba$, $ab \cdot b = (ba \cdot a)b = (ba \cdot b) \cdot ab = aba \cdot ab = a(ba \cdot b) = (b \cdot ba)(ba \cdot b) = b(ba \cdot ab) = ba$ and $b = a \cdot ab$. Also, $aba \cdot a = a(ba \cdot a) = a \cdot ab = b$, $aba \cdot ab = ab \cdot (a \cdot ab) = ab \cdot b = ba$, $aba \cdot b = bab \cdot b = b(ab \cdot b) = b \cdot ba = a$ and $aba \cdot ba = (aba \cdot b)(aba \cdot a) = ab$. So, we have proved that $(H1 \cup \{aba\}) \cdot H1 = H1 \cup \{aba\}$.

Similarly, if $ba \cdot b = b$, then $(H1 \cup \{aba\}) \cdot H1 = H1 \cup \{aba\}$, which is not possible because, if $c \in C$, then $c \in C = \{ca, c \cdot ab, c \cdot ba, cb\}$, a contradiction. So, $ba \cdot a = c$, for some $c \in C$. Then, since $C = \{c, dc, d, cd\}$ for some $d \in C$, we have $aba = c \cdot dc = dc \cdot d = d \cdot cd = cd \cdot c$. So, $aba = (ba \cdot a) \cdot dc$, which implies $dc = b \cdot ba$. Also, $aba = cd \cdot (ba \cdot a)$, which implies $cd = a \cdot ab$. Then, $aba = dc \cdot d = (b \cdot ba)d$, which gives $d = ab \cdot b$. Hence, $C = \{a \cdot ab, ab \cdot b, ba \cdot a, b \cdot ba\} = H2$. □

So, we have proved that a quadratical quasigroup of order 9 must be the quasigroup Q2.

Open question. *Is a finite, idempotent, alterable, cancellative, elastic groupoid of form Qn quadratical?*

Note that we can prove that the answer is affirmative when $n = 1$ or $n = 2$.

Now, if we calculate the Cayley table for $(Q2)^*$, the dual of $Q2$, we see that the table for the dual product $*$ (defined as $a * b = b \cdot a$) is exactly the same as Table 1, where the product is the dot product $\cdot$. (For example, $((b * a) * a) * (b * a) = (a \cdot ab) * ab = ab \cdot (a \cdot ab) = b \cdot ba = (a * b) * b$ and, by Table 1, $(ba \cdot a) \cdot ba = ab \cdot b$). Hence, $Q2 \cong (Q2)^*$. Another way to put this is that the quadratical groupoid $Q2$ must be self-dual. An isomorphism θ between $Q2$ and $(Q2)^*$ is: $\theta a = a$, $\theta b = b$, $\theta(ab) = a * b$, $\theta(ba) = b * a$, $\theta(a \cdot ab) = a * (a * b)$, $\theta(ab \cdot b) = (a * b) * b$, $\theta(ba \cdot a) = (b * a) * a$ and $\theta(b \cdot ba) = b * (b * a)$.

Example 4.3. It is straightforward to calculate the Cayley tables of the quadratical quasigroups, each of order 9, given in [1]. They are each based on the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ of ordered pairs of integers, with product being addition (mod 3). The products are defined as follows:

$(x, y) *_1 (z, u) = (y + z + 2u, x + y + 2z)$,
$(x, y) *_2 (z, u) = (2y + z + u, 2x + y + z)$,
$(x, y) *_3 (z, u) = (x + y + 2u, x + 2z + u)$,
$(x, y) *_4 (z, u) = (x + 2y + u, 2x + z + u)$,
$(x, y) *_5 (z, u) = (2x + y + 2z + 2u, 2x + 2y + z + 2u)$,
$(x, y) *_6 (z, u) = (2x + 2y + 2z + u, x + 2y + 2z + 2u)$.

In each table, if we calculate ab and ba for the ordered pairs $a = (1, 1)$ and $b = (1, 2)$ we see that $Q = \{aba\} \cup H1 \cup H2$ and that $aba \cdot a = ab \cdot b$. Therefore, these six quadratical quasigroups are isomorphic to each other and to $Q2$. We already knew that there is only one quadratical quasigroup of order 9, but these calculations clarify (and reinforce a conviction) that the quadratical quasigroups of order 9 presented in [1] are isomorphic.

Example 4.4. We now calculate the Cayley table for a groupoid $Q1$ and its dual, when $aba \cdot a \in \{ab, b\}$.

$Q1$	a	ab	ba	b	aba
a	a	ba	aba	ab	b
ab	aba	ab	b	a	ba
ba	b	a	ba	aba	ab
b	ba	aba	ab	b	a
aba	ab	b	a	ba	aba

$(Q1)^*$	a	$b*a$	$a*b$	b	aba
a	a	aba	b	$a*b$	$b*a$
$b*a$	$a*b$	$b*a$	a	aba	b
$a*b$	aba	b	$a*b$	$b*a$	a
b	$b*a$	a	aba	b	$a*b$
aba	b	$a*b$	$b*a$	a	aba

Table 2.

Checking these tables shows that each is medial and bookend and that, indeed, these two quadratical quasigroups are dual.

Open question. *Examining Tables* 1 *and* 2 *closely, we can show that any two distinct elements of* $Q1$ *(resp.* $(Q1)^*$, $Q2$*) generate* $Q1$ *(resp.* $(Q1)^*$, $Q2$*). This will later be seen to be the case also for* $Q3$, $Q4$ *and their duals. We conjecture that if* Q *is a quadratical quasigroup of form* Qn*, for some positive integer* n*, then it is generated by any two distinct elements. Such a property does not hold in quadratical quasigroups in general, as we shall now prove.*

Example 4.5. Since $\mathcal{Q}$ is a variety of groupoids, the direct product of quadratical quasigroups is quadratical. Hence, $Q1 \times Q1$ is quadratical. If we choose a base element, (a, b) say, then $Q1 \times Q1$ consists of six disjoint 4-cycles based on (a, b); namely,

$$\{(a,a),(a,aba),(a,ab),(a,ba)\}, \qquad \{(b,ab),(aba,ba),(ba,a),(ab,aba)\},$$
$$\{(ab,b),(b,b),(aba,b),(ba,b)\}, \qquad \{(ab,ab),(b,ba),(aba,a),(ba,aba)\},$$
$$\{(ba,ba),(ab,a),(b,aba),(aba,ab)\}, \quad \{(aba,aba),(ba,ab),(ab,ba),(b,a)\}.$$

If C is any one of these six 4-cycles, then no two distinct elements x and y of C generates $Q1 \times Q1$, because $\{x, y\} \subseteq C$ and C is a proper subquadratical quasigroup of $Q1 \times Q1$, isomorphic to $Q1$.

Example 4.6. $(Q1 \times Q1)^* = (Q1)^* \times (Q1)^*$ and $(Q1 \times (Q1)^*)^* = (Q1)^* \times Q1$. Note that (a, ba) and (ab, b) generate $Q1 \times (Q1)^*$ and (ba, a) and (b, ab) generate $(Q1)^* \times Q1$ while $Q1 \times Q1$ and $(Q1)^* \times (Q1)^*$ are not 2-generated.

5. The elements nk*

The following Theorem is easily proved for $k = 1$ and, by induction on k, is straightforward to prove for all $k \in \{0, 1, 2, \ldots\} = \mathbb{N}_0$. The proof is omitted but we proceed to give an idea of some of the calculations.

For $k = 0$

$$\begin{aligned}((4+4k)4)^* &= 44^* = (34 \cdot 33)^* = ((24 \cdot 23) \cdot (23 \cdot 21))^* \\ &= ((b \cdot ba)(ba \cdot a) \cdot (ba \cdot a)(a \cdot ab))^* = (ba \cdot a)(a \cdot ab) \cdot (a \cdot ab)(ab \cdot b) \\ &= (23 \cdot 21) \cdot (21 \cdot 22) = 33 \cdot 31 = 43 = ((4+4k)3).\end{aligned}$$

Note that we get the same result if we write

$$44^* = [(b * (b * a) * ((b * a) * a))] * [((b * a) * a) * (a * (a * b))].$$

Theorem 5.1. *For all $k \in \mathbb{N}_0$,*

$$\begin{array}{llll} ((1{+}4k)1)^* = (1{+}4k)1, & ((1{+}4k)2)^* = (1{+}4k)3, & ((1{+}4k)3)^* = (1{+}4k)2, & ((1{+}4k)4)^* = (1{+}4k)4, \\ ((2{+}4k)1)^* = (2{+}4k)3, & ((2{+}4k)2)^* = (2{+}4k)4, & ((2{+}4k)3)^* = (2{+}4k)1, & ((2{+}4k)4)^* = (2{+}4k)2, \\ ((3{+}4k)1)^* = (3{+}4k)4, & ((3{+}4k)2)^* = (3{+}4k)2, & ((3{+}4k)3)^* = (3{+}4k)3, & ((3{+}4k)4)^* = (3{+}4k)1, \\ ((4{+}4k)1)^* = (4{+}4k)2, & ((4{+}4k)2)^* = (4{+}4k)1, & ((4{+}4k)3)^* = (4{+}4k)4, & ((4{+}4k)4)^* = (4{+}4k)3. \end{array}$$

Further, for simplicity, elements of the form $(xy)^*$ will be denoted as xy^*.

