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Division on semigroups that are semilattices of groups

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Abstract. The binary products of right, left or double division on semigroups that are semilattices of groups give interesting groupoid structures that are in one-one correspondence with semigroups that are semilattices of groups. This work is inspired by the well-known one-one correspondence between groups and Ward quasigroups.

1. Introduction

It appears in the literature that in 1930 M. Ward was the first to find a set of axioms on $(S, *)$ (a set S with a binary operation $*$, called here a *groupoid*) that ensure the existence of a group binary operation $\circ$ on S such that $x * y = x \circ y^{-1}$ cf. [12]. Such a groupoid was called a *division groupoid* by Polonijo (cf. [10]) and it is clear that division groupoids are quasigroups.

Over the next 63 years many other sets of axioms on a groupoid were found that make it a division groupoid, now commonly known as a Ward quasigroup (see for example: [1, 2, 4, 6, 7, 8, 9, 10, 11]). Perhaps the most impressive of these characterisations of Ward quasigroups is that of Higman and Neumann who found a single law making a groupoid a Ward quasigroup (cf. [6]). It is now known that a quasigroup is a Ward quasigroup if and only if it satisfies the law of right transitivity, $(x * z) * (y * z) = x * y$ (cf. [9]). It follows that a quasigroup is the dual of a Ward quasigroup, which we will call a Ward dual quasigroup, if and only if it satisfies the identity $(z * x) * (z * y) = x * y$.

Starting from any group $(G, \circ)$ we can form a Ward quasigroup $(G, *)$ by defining $x * y = x \circ y^{-1}$; that is, $*$ is the operation of right division in the group $(G, \circ)$. Conversely, any Ward quasigroup $(W, *)$ is unipotent and its only idempotent $e = e * e = x * x$ (for any $x \in W$), is a right identity element. If we then define $(W, \circ)$ as $x \circ y = x * (e * y)$, $(W, \circ)$ is a group, $x^{-1} = e * x$ and $x * y = x \circ y^{-1}$. These mappings, $(G, \circ) \mapsto (G, *)$ and $(W, *) \mapsto (W, \circ)$ are inverse mappings, which implies that groups are in one-to-one correspondence with Ward quasigroups. (This is all well known.) In addition, a Ward quasigroup is an inverse groupoid, with the unique inverse of x being $x^{-1} = e * x$. That is, the inverse of an element of a Ward quasigroup is the inverse element in the group it *induces*.

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In 2007 N.C. Fiala proved (cf. [5]) that a quasigroup $(S,*)$ satisfies the identity $[(e*e)*(x*z)]*[(e*y)*z] = x*y$ (for some $e \in S$) if and only if there is a group $(S,\circ)$ with identity element e such that $x*y = x^{-1}\circ y^{-1}$. Fiala called such groupoids *double Ward quasigroups*. He noted that the binary operation $\circ$ on a double Ward quasigroup S defined by $x\circ y = (e*x)*(e*y)$ is a group operation and that double Ward quasigroups are in one-to-one correspondence with groups. Double Ward quasigroups are also inverse quasigroups, with $x^{-1} = x$.

Our intention here is to explore the operations $x*y = x\cdot y^{-1}$ (called *right division*), $x*y = x^{-1}\cdot y$ (called *left division*) and $x*y = x^{-1}\cdot y^{-1}$ (called *double division*) when $(S,\cdot)$ is a semigroup and a semilattice of groups, where x^{-1} is the inverse of x in the group to which it belongs. We will prove that each collection of all such structures are in one-one correspondence with the collection of all semigroups that are semilattices of groups and, in this sense, we extend the result that Ward quasigroups are in one-to-one correspondence with groups.