Now, considering the quadratical quasigroups of form Qn, from the remarks in the paragraph preceding Example 4.1, we see that there are at most 4 groupoids of the form Qn for any given integer n. Since the dual of a quadratical quasigroup of the form Qn must also have the form Qn, we can tell, from the following Theorem, which values of $aba \cdot a$ may yield groupoids that are duals of each other.

Theorem 5.2. *For all positive integers $n \geqslant 2$, the following identities are valid in a quadratical quasigroup of form Qn, depending on the value of $aba \cdot a$:*

$aba{\cdot}a$	$aba{\cdot}ab$	$aba{\cdot}ba$	$aba{\cdot}b$	$a{\cdot}aba$	$ab{\cdot}aba$	$ba{\cdot}aba$	$b{\cdot}aba$	$n1{\cdot}n2$	$n2{\cdot}n4$	$n3{\cdot}n1$	$n4{\cdot}n3$
$n1$	$n2$	$n3$	$n4$	$n2$	$n4$	$n1$	$n3$	a	ab	ba	b
$n2$	$n4$	$n1$	$n3$	$n4$	$n3$	$n2$	$n1$	ba	a	b	ab
$n3$	$n1$	$n4$	$n2$	$n1$	$n2$	$n3$	$n4$	ab	b	a	ba
$n4$	$n3$	$n2$	$n1$	$n3$	$n1$	$n4$	$n2$	b	ba	ab	a

$aba{\cdot}a$	$11{\cdot}34$	$23{\cdot}14$	$34{\cdot}14$	$14{\cdot}21$	
$n1$	$n3$	$n2$	$n1$	$n1$	$(n-1)2 = 11{\cdot}n1 = n2{\cdot}11$
$n2$	$n1$	$n4$	$n2$	$n2$	$(n-1)4 = 11{\cdot}n2 = n4{\cdot}11$
$n3$	$n4$	$n1$	$n3$	$n3$	$(n-1)1 = 11{\cdot}n3 = n1{\cdot}11$
$n4$	$n2$	$n3$	$n4$	$n4$	$(n-1)3 = 11{\cdot}n4 = n3{\cdot}11$

Proof. We prove only the identities for when $aba{\cdot}a = n2$, as the proofs of the other three cases are similar. We have $aba{\cdot}n2$. Then, $aba = (a{\cdot}aba)(aba{\cdot}a) = (a{\cdot}aba){\cdot}n2$. By Proposition 3.11 and Theorem 2.1, $a \cdot aba = n4 = a \cdot bab = aba \cdot ab$. Then,

$n4 = aba \cdot ab = (aba \cdot a)(aba \cdot b) = n2 \cdot (aba \cdot b)$. By Proposition 3.5, $aba \cdot b = n3$. So $aba \cdot ba = (aba \cdot b)(aba \cdot a) = n3 \cdot n2 = n1$ (by Proposition 3.5). Then, $aba = (b \cdot aba)(aba \cdot b) = (b \cdot aba) \cdot n3$, which by Proposition 3.11 implies $b \cdot aba = n1$. Then, using Proposition 3.5, $ab \cdot aba = (a \cdot aba)(b \cdot aba) = n4 \cdot n1 = n3$ and $ba \cdot aba = (b \cdot aba)(a \cdot aba) = n1 \cdot n4 = n2$. We also have $n1 \cdot n2 = (aba \cdot ba)(ba \cdot aba) = ba$, $n2 \cdot n4 = (aba \cdot a)(a \cdot aba)) = a$, $n3 \cdot n1 = (aba \cdot b)(b \cdot aba) = b$ and $n4 \cdot n3 = (aba \cdot ab)(ab \cdot aba) = ab$. Now, $11 \cdot 34 = a \cdot (b \cdot ba)(ba \cdot a) = a(b \cdot ba) \cdot a(ba \cdot a) = (ab \cdot aba)(aba \cdot a) = n3 \cdot n2 = n1$, $34 \cdot 14 = (b \cdot ba)(ba \cdot a) \cdot b = (b \cdot ba)b \cdot (ba \cdot a)b = (b \cdot bab)(bab \cdot ab) = (b \cdot aba)(aba \cdot ab) = n1 \cdot n4 = n2$, $14 \cdot 21 = b(a \cdot ab) = ba \cdot bab = ba \cdot aba = n2$ and $23 \cdot 14 = (ba \cdot a)b = bab \cdot ab = aba \cdot ab = n4$.

Finally, $a \cdot n2 = a \cdot aba \cdot a = aba \cdot a \cdot aba = aba \cdot n4 = (n-1)4 = 11 \cdot n2$ and $n4 \cdot a = a \cdot aba \cdot a = aba \cdot a \cdot aba = (n-1)4 = n4 \cdot 11$.

This completes the proof of the validity of the identities indicated in row 3 of the two tables in Theorem 5.2, when $aba \cdot a = n2$. □

As mentioned above, Theorem 5.2 will be useful when we look for the duals of the quadratical quasigroups that we will call $Q3$ and $Q4$, as will the following concept.

Definition 5.3. If a quadratical quasigroup of form Qn exists for some integer n then the identity generated on the left (on the right) by an identity $kr \cdot ls = mt$, where $r, s, t \in \{1, 2, 3, 4\}$ and $k, l, m \leqslant n$, is defined as the identity

$$(aba \cdot kr)(aba \cdot ls) = aba \cdot mt \quad \text{(resp. } (kr \cdot aba)(ls \cdot aba) = mt \cdot aba)$$

and $kr \cdot ls = mt$ is called the *generating identity*.

Note that Propositions 3.6 and 3.7, along with Theorem 5.2, give the means of calculating identities generated on the left and right by a given identity. Multiplying on the left (or on the right) repeatedly n-times gives n distinct identities. These methods will later be used to prove that quadratical quasigroups of the form $Q6$ do not exist.

6. Quadratical quasigroups of forms Q3 and Q4

We give the Cayley tables of quadratical quasigroups of orders 13 and 17.

First we note that for a quadratical quasigroup of form $Q3$, if $aba \cdot a = n3 = 33 = (ba \cdot a)(a \cdot ab)$, then $aba \cdot a = a(ba \cdot a) = (a \cdot ab) \cdot aba = ab$, which implies, by cancellation, $ba \cdot a = b$, a contradiction because $H1 \cap H2 = \emptyset$. If $aba \cdot a = n4 = 34 = (b \cdot ba)(ba \cdot a)$, then $ab \cdot aba = a(b \cdot ba) = (ba \cdot a) \cdot aba = a$, which implies $b \cdot ba = a$, a contradiction. Hence, $aba \cdot a \in \{31, 32\} = \{(a \cdot ab)(ab \cdot b), (ab \cdot b)(b \cdot ba)\}$. Setting $aba \cdot a = a \cdot ab$ and using the properties of quadratical quasigroups (Theorem 2.1) we obtain the Cayley Table 3. It can be checked that it is medial and bookend and so, by Theorem 2.2, this groupoid is a quadratical quasigroup.

$Q3$	11	12	13	14	aba	21	22	23	24	31	32	33	34
11	11	21	aba	12	32	14	23	31	34	22	13	24	33
12	aba	12	14	22	34	32	13	33	21	23	24	31	11
13	23	11	13	aba	31	24	32	12	33	14	34	21	22
14	13	aba	24	14	33	31	34	22	11	32	21	12	23
aba	31	32	33	34	aba	11	12	13	14	21	22	23	24
21	32	23	34	13	12	21	31	aba	22	24	33	11	14
22	33	34	11	21	14	aba	22	24	32	12	23	13	31
23	24	14	31	32	11	33	21	23	aba	34	12	22	13
24	12	31	22	33	13	23	aba	34	24	11	14	32	21
31	34	13	21	24	22	12	33	14	23	31	11	aba	32
32	22	33	23	11	24	13	14	21	31	aba	32	34	12
33	14	22	32	23	21	34	24	11	12	13	31	33	aba
34	21	24	12	31	23	22	11	32	13	33	aba	14	34

Table 3.

There are then two ways to obtain the Cayley table for $(Q3)^*$. Firstly, we can use $aba * a = 32^* = [(a*b)*b]*[b*(b*a)]$ and, using the properties of quadratical quasigroups, we can then calculate the remaining products in Table 4.

Alternatively, we can calculate the products directly from Table 3, using our Theorem 5.1. For example, $23^* = (b*a)*a = a \cdot ab = 21$, and similarly $32^* = ((a*b)*b)*(b*(b*a)) = (ab \cdot b)(b \cdot ba) = 32$. Hence, $32^* * 23^* = 32 * 21 = 21 \cdot 32$. From Table 3, $21 \cdot 32 = 33$. But from Theorem 5.1, $33 = 33^*$. So, we obtain $32^* * 23^* = 33 = 33^*$. The remaining products in Table 4 can be calculated in similar fashion. Having already checked that Table 3 is quadratical, Table 4 also produces a quadratical quasigroup, the dual groupoid.