2. Preliminary definitions and results

The set of all idempotent elements of a groupoid $(S,*)$ is denoted by $E(S,*)$, i.e., $E(S,*) = \{x \in S \mid x*x = x\}$. Note that the set $E(S,*)$ may not be closed under the operation $*$. The groupoid $(S,*)$ is called an *idempotent groupoid* (a *semilattice groupoid*) if all of its elements are idempotent (idempotent and commute). A semilattice groupoid $(S,*)$ is called a *semigroup semilattice* if it is a semigroup. A groupoid $(S,*)$ is called an (*idempotent*) *groupoid* $(T,\cdot)$ *of groupoids* $(S_\alpha, *_{|S_\alpha})$ $(\alpha \in T)$ if S is a disjoint union of the S_α $(\alpha \in T)$ and $S_\alpha * S_\beta \subseteq S_{\alpha\cdot\beta} = S_{\alpha\beta}$ for all $\alpha,\beta \in T$. Note that this definition does not require either of the binary operations $\cdot$ or $*$ to be associative.

We call the groupoid $(S,*)$ *right* (*left*) *solvable* if for any $a,b \in S$ there exists a unique $x \in S$ such that $a*x = b$ $(x*a = b)$. The groupoid $(S,*)$ is a *quasigroup* if it is right and left solvable, in which case it is right and left cancellative. We call the quasigroup $(S,*)$ a *Ward quasigroup* (*Ward dual quasigroup*) if it satisfies the identity $(x*z)*(y*z) = x*y$ $((z*x)*(z*y) = x*y)$. The quasigroup $(S,*)$ is called a *double Ward quasigroup* if it satisfies the identity $((e*e)*(x*z))*((e*y)*z) = x*y$ for some fixed $e \in S$. A groupoid $(S,*)$ is called an *inverse groupoid* if for all $x \in S$ there exists a unique element x^{-1} such that $(x*x^{-1})*x = x$ and $(x^{-1}*x)*x^{-1} = x^{-1}$. The fact that $(S,*)$ and $(T,\circ)$ are isomorphic groupoids is denoted by $(S,*) \cong (T,\circ)$. The groupoid $(S,\bar{*})$ is dual to the groupoid $(S,*)$ if $x\bar{*}y = y*x$. The collection $\overline{\mathbf{C}}$ is the collection of all groupoids $(S,\bar{*})$, where $(S,*) \in \mathbf{C}$. Clearly, $\mathbf{C}$ is in one-one correspondence with $\overline{\mathbf{C}}$.

Below we list a few identities that we will use later. The proofs of these identities one can find in [2] and [9].

A Ward quasigroup $(S,*)$ satisfies the following identities:

(1) $x*x = y*y = r$,

(2) $x * r = x$,

(3) $r * (x * y) = y * x$,

(4) $r * (r * x) = x$,

(5) $(x * y) * z = x * (z * (r * x))$.

Note that a Ward (Ward dual) quasigroup $(S, *)$ has a unique right (left) identity element r. So, we will denote this by $(W, *, r)$ (resp. $(WD, *, r)$). We will denote a double Ward quasigroup by $(DW, *, e)$, although we note that the element e may not be unique.

A double Ward quasigroup $(S, *)$ satisfies the following identities:

(6) $e * e = e$,

(7) $(e * (x * z)) * ((e * y) * z) = x * y$,

(8) $(y * x) * y = y * (x * y) = x$,

(9) $e * x = x * e$,

(10) $x * (x * e) = (e * x) * x = e$,

(11) $x * y = e * ((e * y) * (e * x))$.

The following facts on connections of groups with various types of Ward quasigroups are well-known, or follow readily from [2], [5] and [9].

($F1$) $(W, *, r)$ is a Ward quasigroup if and only if there is a group $(W, \circ, r)$ such that $x * y = x \circ y^{-1}$ for all $x, y \in W$.

($F2$) $(WD, *, r)$ is a Ward dual quasigroup if and only if there is a group $(WD, \circ, r)$ such that $x * y = y \circ x^{-1}$ for all $x, y \in WD$.

($F3$) $(DW, *, e)$ is a double Ward quasigroup if and only if there is a group $(DW, \circ, e)$ such that $x * y = x^{-1} \circ y^{-1}$ for all $x, y \in DW$.