$(Q3)^*$	11^*	12^*	13^*	14^*	aba	21^*	22^*	23^*	24^*	31^*	32^*	33^*	34^*
11^*	11^*	21^*	aba	12^*	34^*	22^*	13^*	32^*	33^*	23^*	24^*	14^*	31^*
12^*	aba	12^*	14^*	22^*	33^*	34^*	24^*	31^*	11^*	13^*	21^*	32^*	23^*
13^*	23^*	11^*	13^*	aba	32^*	14^*	34^*	21^*	31^*	22^*	33^*	24^*	12^*
14^*	13^*	aba	24^*	14^*	31^*	32^*	33^*	12^*	23^*	34^*	11^*	21^*	22^*
aba	32^*	34^*	31^*	33^*	aba	11^*	12^*	13^*	14^*	21^*	22^*	23^*	24^*
21^*	34^*	13^*	33^*	24^*	12^*	21^*	31^*	aba	22^*	32^*	23^*	11^*	14^*
22^*	31^*	33^*	23^*	11^*	14^*	aba	22^*	24^*	32^*	12^*	34^*	13^*	21^*
23^*	14^*	22^*	32^*	34^*	11^*	33^*	21^*	23^*	aba	23^*	12^*	31^*	13^*
24^*	21^*	32^*	12^*	31^*	13^*	23^*	aba	34^*	24^*	11^*	14^*	22^*	33^*
31^*	33^*	24^*	11^*	21^*	22^*	12^*	23^*	14^*	34^*	31^*	13^*	aba	32^*
32^*	12^*	31^*	22^*	23^*	24^*	13^*	14^*	33^*	21^*	aba	32^*	34^*	11^*
33^*	22^*	23^*	34^*	13^*	21^*	24^*	32^*	11^*	12^*	14^*	31^*	33^*	aba
34^*	24^*	14^*	21^*	32^*	23^*	31^*	11^*	22^*	13^*	33^*	aba	12^*	34^*

Table 4.

Similarly, we can calculate the Cayley tables for $Q4$ and its dual $(Q4)^*$:

$Q4$	11	12	13	14	aba	21	22	23	24	31	32	33	34	41	42	43	44
11	11	21	aba	12	44	24	32	42	43	14	23	31	41	33	34	13	22
12	aba	12	14	22	43	44	23	41	34	32	13	42	21	11	31	24	33
13	23	11	13	aba	42	31	44	22	41	24	43	12	33	32	21	34	14
14	13	aba	24	14	41	42	43	33	21	44	34	22	11	23	12	31	32
aba	42	44	41	43	aba	11	12	13	14	21	22	23	24	31	32	33	34
21	44	32	43	23	12	21	31	aba	22	34	42	11	14	24	33	41	13
22	41	43	21	34	14	aba	22	24	32	12	33	13	44	42	23	11	31
23	31	24	42	44	11	33	21	23	aba	41	12	32	13	34	14	22	43
24	22	42	33	41	13	23	aba	34	24	11	14	43	31	12	44	32	21
31	43	23	34	13	22	12	42	14	33	31	41	aba	32	44	11	21	24
32	33	41	11	21	24	13	14	31	44	aba	32	34	42	22	43	23	12
33	24	14	44	32	21	41	34	11	12	43	31	33	aba	13	22	42	23
34	12	31	22	42	23	32	11	43	13	33	aba	44	34	21	24	14	41
41	21	34	12	31	32	14	33	44	23	22	11	24	43	41	13	aba	42
42	14	22	32	33	34	43	13	21	31	23	24	41	12	aba	42	44	11
43	32	33	23	11	31	34	24	12	42	13	44	21	22	14	41	43	aba
44	34	13	31	24	33	22	41	32	11	42	21	14	23	43	aba	12	44

Table 5.

$(Q4)^*$	11*	12*	13*	14*	aba	21*	22*	23*	24*	31*	32*	33*	34*	41*	42*	43*	44*
11*	11*	21*	aba	12*	41*	34*	24*	43*	42*	13*	33*	22*	44*	14*	23*	31*	32*
12*	aba	12*	14*	22*	42*	41*	33*	44*	23*	24*	11*	43*	31*	32*	13*	34*	21*
13*	23*	11*	13*	aba	43*	22*	41*	32*	44*	34*	42*	14*	21*	24*	31*	12*	33*
14*	13*	aba	24*	14*	44*	43*	42*	21*	31*	41*	23*	32*	12*	33*	34*	22*	11*
aba	43*	41*	44*	42*	aba	11*	12*	13*	14*	21*	22*	23*	24*	31*	32*	33*	34*
21*	41*	24*	42*	33*	12*	21*	31*	aba	22*	44*	34*	11*	14*	23*	43*	32*	13*
22*	44*	42*	31*	23*	14*	aba	22*	24*	32*	12*	43*	13*	33*	34*	21*	11*	41*
23*	22*	34*	43*	41*	11*	33*	21*	23*	aba	32*	12*	42*	13*	44*	14*	24*	31*
24*	32*	43*	21*	44*	13*	23*	aba	34*	24*	11*	14*	31*	41*	12*	33*	42*	22*
31*	42*	33*	23*	11*	22*	12*	34*	14*	43*	31*	41*	aba	32*	13*	44*	21*	24*
32*	21*	44*	12*	31*	24*	13*	14*	41*	33*	aba	32*	34*	42*	22*	11*	23*	43*
33*	34*	13*	41*	24*	21*	32*	44*	11*	12*	43*	31*	33*	aba	42*	22*	14*	23*
34*	14*	22*	32*	43*	23*	42*	11*	31*	13*	33*	aba	44*	34*	21*	24*	41*	12*
41*	31*	23*	34*	13*	32*	14*	43*	33*	21*	22*	44*	24*	11*	41*	12*	aba	42*
42*	33*	32*	11*	21*	34*	31*	13*	22*	41*	23*	24*	12*	43*	aba	42*	44*	14*
43*	24*	14*	33*	32*	31*	44*	23*	12*	34*	42*	13*	21*	22*	11*	41*	43*	aba
44*	12*	31*	22*	34*	33*	24*	32*	42*	11*	14*	21*	41*	23*	43*	aba	13*	44*

Table 6.

Groups of orders 13 and 17 are isomorphic to the additive groups $\mathbb{Z}_{13}$ and $\mathbb{Z}_{17}$, respectively. So, by Theorem 2.5, quasigroups $Q3$ and $Q4$ are isomorphic to quadratical quasigroups induced by $\mathbb{Z}_{13}$ and $\mathbb{Z}_{17}$, respectively. Direct computations show that $Q3$ is isomorphic to the quadratical quasigroup $(\mathbb{Z}_{13}, \cdot)$ with the operation $x \cdot y = 11x + 3y (\mathrm{mod}\, 13)$; the dual quasigroup $(Q3)^*$ is isomorphic to the quasigroup $(\mathbb{Z}_{13}, \circ)$ with the operation $x \circ y = 3x + 11y (\mathrm{mod}\, 13)$. Similarly,

$Q4$ is isomorphic to $(\mathbb{Z}_{17}, \cdot)$ with the operation $x \cdot y = 11x + 7y (\operatorname{mod} 17)$. Its dual quasigroup $(Q4)^*$ is isomorphic to the quasigroup $(\mathbb{Z}_{17}, \circ)$ with the operation $x \circ y = 7x + 11y (\operatorname{mod} 17)$.

7. No quadratical quasigroup of form Q6 exists

The quasigroup $x \cdot y = [9x + 21y]_{29}$ is clearly idempotent, medial and bookend. Therefore, by Theorem 2.2, it is quadratical. Set $a = 1$ and $b = 2$. Then we can calculate that $aba = 16$, $H1 = \{1, 22, 10, 2\}$, $H2 = \{7, 8, 24, 25\}$, $H3 = \{28, 17, 15, 4\}$, $H4 = \{29, 5, 27, 3\}$, $H5 = \{18, 21, 11, 14\}$, $H6 = \{23, 19, 13, 9\}$ and $H7 = \{26, 12, 20, 6\}$. Hence, this quasigroup and its dual are of the form $Q7$. So, we have so far shown that there are quadratical quasigroups of the form $Q1$, $Q2$, $Q3$, $Q4$ and $Q7$.

It follows from Theorem 4.11 [3] that there are no quadratical quasigroups of order 21 or 33, so there are no quadratical quasigroups of the form $Q5$ or $Q8$.

Theorem 7.1. *There is no quadratical quasigroup of form* $Q6$.

Proof. Case 1: $aba \cdot a = 61$. Using Propositions 3.6, 3.7, Theorem 5.2 and Theorem 2.1, we see that $aba \cdot a = 61$, by (10), implies

$$a \cdot 61 = 61 \cdot aba = 52 = 62 \cdot 11 \overset{(10)}{=} aba \cdot 62 = 52 \cdot 52. \tag{12}$$

Then, $62 = 61 \cdot 64$, by Proposition 3.5. This, by Proposition 3.6, gives $52 = 51 \cdot 54$. Also, $62 = 52 \cdot 54$, by Definition 3.1, whence $52 = 42 \cdot 44$, by Proposition 3.6 and (10), and so

$$52 = 51 \cdot 54 = 42 \cdot 44. \tag{13}$$

Theorem 5.2 implies $61 = 14 \cdot 21 = 34 \cdot 14$, $62 = 23 \cdot 14$ and $63 = 11 \cdot 34$. So, these identities generate the following:

$$52 = 63 \cdot 12 = 13 \cdot 64 = 64 \cdot 21 = 23 \cdot 63. \tag{14}$$

As a consequence of (12), (13), (15), Proposition 3.5 and Proposition 3.10 we can see that the solutions to the equation $52 = 12 \cdot x$ must be in the set $\{14, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 51, 53\}$. Now, by Definition 3.1, we obtain $22 = 12 \cdot 14 \neq 52$ and so $x \neq 14$.

To eliminate the other possibilities for x we now use the generating identities (15) through (25), indicated in the Table 7 below.

	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)
	$(n{-}1)2$ $=11\cdot n1$	$(n{-}1)2$ $=n2\cdot 11$	$aba\cdot 11$ $=61$	$11\cdot aba$ $=62$	*Prop.* 3.5	*Def.* 3.1	$n1=$ $14\cdot 21$	$n2=$ $23\cdot 14$	$n3=$ $11\cdot 34$	$n1=$ $34\cdot 14$	*idem.*
52	11·61	62·11	*aba*·62	61·*aba*	51·54	42·44	63·12	13·64	64·21	23·13	52·52
44	62·52	54·62	*aba*·54	52·*aba*	42·43	34·33	51·64	61·22	53·12	11·61	44·44
33	54·44	43·54	*aba*·43	44·*aba*	34·31	23·21	42·53	52·41	41·64	62·52	33·33
21	43·33	31·43	*aba*·31	33·*aba*	23·22	11·12	34·41	44·32	32·53	54·34	21·21
12	31·21	22·31	*aba*·22	21·*aba*	11·14	62·64	23·32	33·24	24·41	43·33	12·12
64	22·12	14·22	*aba*·14	12·*aba*	62·63	54·53	11·24	21·13	13·32	31·21	64·64
53	14·64	63·14	*aba*·63	64·*aba*	54·51	43·41	62·13	12·61	61·24	22·12	53·53
41	63·53	51·63	*aba*·51	53·*aba*	43·42	31·32	54·61	64·52	52·13	14·54	41·41
32	51·41	42·51	*aba*·42	41·*aba*	31·34	22·24	43·52	53·44	44·61	63·53	32·32
24	42·32	34·42	*aba*·34	32·*aba*	22·23	14·13	31·44	41·33	33·52	51·41	24·24
13	34·24	23·34	*aba*·23	24·*aba*	14·11	63·61	22·33	32·21	21·44	42·32	13·13
61	23·13	11·23	*aba*·11	13·*aba*	63·62	51·52	14·21	24·12	12·33	34·14	61·61
51	13·63	61·13	*aba*·61	63·*aba*	53·52	41·42	64·11	14·62	62·23	24·64	51·51
31	53·43	41·53	*aba*·41	43·*aba*	33·32	21·22	44·51	54·42	42·63	64·44	31·31
11	33·23	21·33	*aba*·21	23·*aba*	13·12	61·62	24·31	34·22	22·43	44·24	11·11
62	21·11	12·21	*aba*·12	11·*aba*	61·64	52·54	13·22	23·14	14·31	33·23	62·62
54	12·62	64·12	*aba*·64	62·*aba*	52·53	44·43	61·14	11·63	63·22	21·11	54·54
43	64·54	53·64	*aba*·53	54·*aba*	44·41	33·31	52·63	62·51	51·14	12·62	43·43
34	52·42	44·52	*aba*·44	42·*aba*	32·33	24·23	41·54	51·43	43·62	61·51	34·34
23	44·34	33·44	*aba*·33	34·*aba*	24·21	13·11	32·43	42·31	31·54	52·42	23·23
14	32·22	24·32	*aba*·24	22·*aba*	12·13	64·63	21·34	31·23	23·42	41·31	14·14
63	24·14	13·24	*aba*·13	14·*aba*	64·61	53·51	12·23	22·11	11·34	32·22	63·63
42	61·51	52·61	*aba*·52	51·*aba*	41·44	32·34	53·62	63·54	54·11	13·63	42·42
22	41·31	32·41	*aba*·32	31·*aba*	21·24	12·14	33·42	43·34	34·51	53·53	22·22

Table 7.

Assuming that $Q6$ is quadratical, using the properties of a quadratical quasigroup we will prove that all the remaining possible values of x lead to a contradiction.

When we use a particular value of an element we will refer to the column in which this value appears in Table 7. For example, we will use the fact that $52 = 63\cdot 12$, from (21), henceforth without mention

By (21), if $52 = 12\cdot 53 = 63\cdot 12$, then $12 = 53\cdot 63$, and, multiplying on the right by aba gives $64 = 41\cdot 51$, which, along with $51\cdot 41 = 24$, (from (24)) gives $51 = 64\cdot 24$. This contradicts $51 = 64\cdot 11$, from (21).

If $52 = 12\cdot 51 = 63\cdot 12$ then $12 = 51\cdot 63 = 62\cdot 64$, from (20). Hence, by (19) and (20), $61 = 63\cdot 62 = 64\cdot 51 = 51\cdot 52$. Therefore, using (24), $51 = 52\cdot 64 = 24\cdot 64$, a contradiction.

If $52 = 12\cdot 43 = 63\cdot 12$ then, by (23), $12 = 43\cdot 63 = 24\cdot 41$. By Proposition 3.11 we have $63\cdot 24 = 41\cdot 43 = aba = 23\cdot 24$, contradiction.

If $52 = 12\cdot 42 = 63\cdot 12$ then, by (23), is $12 = 42\cdot 63 = 24\cdot 41$. By Proposition 3.11 and (24), $51 = 41\cdot 42 = 63\cdot 24 = 24\cdot 64$. So, by (20), $24 = 64\cdot 63 = 14$, contradiction.

If $52 = 12 \cdot 41 = 63 \cdot 12$ then, by (23), $12 = 41 \cdot 63 = 24 \cdot 41$ and so, using (15), $41 = 63 \cdot 24 = 63 \cdot 53$, contradiction.

If $52 = 12 \cdot 34 = 63 \cdot 12$ then, by (21), $12 = 34 \cdot 63 = 23 \cdot 32$ and, by Proposition 3.11 and (22), $42 = 32 \cdot 34 = 63 \cdot 23 = 63 \cdot 54$, contradiction.

If $52 = 12 \cdot 33 = 63 \cdot 12$ then, by (21), $12 = 33 \cdot 63 = 23 \cdot 32$ and so, by Propositions 3.11 and 3.5, $34 = 32 \cdot 33 = 63 \cdot 23 = 24 \cdot 23$, contradiction.

If $52 = 12 \cdot 32 = 63 \cdot 12$ then, by (21), $12 = 32 \cdot 63 = 23 \cdot 32$ and so, by (24), $32 = 63 \cdot 33 = 63 \cdot 53$, contradiction.

If $52 = 12 \cdot 31 = 63 \cdot 12$ then, by (15), $12 = 31 \cdot 63 = 31 \cdot 21$, contradiction.

If $52 = 12 \cdot 24 = 63 \cdot 12$ then, by (15), $12 = 24 \cdot 63 = 24 \cdot 41$, contradiction.

If $52 = 12 \cdot 23 = 63 \cdot 12$ then, by (21), $12 = 23 \cdot 63 = 23 \cdot 32$, contradiction.

If $52 = 12 \cdot 22 = 63 \cdot 12$ then, by (26), $12 = 22 \cdot 63 = 22 \cdot 31$, contradiction.

If $52 = 12 \cdot 14 = 63 \cdot 12$ then, by Proposition 3.11, $52 = 12 \cdot 14 = 22$, contradiction.

In this way we have proved that when $aba \cdot a = 61$, there is no right solvability, a contradiction.

The proof that there is no right solvability in Case 2 ($aba \cdot a = 62$), Case 3 ($aba \cdot a = 63$) and Case 4 ($aba \cdot a = 64$) are similar, where the values in Table 7 are different, according to Theorem 5.2. We omit these detailed calculations. □

There are 32 quadratical quasigroups of order 25 (cf. [3]). Some of them are isomorphic to quasigroups $Q1 \times Q1$, $Q1 \times (Q1)^*$, $(Q1)^* \times Q1$, $(Q1)^* \times (Q1)^*$.

Theorem 7.2. *Quadratical quasigroups induced by* $\mathbb{Z}_{25}$ *are not isomorphic to* $Q1 \times Q1$, $Q1 \times (Q1)^*$, $(Q1)^* \times Q1$, $(Q1)^* \times (Q1)^*$.

Proof. There are only two quadratical quasigroups induced by $\mathbb{Z}_{25}$ (cf. [3]). Their operations are given by $x \cdot y = 22x + 4y (\mathrm{mod}\, 25)$ and $x \circ y = 4x + 22y (\mathrm{mod}\, 25)$. Quasigroups $Q1$ and $(Q1)^*$ are isomorphic, respectively, to quasigroups $(\mathbb{Z}_5, \cdot)$ and $(\mathbb{Z}_5, \circ)$, where $x \cdot y = 4x + 2y (\mathrm{mod}\, 5)$ and $x \circ y = 2x + 4y (\mathrm{mod}\, 5)$.