($F4$) If $(W, *, r)$ is a Ward quasigroup, then $(W, \circ)$ defined as $x \circ y = x * (r * y)$ is a group with identity r and $x^{-1} = r * x$.

($F5$) If $(WD, *, r)$ is a Ward dual quasigroup, then $(WD, \circ)$ defined as $x \circ y = (x * r) * y$ is a group with identity r and $x^{-1} = x * r$.

($F6$) If $(DW, *, e)$ is a double Ward quasigroup, then $(DW, \circ)$ defined as $x \circ y = (e * x) * (e * y)$ is a group with identity e and $x^{-1} = e * x$.

The fact $(F6)$ was noted in [5] without proof. Below we give a short proof.

By definition we have

$$(x \circ y) \circ z = (e * ((e * x) * (e * y)) * (e * z) \overset{(11)}{=} (y * x) * (e * z)$$

and

$$x \circ (y \circ z) = (e * x) * (e * ((e * y) * (e * z))) \overset{(11)}{=} (e * x) * (z * y).$$

Since, by (8) and (9), $x = (y * x) * y$ and $z = e * (e * z)$, we have

$$\begin{aligned} x \circ (y \circ z) &= (e * x) * (z * y) = (e * ((y * x) * y)) * ((e * (e * z)) * y) \\ &\overset{(7)}{=} (y * x) * (e * z) = (x \circ y) \circ z. \end{aligned}$$

So, $(DW, *)$ is a semigroup.

Now suppose that $a, b \in DW$. Since $(DW, *, e)$ is a quasigroup, there exists a unique $x \in DW$ such that $x * a = e * b$. So

$$a \circ x = (e * a) * (e * x) \overset{(8,9,11)}{=} e * (x * a) = e * (e * b) \overset{(8,9)}{=} b.$$

If $a \circ y = b$, then $b = (e * a) * (e * y) \overset{(8,9,11)}{=} e * (y * a)$ and so, by (8) and (9), $e * b = b * e = (e * (y * a)) * e = y * a$. But x was unique, so $x = y$. Similarly, there exists a unique element $z \in DW$ such that $z \circ a = b$. So, $(DW, \circ)$ is a group. The facts that e is the identity and $x^{-1} = e * x$ follow from (8) and (10).

As a consequence of $(F1) - (F6)$, we have the following corollaries:

Corollary 2.1. (cf. [2] and [9]) *The collection of all Ward quasigroups is in one-to-one correspondence with the collection of all groups.*

Corollary 2.2. *The collection of all Ward dual quasigroups is in one-to-one correspondence with the collection of all groups.*

Corollary 2.3. (cf. [5]) *The collection of all double Ward quasigroups is in one-to-one correspondence with the collection of all groups.*

The following two facts follow readily from $(F1)$ and $(F3)$ and proofs are omitted.

$(F10)$ If $(W, *, r)$ is a Ward quasigroup, then $(W, \cdot, r)$, where $x \cdot y = (r * x) * y$, is a double Ward quasigroup.

$(F11)$ If $(DW, *, e)$ is a double Ward quasigroup, then $(DW, \cdot, e)$, where $x \cdot y = (e * x) * y$, is a Ward quasigroup.

Also the following fact is true.

$(F12)$ If $(S,*)$ is a semigroup semilattice V of Ward quasigroups $(W_\alpha, *_{|_{W_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and satisfies the identity $(x*y)*(z*w) = (x*(w^{-1}*y^{-1}))*z$, then $E(S,*) = \{e_\alpha \,|\, \alpha \in V\}$, $e_\alpha * e_\beta = e_{\alpha\beta}$ and the mapping $\Psi(e_\alpha) = \alpha$ restricted to $E(S,*)$ is an isomorphism between $(E(S,*), *_{|_{E(S,*)}})$ and V.