Suppose that $(\mathbb{Z}_{25}, \cdot)$ is isomorphic to $Q1 \times Q1$ or to $Q1 \times (Q1)^*$. Since in $(\mathbb{Z}_5, \cdot)$ we have $x \cdot xy = yx$, in $Q1 \times Q1$ and $Q1 \times (Q1)^*$ for all $\overline{x} = (x, a) \neq \overline{y} = (y, a)$, $\bar{x} \cdot \bar{x}\bar{y} = \bar{y}\bar{x}$. But in $(\mathbb{Z}_{25}, \cdot)$ we have $22\bar{y} + 4\bar{x} = \bar{y}\bar{x} = \bar{x} \cdot \bar{x}\bar{y} = 10\bar{x} + 16\bar{y}$, which implies $\bar{x} = \bar{y}$. So, $(\mathbb{Z}_{25}, \cdot)$ cannot be isomorphic to $Q1 \times Q1$ or $Q1 \times (Q1)^*$.

In $(Q1)^* \times Q1$ and $(Q1)^* \times (Q1)^*$ for all $\overline{x} = (x, a) \neq \overline{y} = (y, a)$, we have $\bar{y}\bar{x} \cdot \bar{x} = \bar{x}\bar{y}$. But in $(\mathbb{Z}_{25}, \cdot)$ we have $22\bar{x} + 4\bar{y} = \bar{x}\bar{y} = \bar{y}\bar{x} \cdot \bar{x} = 9\bar{y} + 17\bar{x}$, which implies $\bar{x} = \bar{y}$. So, $(\mathbb{Z}_{25}, \cdot)$ also cannot be isomorphic to $(Q1)^* \times Q1$ or $(Q1)^* \times (Q1)^*$.

In the same manner we can prove that $(\mathbb{Z}_{25}, \circ)$ is not isomorphic to $Q1 \times Q1$, $Q1 \times (Q1)^*$, $(Q1)^* \times Q1$, $(Q1)^* \times (Q1)^*$. □

8. Translatable groupoids

Patterns of *translatability* can be hidden in the Cayley tables of quadratical quasigroups. One can assume the properties of quadratical quasigroups and then calculate whether translatable groupoids of various orders exist with these properties. We proceed to prove that the quadratical quasigroups $Q1$, $(Q1)^*$, $Q3$, $(Q3)^*$, $Q4$ and $(Q4)^*$ are translatable and that $Q2$ is not translatable.

Definition 8.1. A finite groupoid $Q = \{1, 2, \ldots, n\}$ is called *k-translatable*, where $1 \leqslant k < n$, if its Cayley table is obtained by the following rule: If the first row of the Cayley table is $a_1, a_2, \ldots, a_n$, then the q-th row is obtained from the $(q-1)$-st row by taking the last k entries in the $(q-1)-$st row and inserting them as the first k entries of the q-th row and by taking the first $n-k$ entries of the $(q-1)$-st row and inserting them as the last $n-k$ entries of the q-th row, where $q \in \{2, 3, \ldots, n\}$. Then the (ordered) sequence $a_1, a_2, \ldots, a_n$ is called a *k-translatable sequence* of Q with respect to the ordering $1, 2, \ldots, n$. A groupoid is called a *translatable groupoid* if it has a k-translatable sequence for some $k \in \{1, 2, \ldots, n\}$.

It is important to note that a k-translatable sequence of a groupoid Q depends on the ordering of the elements in the Cayley table of Q. A groupoid may be k-translatable for one ordering but not for another (see Example 8.13 below). Unless otherwise stated we will assume that the ordering of the Cayley table is $1, 2, \ldots, n$ and the first row of the table is $a_1, a_2, \ldots, a_n$.

Proposition 8.2. *The additive group $\mathbb{Z}_n$ is $(n-1)$-translatable.*

The example below shows that there are $(n-1)$-translatable quasigroups of order n which are not a cyclic group.

Example 8.3. Consider the following three groupoids of order $n = 5$.

·	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
5	3	1	4	2	5

·	1	2	3	4	5
1	2	1	3	4	5
2	1	3	4	5	2
3	3	4	5	2	1
4	4	5	2	1	3
5	5	2	1	3	4

·	1	2	3	4	5
1	3	1	5	2	4
2	1	5	2	4	3
3	5	2	4	3	1
4	2	4	3	1	5
5	4	3	1	5	2

These groupoids are 4-translatable quasigroups but they are not groups. The first is idempotent, the second is without idempotents, the third is a cyclic quasigroup generated by 1 or by 5.

Proposition 8.4. *Any $(n-1)$-translatable groupoid of order n is commutative.*

Proof. In a k-translatable groupoid $i \cdot j = a_{(i-1)(n-k)+j}$, where the subscript is calculated modulo n. If $k = n-1$, then $i \cdot j = a_{i+j-1} = j \cdot i$. □

Theorem 8.5. *There are no $(m-1)$-translatable quadratical quasigroups of order m.*

Proof. By Proposition 8.4 such a quasigroup is commutative. Since it also is bookend and idempotent, $x = (y \cdot x) \cdot (x \cdot y) = (x \cdot y) \cdot (x \cdot y) = x \cdot y$, so it cannot be a quasigroup. □

The following proposition is obvious.

Proposition 8.6. *Every* 1*-translatable groupoid is unipotent, i.e., in such groupoid there exists an element* a *such that* $x^2 = a$ *for every* x*.*

Corollary 8.7. *There is no idempotent* 1*-translatable groupoid of order* $n > 1$*.*

Proposition 8.8. *A* k*-translatable groupoid of order* n *containing a cancellable element is a quasigroup if and only if* $(k, n) = 1$*.*

Proof. Let Q be a k-translatable groupoid of order n and let a be its cancellable element. Then in the Cayley table $[x_{ij}]_{n\times n}$ corresponding to this groupoid the a-row contains all elements of Q. Without loss of generality we can assume that this is the first row. If this row has the form $a_1, a_2, \dots, a_n$, then other entries have the form $x_{ij} = a_{(i-1)(n-k)+j}$, where the subscript $(i-1)(n-k)+j$ is calculated modulo n. Obviously, for fixed $i = 1, 2, \dots, n$, all entries $x_{i1}, x_{i2}, \dots, x_{in}$ are different.

If $(n, k) = 1$, then also $(n, n-k) = 1$. So, in this case, also all $x_{1j}, x_{2j}, \dots, x_{nj}$ are different. Hence, this table determines a quasigroup.

If $(n, k) = t > 1$, then $(n, n-k) = t$ and the equation $(i-1)(n-k) = 0$ has at least two solutions in the set $\{1, 2, \dots, n\}$. Thus, in the Cayley table of such groupoid at least two rows are identical. Hence such groupoid cannot be a quasigroup. □

Theorem 8.9. *For every odd* n *and every* $k > 1$ *such that* $(k, n) = 1$ *there is at most one idempotent* k*-translatable quasigroup. For even* n *there are no such quasigroups.*

Proof. Let $a_1, a_2, a_3, \dots, a_n$ be the first row of a k-translatable quasigroup Q.

This quasigroup is idempotent only in the case when in its Cayley table we have $1 = x_{11}$, $2 = x_{22} = a_{(n-k)+2}$, $3 = x_{33} = a_{2(n-k)+3}$, $4 = x_{44} = a_{3(n-k)+4}$, and so on. This means that the main diagonal of the table $[x_{ij}]_{n\times n}$ should contains elements $a_1, a_{(n-k)+2}, a_{2(n-k)+3}, \dots, a_{(n-1)(n-k)+n}$, where all subscripts are calculated modulo n. Obviously, $a_{t(n-k)+t} = a_{t'(n-k)+t'}$ only in the case when $t - tk \equiv t' - t'k (\bmod n)$, i.e., $(t-t')(k-1) \equiv 0 (\bmod n)$. If n is odd and $(n, k) = 1$, then for some k also is possible $(n, k-1) = 1$. In this case the equation $z(k-1) \equiv 0 (\bmod n)$ has only one solution $z = 0$, so $t = t'$. Hence the diagonal of the table $[x_{ij}]_{n\times n}$ contains n different elements.

If n is even and $(n, k) = 1$, then k is odd. Thus, $k-1$ is even and $(n, k-1) \neq 1$. Hence, the equation $z(k-1) \equiv 0 (\bmod n)$ has at least two solutions. Consequently, the diagonal of the table $[x_{ij}]_{n\times n}$ contains at least two equal elements. This contradicts to the fact that this quasigroup is idempotent. Therefore, for even n there are no idempotent k-translatable quasigroups. □

Corollary 8.10. *For every odd* n *and every* $k > 1$ *such that* $(n, k) = (n, k-1) = 1$ *there is exactly one idempotent* k*-translatable quasigroup of order* n*.*

Corollary 8.11. *The first row of an idempotent k-translatable quasigroup $Q = \{1, 2, \dots, n\}$ has the form $1, a_2, a_3, \dots, a_n$, where $a_{(i-1)(n-k)+i(\mathrm{mod}\, n)} = i$ for every $i \in Q$.*

Example 8.12. Consider an idempotent quasigroup $Q = \{1, 2, \dots, 7\}$. From the proof of Theorem 8.9 it follows that if this quasigroup is 3-translatable, then the first row of its Cayley table has the form $1, 4, 7, 3, 6, 2, 5$. If it is 4-translatable, then the first row has the form $1, 3, 5, 7, 2, 4, 6$.

Example 8.13. The following example shows that for $Q1 = \{a, ab, ba, b, aba\}$ the sequence a, ba, aba, ab, b is 3-translatable, but $Q1$ presented in the form $Q1' = \{a, b, ab, ba, aba\}$ has no translatable sequences.

$Q1$	a	ab	ba	b	aba
a	a	ba	aba	ab	b
ab	aba	ab	b	a	ba
ba	b	a	ba	aba	ab
b	ba	aba	ab	b	a
aba	ab	b	a	ba	aba

$Q1'$	a	b	ab	ba	aba
a	a	ab	ba	aba	b
b	ba	b	aba	ab	a
ab	aba	a	ab	b	ba
ba	b	aba	a	ba	ab
aba	ab	ba	b	a	aba

The sequence $a, aba, b, a*b, b*a$ is 2-translatable for $(Q1)^* = \{a, b*a, a*b, b, aba\}$. $(Q1')^* = \{a, b, b*a, a*b, aba\}$ has no translatable sequence.

$(Q1)^*$	a	$b*a$	$a*b$	b	aba
a	a	aba	b	$a*b$	$b*a$
$b*a$	$a*b$	$b*a$	a	aba	b
$a*b$	aba	b	$a*b$	$b*a$	a
b	$b*a$	a	aba	b	$a*b$
aba	b	$a*b$	$b*a$	a	aba

$(Q1')^*$	a	b	$b*a$	$a*b$	aba
a	a	$a*b$	aba	b	$b*a$
b	$b*a$	b	a	aba	$a*b$
$b*a$	$a*b$	aba	$b*a$	a	b
$a*b$	aba	$b*a$	b	$a*b$	a
aba	b	a	$a*b$	$b*a$	aba

By Corollary 8.10, the quasigroup $Q1$ is isomorphic to a 3-translatable quasigroup $(\mathbb{Z}_5, \circ)$ with the operation $x \circ y = 4x + 2y(\mathrm{mod}\, 5)$. The dual quasigroup $(Q1)^*$ is isomorphic to a 2-translatable quasigroup $(\mathbb{Z}_5, \diamond)$ with the operation $x \diamond y = 2x + 4y(\mathrm{mod}\, 5)$.

Theorem 8.14. *A groupoid isomorphic to a k-translatable groupoid also has a k-translatable sequence.*

Proof. Let α be an isomorphism from a k-translatable groupoid $(Q, \cdot)$ to a groupoid $(S, \circ)$. If Q is with ordering $1, 2, \dots, n$, then on S we consider ordering induced by α, namely $\alpha(1), \alpha(2), \dots, \alpha(n)$. Suppose that the first row of the Cayley table of Q has the form $a_1, a_2, \dots, a_n$. Then in the i-th row and j-th column of this table is $x_{ij} = a_{(i-1)(n-k)+j(\mathrm{mod}\, n)}$. Consequently, in the $\alpha(i)$-row and $\alpha(j)$-th column of the Cayley table $[z_{ij}]$ of S we have $z_{\alpha(i),\alpha(j)} = \alpha(i) \circ \alpha(j) = \alpha(i \cdot j) = \alpha(x_{ij})$. Since Q is k-translatable, for every $1 \leqslant t \leqslant k$, we have $a_{i,n-k+t} = a_{i+1,t}$. Thus, $z_{\alpha(i),\alpha(n-k+t)} = \alpha(i) \circ \alpha(n-k+t) = \alpha(x_{i,n-k+t}) = \alpha(x_{i+1,t}) = \alpha((i+1) \cdot t) =$

$\alpha(i+1)\circ\alpha(t) = z_{\alpha(i+1),\alpha(t)}$. This shows that S also is k-translatable (for ordering $\alpha(1), \alpha(2), \dots, \alpha(n)$). □

Theorem 8.15. *An idempotent cancellable groupoid of order* 9 *is not translatable.*

Proof. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$ be the first row of the Cayley table of an idempotent cancellable groupoid Q. Then obviously $a_i \neq a_j$ for $i \neq j$. If Q is k-translatable, then $x_{44} = 4 = a_{3(9-k)+4}$. Since $3(9-k)+4 \equiv 4 (\mathrm{mod}\, 9)$ only for $k = 3$ and $k = 6$, this groupoid can be 3-translatable or 6-translatable. But in this case the fourth row coincides with the first, so Q cannot be cancellable. □

Corollary 8.16. *The quadratical quasigroups of order* 9 *are not translatable.*

Theorem 8.17. *An idempotent, bookend quasigroup Q, where $Q = \{1, 2, \dots, n\}$, is k-translatable if and only if for every $i \in Q$ we have $i = a_{(s-1)(n-k)+t(\mathrm{mod}\, n)}$, where $s, t \in Q$ are such that*

$$\begin{cases} k-2 \equiv s(k-1)(\mathrm{mod}\, n), \\ ik-1 \equiv t(k-1)(\mathrm{mod}\, n). \end{cases} \tag{15}$$

Proof. Let $1, a_2, a_3, \dots, a_n$ be the first row of the Cayley table $[x_{ij}]$ of an idempotent, bookend quasigroup $Q = \{1, 2, 3, \dots, n\}$. If it is k-translatable, then, by Corollary 8.11, we have $a_{(i-1)(n-k)+i(\mathrm{mod}\, n)} = i$ for each $i \in Q$.

Moreover, in this quasigroup for every $i \in Q$ should be

$$\begin{aligned} i = (1\cdot i)\cdot(i\cdot 1) &= a_i \cdot x_{i1} = a_i \cdot a_{(i-1)(n-k)+1(\mathrm{mod}\, n)} \\ &= s\cdot t = x_{st} = a_{(s-1)(n-k)+t(\mathrm{mod}\, n)}, \end{aligned}$$

where

$$\begin{cases} a_i = a_{(s-1)(n-k)+s(\mathrm{mod}\, n)} = s, \\ a_{(i-1)(n-k)+1(\mathrm{mod}\, n)} = a_{(t-1)(n-k)+t(\mathrm{mod}\, n)} = t \end{cases}$$

for some $s, t \in \{1, 2, \dots, n\}$ satisfying (15).

The converse statement is obvious. □

Corollary 8.18. *A quadratical quasigroup of order* 25 *can be k-translatable only for $k = 7$ or $k = 18$.*

Proof. Let $Q = \{1, 2, \dots, 25\}$ be a quadratical quasigroup. By Theorem 8.17, in this quasigroup for $i = 2$ should be

$$a_{27-k(\mathrm{mod}\, 25)} = x_{st} = a_{(s-1)(25-k)+t(\mathrm{mod}\, 25)},$$

where $s, t \in \{1, 2, \dots, 25\}$ satisfy the equations

$$\begin{cases} k-2 \equiv s(k-1)(\mathrm{mod}\, 25), \\ 2k-1 \equiv t(k-1)(\mathrm{mod}\, 25). \end{cases}$$

To reduce the number of solutions of these equations observe that

$$x_{i1} \neq 1 \longleftrightarrow a_{(i-1)(25-k)+1(\text{mod}\,25)} \neq 1 = a_1 \longleftrightarrow (i-1)k \not\equiv 0(\text{mod}\,25).$$

The last, for $i = 6$, is possible only for $k \neq 5, 10, 15, 20$.

Also

$$x_{ii} \neq 1 \longleftrightarrow a_{(i-1)(25-k)+i(\text{mod}\,25)} \neq 1 = a_1 \longleftrightarrow (i-1)(k-1) \not\equiv 0(\text{mod}\,25),$$

which for $i = 6$ is possible only for $k \neq 6, 11, 16, 21$.

Hence Q cannot be k-translatable for $k \in \{5, 6, 10, 11, 15, 16, 20, 21\}$. By Theorem 8.5 and Corollary 8.7 it also cannot be k-translatable for $k \in \{1, 24, 25\}$.

In other cases, for $i = 2$, we obtain

k	2	3	4	7	8	9	12	13	14	17	18	19	22	23
s	25	13	9	5	8	4	10	3	24	15	23	19	20	18
t	3	15	19	23	20	24	18	25	4	12	5	9	8	10
x_{st}	a_5	a_4	a_{12}	a_{20}	a_{14}	a_{22}	a_{10}	a_{24}	a_7	a_{24}	a_9	a_{17}	a_{15}	a_{19}
a_{27-k}	a_{25}	a_{24}	a_{23}	a_{20}	a_{19}	a_{18}	a_{15}	a_{14}	a_{13}	a_{10}	a_9	a_8	a_5	a_4

Since $x_{st} = a_{27-k}$ only for $k = 7$ and $k = 18$, a quasigroup of order 25 can be k-translatable only for $k = 7$ and $k = 18$.

Direct computations shows that $\mathbb{Z}_{25}$ with the operation $x \cdot y = 22x + 4y(\text{mod}\,25)$ is an example of a 7-translatable quadratical quasigroup of order 25. Its dual quasigroup is a 18-translatable. □

By changing the order of rows and columns in Tables 3, 4, 5 and 6 we obtain the following two theorems.

Theorem 8.19. *The sequence* $11, 12, 33, 21, 31, 34, 24, 32, 13, 14, 13, aba, 22$ *is* 5-*translatable for* $Q3 = \{11, 14, 34, 12, 23, 24, 33, aba, 32, 21, 22, 13, 31\}$.