Proof. First, we note that each $(W_\alpha, *_{|_{W_\alpha}}, e_\alpha)$ is an inverse groupoid, with $x_\alpha^{-1} = e_\alpha * x_\alpha$. Since a semigroup semilattice groupoid of inverse groupoids is an inverse groupoid, $(S,*)$ is an inverse groupoid. Hence, the identity $(x*y)*(z*w) = (x*(w^{-1}*y^{-1}))*z$ has a clear meaning. We call this identity $(\star)$.

Now, by definition, $e_\alpha * e_\beta \in W_{\alpha\beta}$. Therefore, $e_{\alpha\beta} \overset{(1)}{=} (e_\alpha * e_\beta)*(e_\alpha * e_\beta) \overset{(\star)}{=} (e_\alpha * e_\beta) * e_\alpha$. Then, $(e_\alpha * e_\beta) * e_\beta \overset{(\star)}{=} (e_\alpha * (e_\beta * e_\alpha)) * e_\beta \overset{(\star)}{=} (e_\alpha * e_\alpha) * (e_\beta * e_\beta) = e_\alpha * e_\beta = (e_\alpha * e_\beta) * e_{\alpha\beta} = (e_\alpha * e_\beta) * ((e_\alpha * e_\beta) * e_\alpha) \overset{(\star)}{=} (e_\alpha * (e_\alpha * e_\beta)) * (e_\alpha * e_\beta) \overset{(\star)}{=} ((e_\alpha * (e_\beta * e_\alpha)) * e_\alpha) * (e_\alpha * e_\beta) \overset{(\star)}{=} ((e_\alpha * e_\beta) * e_\alpha) * (e_\alpha * e_\beta) = e_{\alpha\beta} * (e_\alpha * e_\beta)$. So, $(e_\alpha * e_\beta) * e_\beta = e_\alpha * e_\beta = e_{\alpha\beta} * (e_\alpha * e_\beta) = ((e_\alpha * e_\beta) * e_\alpha) * ((e_\alpha * e_\beta) * e_\beta) \overset{(\star)}{=} ((e_\alpha * e_\beta) * (e_\beta * e_\alpha)) * (e_\alpha * e_\beta)$. But, since $(e_\alpha * e_\beta) * (e_\beta * e_\alpha) \in W_{\alpha\beta}$, $W_{\alpha\beta}$ is a Ward quasigroup and $e_{\alpha\beta} * (e_\alpha * e_\beta) = ((e_\alpha * e_\beta) * (e_\beta * e_\alpha)) * (e_\alpha * e_\beta)$, $e_{\alpha\beta} = (e_\alpha * e_\beta) * (e_\beta * e_\alpha) \overset{(1)}{=} (e_\alpha * e_\beta) * (e_\alpha * e_\beta)$ and $e_\alpha * e_\beta = e_\beta * e_\alpha$. But this implies $e_\alpha * (e_\beta * e_\sigma) = e_\alpha * (e_\sigma * e_\beta) = (e_\alpha * e_\alpha) * (e_\sigma * e_\beta) \overset{(\star)}{=} (e_\alpha * (e_\beta * e_\alpha)) * e_\sigma \overset{(\star)}{=} (e_\alpha * e_\beta) * e_\sigma$. Then, $e_{\alpha\beta} = (e_\alpha * e_\beta) * (e_\alpha * e_\beta) \overset{(\star)}{=} (e_\alpha * e_\beta) * e_\alpha = e_\alpha * (e_\beta * e_\alpha) = e_\alpha * (e_\alpha * e_\beta) = (e_\alpha * e_\alpha) * e_\beta = e_\alpha * e_\beta$. It follows that the mapping $e_\alpha \mapsto \alpha$ is an isomorphism between $E(S,*)$ and V. □

Dually, we have

$(F13)$ If $(S,*)$ is a semigroup semilattice V of Ward dual quasigroups $(W_\alpha, *_{|_{W_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and satisfies the identity $(x*y)*(z*w) = y*((z^{-1}*x^{-1})*w)$, then $E(S,*) = \{e_\alpha \,|\, \alpha \in V\}$, $e_\alpha * e_\beta = e_{\alpha\beta}$ and the mapping $\Psi(e_\alpha) = \alpha$ restricted to $E(S,*)$ is an isomorphism between $(E(S,*), *_{|_{E(S,*)}})$ and V.