The sequence $11^*, 12^*, 23^*, aba^*, 22^*, 13^*, 14^*, 34^*, 24^*, 32^*, 33^*, 21^*, 31^*$ *is* 8-*translatable for* $(Q3)^* = \{11^*, 14^*, 31^*, 13^*, 21^*, 22^*, 33^*, aba^*, 32^*, 23^*, 24^*, 12^*, 34^*\}$.

Theorem 8.20. *The sequence*

$$11, 12, 42, 43, 13, 14, 33, 21, 31, 44, 23, aba, 22, 41, 34, 24, 32$$

is 13-*translatable for*

$$Q4 = \{11, 14, 23, 24, 43, 31, 41, 12, 33, aba, 32, 13, 44, 34, 42, 21, 22\}.$$

The sequence

$$11^*, 12^*, 34^*, 24^*, 32^*, 44^*, 23^*, aba^*, 22^*, 41^*, 33^*, 21^*, 31^*, 13^*, 14^*, 43^*, 42^*$$

is 4-*translatable for*

$$(Q4)^* = \{11^*, 14^*, 21^*, 22^*, 44^*, 34^*, 42^*, 13^*, 33^*, aba^*, 32^*, 12^*, 43^*, 31^*, 41^*, 23^*, 24^*\}.$$

Quasigroups $Q3$ and $(Q3)^*$ are isomorphic, respectively, to quasigroups $(\mathbb{Z}_{13}, \cdot)$ and $(\mathbb{Z}_{13}, \circ)$, where $x \cdot y = 11x + 3y(\text{mod}\,13)$ and $x \circ y = 3x + 11y(\text{mod}\,13)$.

Quasigroups $Q4$ and $(Q4)^*$ are isomorphic, respectively, to quasigroups $(\mathbb{Z}_{17}, \cdot)$ and $(\mathbb{Z}_{17}, \circ)$, where $x \cdot y = 11x + 7y(\text{mod}\,17)$ and $x \circ y = 7x + 11y(\text{mod}\,17)$.

9. Translatable quasigroups induced by groups $\mathbb{Z}_m$

In this section we describe quadratical quasigroups induced by groups $\mathbb{Z}_m$. We start with some general results.

Lemma 9.1. *A quasigroup of the form $x * y = ax + by + c$ induced by a group $\mathbb{Z}_m$ is k-translatable if and only if $a + kb \equiv 0 (\bmod\, m)$.*

Proof. The i-th row of the Cayley table of this quasigroup has the form

$$a(i-1)+c, a(i-1)+b+c, a(i-1)+2b+c, \ldots, a(i-1)+(m-1)b+c,$$

the $(i+1)$-row has the form

$$ai+c, ai+b+c, ai+2b+c, \ldots, ai+(m-1)b+c.$$

So, this quasigroup is k-translatable if and only if

$$ai+c = a(i-1)+(m-k)b+c (\bmod\, m),$$

i.e., if and only if $a + kb \equiv 0(\bmod\, m)$. □

Corollary 9.2. *A quasigroup $(\mathbb{Z}_m, \diamond)$, where $x \diamond y = ax + y + c$, is $(m-a)$-translatable.*

Theorem 9.3. *Each quadratical quasigroup induced by group $\mathbb{Z}_m$ is k-translatable for some $1 < k < m-1$, namely for k such that $(a-1)k \equiv a(\bmod\, m)$. This is valid for exactly one value of k.*

Proof. By Theorem 2.5 and Lemma 9.1 a quadratical quasigroup induced by $\mathbb{Z}_m$ is k-translatable if and only if there exist k such that $a \equiv (1-a)k(\bmod\ m)$, i.e., $(a-1)k \equiv a(\bmod\ m)$. Since $(a-1, m) = 1$, the last equation has exactly one solution in $\mathbb{Z}_m$ (cf. [8]). □

Theorem 9.4. *A quadratical quasigroup $(\mathbb{Z}_m, \cdot)$ with $x \cdot y = ax + (1-a)y$ is k-translatable if and only if its dual quasigroup $(\mathbb{Z}_m, \circ)$, where $x \circ y = (1-a)x + ay$, is $(m-k)$-translatable.*

Proof. Let $(\mathbb{Z}_m, \cdot)$ be k-translatable, then $(a-1)k \equiv a(\bmod\, m)$, i.e., $k \equiv \frac{a}{a-1}(\bmod\ m)$. If $(\mathbb{Z}_m, \circ)$ is t-translatable, then $ak \equiv (a-1)(\bmod\, m)$, i.e., $t \equiv \frac{a-1}{a}(\bmod\ m)$. ($\frac{a}{a-1}$ and $\frac{a1}{a}$ are well defined in $\mathbb{Z}_m$ because $(a, m) = (a-1, m) = 1$.) Thus $k + t = \frac{2a^2-2a+1}{a(a-1)} = 0(\bmod\ m)$, by Theorem 2.5. Hence $k + t = m$. □

Note that this theorem is not valid for quasigroups which are not quadratical. Indeed, a quasigroup $(\mathbb{Z}_7, \cdot)$ with $x \cdot y = 4x + y(\bmod 7)$ is 3-translatable, but its dual quasigroup $(\mathbb{Z}_7, *)$, where $x * y = x + 4y(\bmod 7)$, is 5-translatable.

Corollary 9.5. *There are no self-dual quadratical quasigroups induced by groups $\mathbb{Z}_m$.*

Using Theorem 9.3 we can calculate all k-translatable quadratical quasigroups induced by groups $\mathbb{Z}_m$. For this, it is better to rewrite the condition given in Theorem 9.3 in the form $(k-1)a \equiv k(\mathrm{mod}\, m)$.

2-TRANSLATABLE QUADRATICAL QUASIGROUPS

In this case $a \equiv 2(\mathrm{mod}\, m)$, where a satisfies (5). So, $5 \equiv 0(\mathrm{mod}\, m)$. Thus $m = 5$. Therefore there is only one 2-translatable quadratical quasigroup induced by $\mathbb{Z}_m$. It is induced by $\mathbb{Z}_5$ and has the form $x \cdot y = 2x + 4y(\mathrm{mod}\, 5)$.

3-TRANSLATABLE QUADRATICAL QUASIGROUPS

Then $2a \equiv 3(\mathrm{mod}\ m)$. Since (5) can be written in the form $2a(a-1)+1 = 0$, we also have $3a \equiv 2(\mathrm{mod}\ m)$. This, together with $4a \equiv 6(\mathrm{mod}\ m)$, implies $a = 4$. Hence $8 \equiv 3(\mathrm{mod}\ m)$. Thus $m = 5$. Therefore there is only one 3-translatable quadratical quasigroup induced by $\mathbb{Z}_m$. It is induced by $\mathbb{Z}_5$ and has the form $x \cdot y = 4x + 2y(\mathrm{mod}\, 5)$.

4-TRANSLATABLE QUADRATICAL QUASIGROUPS

Now $3a \equiv 4(\mathrm{mod}\, m)$ and $6a \equiv 8(\mathrm{mod}\, m)$. From (5) we obtain $6a(a-1)+3 = 0$, which together with the last equation gives $8a \equiv 5(\mathrm{mod}\, m)$. This, with $9a \equiv 12(\mathrm{mod}\, m)$, implies $a = 7$. Hence $21 \equiv 4(\mathrm{mod}\, m)$. Thus $m = 17$. Therefore there is only one 4-translatable quadratical quasigroup induced by $\mathbb{Z}_m$. It is induced by $\mathbb{Z}_{17}$ and has the form $x \cdot y = 7x + 11y(\mathrm{mod}\, 17)$.

5-TRANSLATABLE QUADRATICAL QUASIGROUPS

Now $4a \equiv 5(\mathrm{mod}\, m)$ and $5a \equiv 3(\mathrm{mod}\, m)$, by (5). Thus, $16a \equiv 20(\mathrm{mod}\, m)$ and $15a \equiv 9(\mathrm{mod}\, m)$, which implies $a = 11$. Hence $44 \equiv 5(\mathrm{mod}\, m)$. Thus $m = 13$. Therefore a 5-translatable quadratical quasigroup is induced by $\mathbb{Z}_{13}$ and has the form $x \cdot y = 11x + 4y(\mathrm{mod}\, 13)$.

6-TRANSLATABLE QUADRATICAL QUASIGROUPS

Now $5a \equiv 6(\mathrm{mod}\, m)$ and $12a \equiv 7(\mathrm{mod}\, m)$, by (5). Thus, $25a \equiv 30(\mathrm{mod}\, m)$ and $24a \equiv 14(\mathrm{mod}\, m)$, which implies $a = 16$. Hence $80 \equiv 6(\mathrm{mod}\, m)$. Thus $m = 37$. Therefore a 6-translatable quadratical quasigroup is induced by $\mathbb{Z}_{37}$ and has the form $x \cdot y = 16x + 22y(\mathrm{mod}\, 37)$.

7-TRANSLATABLE QUADRATICAL QUASIGROUPS

Now $6a \equiv 7(\mathrm{mod}\, m)$ and $7a \equiv 4(\mathrm{mod}\, m)$, by (5). Thus, $a \equiv (-3)(\mathrm{mod}\, m)$ and $(-18) \equiv 7(\mathrm{mod}\, m)$. Consequently, $25 \equiv 0(\mathrm{mod}\, m)$. Hence $m = 25$. (The case $m = 5$ is impossible because must be $m > k = 7$.) Therefore $a = 22$. So, a 7-translatable quadratical quasigroup is induced by $\mathbb{Z}_{25}$ and has the form $x \cdot y = 22x + 4y(\mathrm{mod}\, 25)$.

8-TRANSLATABLE QUADRATICAL QUASIGROUPS

Now $7a \equiv 8(\mathrm{mod}\, m)$ and $16a \equiv 9(\mathrm{mod}\, m)$, by (5). Thus, $49a \equiv 56(\mathrm{mod}\, m)$ and $48a \equiv 27(\mathrm{mod}\, m)$ shows that $a \equiv 29(\mathrm{mod}\, m)$. Hence $7 \cdot 29 \equiv 8(\mathrm{mod}\, m)$ and $16 \cdot 29 \equiv 9(\mathrm{mod}\, m)$ imply $195 \equiv 0(\mathrm{mod}\, m)$ and $455 \equiv 0(\mathrm{mod}\, m)$. Therefore, $65 \equiv 0(\mathrm{mod}\, m)$. Since $m > k = 8$, the last means that $m = 65$ or $m = 13$. So, a

8-translatable quadratical quasigroup is induced by $\mathbb{Z}_{13}$ or by $\mathbb{Z}_{65}$. In the first case it has the form $x \cdot y = 3x + 11y (\mathrm{mod}\, 13)$, in the second $x \cdot y = 29x + 37y (\mathrm{mod}\, 65)$.

9-TRANSLATABLE QUADRATICAL QUASIGROUPS

In this case $8a \equiv 9 (\mathrm{mod}\, m)$ and $9a \equiv 5 (\mathrm{mod}\, m)$, by (5). So, $a \equiv (-4) (\mathrm{mod}\, m)$, and consequently $41 \equiv 0 (\mathrm{mod}\, m)$. Thus, $m = 41$. Hence a 9-translatable quadratical quasigroup is induced by $\mathbb{Z}_{41}$ and has the form $x \cdot y = 37x + 5y (\mathrm{mod}\, 41)$.

10-TRANSLATABLE QUADRATICAL QUASIGROUPS

In a similar way we can see that there is only one 10-translatable quasigroup induced by $\mathbb{Z}_m$. It is induced by $\mathbb{Z}_{101}$ and has the form $x \cdot y = 46x + 56y (\mathrm{mod}\, 101)$.

As a consequence of the above calculations and Theorem 9.4 we obtain the following list of $(m-k)$-translatable quadratical quasigroups induced by $\mathbb{Z}_m$.

$(m-2)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It is induced by $\mathbb{Z}_5$ and has the form $x \cdot y = 4x + 2y (\mathrm{mod}\, 5)$.

$(m-3)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 2x + 4y (\mathrm{mod}\, 5)$.

$(m-4)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 11x + 7y (\mathrm{mod}\, 17)$.

$(m-5)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 3x + 11y (\mathrm{mod}\, 13)$.

$(m-6)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 22x + 16y (\mathrm{mod}\, 37)$.

$(m-7)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 4x + 22y (\mathrm{mod}\, 25)$.

$(m-8)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There are only two such quasigroups. The first has the form $\cdot x \cdot y = 11x + 3y (\mathrm{mod}\, 13)$, the second $x \cdot y = 37x + 29y (\mathrm{mod}\, 65)$.

$(m-9)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

There is only one such quasigroup. It has the form $x \cdot y = 5x + 37y (\mathrm{mod}\, 41)$.

$(m-10)$-TRANSLATABLE QUADRATICAL QUASIGROUPS

Such a quasigroup is induced by $\mathbb{Z}_{101}$ and has the form $x \cdot y = 56x + 46y (\mathrm{mod}\, 101)$.

Below, for $k < 40$, we list all k-translatable quadratical quasigroups of order

$m \leqslant 1200$ defined on $\mathbb{Z}_m$.

k	m	a	b
2	5	2	4
3	5	4	2
4	17	7	11
5	13	11	7
6	37	16	22
7	25	22	4
8	13	3	11
	65	29	37
9	41	37	5
10	101	46	56
11	61	56	6
12	29	9	21
	145	67	79
13	17	11	7
	85	79	7
14	197	92	106
15	113	106	8
16	257	121	137

k	m	a	b
17	29	21	9
	145	137	9
18	25	4	22
	65	24	42
	325	154	172
19	181	172	10
20	401	191	211
21	221	211	11
22	97	38	60
	485	232	254
23	53	42	12
	265	254	12
24	577	277	301
25	313	301	13
26	677	326	352
27	73	60	14
	365	352	14
28	157	65	93
	785	379	407

k	m	a	b
29	421	407	15
30	53	12	42
	901	436	466
31	37	22	16
	481	466	16
32	41	5	37
	205	87	119
	1025	497	529
33	109	93	17
	545	529	17
34	89	28	62
	1157	562	596
35	613	596	18
36	1297	631	667
37	137	119	19
	685	667	198
38	85	24	62
	289	126	164
39	761	742	20

10. Classification of quadratical quasigroups

We have classified translatable quadratical quasigroups in several ways. Firstly, all k-translatable quadratical quasigroups induced by $\mathbb{Z}_m$ were calculated for $k \in \{2.3.\dots,10\}$. Secondly, for a quadratical quasigroup of order m we calculated all $(m-t)$-translatable quadratical quasigroups for $t \in \{2,3,\dots,10\}$. Then we calculated all k-translatable quadratical quasigroups ($k < 40$) on $\mathbb{Z}_m$ of order $m < 1200$. We now list all k-translatable quadratical quasigroups induced by $\mathbb{Z}_m$, for $m < 500$. A list of all translatable quadratical quasigroups of the form Qn, up to a certain order, remains uncalculated.

Below are listed all k-translatable quadratical quasigroups of the form $x \cdot y = ax + by (\mathrm{mod}\, m)$, where $a < b$, defined on the group $\mathbb{Z}_m$ for $m < 500$. Dual quasigroups $x \circ y = bx + ay (\mathrm{mod}\, m)$ are omitted.

For example, the group $\mathbb{Z}_{65}$ induces four quadratical quasigroups: $x \cdot y = 24x + 42y (\mathrm{mod}\, 65)$, $x \cdot y = 29x + 37y (\mathrm{mod}\, 65)$ and two duals to these two. The first is 18-translatable, the second 8-translatable. In the table below these dual quasigroups $x \cdot y = 42x + 24y (\mathrm{mod}\, 65)$ and $x \cdot y = 37x + 29y (\mathrm{mod}\, 65)$ are not listed.

m	a	b	k
5	2	4	2
13	3	11	8
17	7	11	4
25	4	22	18
29	9	21	12
37	16	22	6
41	5	37	32
53	12	42	30
61	6	56	50
65	24	42	18
	29	37	8
73	14	60	46
85	7	79	72
	24	62	38
89	28	62	34
97	38	60	22
101	46	56	10
109	17	93	76
113	8	106	98
125	29	97	68
137	19	119	100
145	9	137	128
	67	79	12
149	53	97	44
157	65	93	28
169	50	120	70

m	a	b	k
173	47	127	80
181	10	172	162
185	22	164	142
	59	127	68
193	41	153	112
197	92	106	14
205	37	169	132
	87	119	32
221	11	211	200
	24	198	174
229	54	176	122
233	45	189	144
241	89	153	64
257	121	137	16
265	12	254	242
	42	224	182
269	94	176	82
277	109	169	60
281	27	255	228
289	126	164	38
293	78	216	138
305	67	239	172
	117	189	72
313	13	301	288
317	102	216	114
325	29	297	268
	154	172	18

m	a	b	k
337	95	243	148
349	107	243	136
353	156	198	42
365	14	352	338
	87	279	192
373	135	239	104
377	50	328	278
	154	224	70
389	58	332	274
397	32	366	334
401	191	211	20
409	72	338	266
421	15	407	392
425	79	347	268
	147	279	132
433	90	344	254
445	62	384	322
	117	329	212
449	34	416	382
457	55	403	348
461	207	255	48
481	16	466	450
	133	349	216
485	157	329	172
	232	254	22
493	79	415	336
	96	398	302

10. Open questions and problems

Problem 1. *For which values of n are there quadratical quasigroups of form Qn?*

Note that $n \notin \{5, 6, 8, 14, 17, 19, 33, 26, 32, \ldots\}$. Moreover, from Theorem 4.11 in [3] it follows that there are no such quasigroups if there is a prime $p|4n+1$ such that $p \equiv 3 (\text{mod}\, 4)$.

Problem 2. *Is every quadratical quasigroup Q of form Qn translatable ($n \neq 2$)?*

The answer is positive if Q is isomorphic to a quasigroup induced by $\mathbb{Z}_{4m+1}$.

Problem 3. *Are there self-dual, quadratical groupoids of order greater than 9?*

Such quasigroups cannot be induced by $\mathbb{Z}_m$.

Problem 4. *Is every quadratical groupoid of order greater than 9 and of form Qn ($n \geqslant 3$) generated by any two of its distinct elements?*

Problem 5. *If a quadratical quasigroup Q of order m is k-translatable, then is Q^* $(m-k)$-translatable?*

For quadratical quasigroups induced by $\mathbb{Z}_m$ the answer is positive.

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