$(F10)$ and $(F11)$ are easily proved using $(F1)$ and $(F3)$. For example, if $(W,*,r)$ is a Ward quasigroup then by $(F1)$ $x \cdot y = (r*x)*y = (r \circ x^{-1}) \circ y^{-1} = x^{-1} \circ y^{-1}$ and so $(W,\cdot,r)$ is a double Ward quasigroup.

Proposition 2.4. *If $(S,*)$ is a semigroup semilattice V of double Ward quasigroups $(DW_\alpha, *_{|_{DW_\alpha}}, e_\alpha)$, $(\alpha \in V)$, then the following conditions are equivalent*

(*i*) $\{e_\alpha \,|\, \alpha \in V\} \cong V$,

(*ii*) *for all* $\alpha, \beta, \gamma, \sigma \in V$, $(e_\alpha * e_\beta) * (e_\gamma * e_\sigma) = e_\beta * ((e_\gamma * e_\sigma) * e_\alpha)$,

(*iii*) *the mapping* $e_\alpha \mapsto \alpha$ *is an isomorphism from* $(\{e_\alpha \,|\, \alpha \in V\}, *_{|_{\{e_\alpha \,|\, \alpha \in V\}}})$ *to* V.

Proof. $(i) \Rightarrow (ii)$: Let $\Psi\colon (\{e_\alpha \mid \alpha \in V\}, *|_{\{e_\alpha \mid \alpha \in V\}}) \to V$ be an isomorphism. Then $\Psi((e_\alpha * e_\beta) * (e_\gamma * e_\sigma)) = \Psi(e_\alpha)\Psi(e_\beta)\Psi(e_\gamma)\Psi(e_\sigma) = \Psi(e_\beta)[[\Psi(e_\sigma)\Psi(e_\gamma)]\Psi(e_\alpha)] = \Psi(e_\beta * ((e_\sigma * e_\gamma) * e_\alpha))$, because V is a semigroup semilatice. Since Ψ is one-one, the last implies (ii).

$(ii) \Rightarrow (iii)$: First, we prove that $e_\alpha * e_\beta = e_\beta * e_\alpha$. By hypothesis, we have

$$(12)\quad (e_\alpha * e_\beta) * (e_\gamma * e_\sigma) = e_\beta * ((e_\sigma * e_\gamma) * e_\alpha).$$

From this we obtain

$$(13)\quad (e_\alpha * e_\beta) * e_\sigma = e_\beta * (e_\sigma * e_\alpha),$$

which implies

$$(14)\quad (e_\beta * e_\alpha) * e_\beta = e_\alpha * e_\beta = e_\alpha * (e_\beta * e_\alpha).$$

Now, $e_{\alpha\beta} \overset{(8)}{=} ((e_\alpha * e_\beta) * e_{\alpha\beta}) * (e_\alpha * e_\beta) \overset{(13)}{=} (e_\beta * (e_{\alpha\beta} * e_\alpha)) * (e_\alpha * e_\beta) \overset{(12)}{=} (e_\beta * ((e_\alpha * e_{\alpha\beta}) * e_\beta)) * (e_\alpha * e_\beta) \overset{(13)}{=} (e_\beta * (e_{\alpha\beta} * (e_\beta * e_\alpha))) * (e_\alpha * e_\beta) \overset{(12)}{=} (e_\beta * (e_{\alpha\beta} * (e_\beta * e_\alpha))) * (e_\alpha * e_\beta)) \overset{(12)}{=} (e_\beta * ((e_{\alpha\beta} * (e_\alpha * e_\beta)) * e_{\alpha\beta})) * (e_\alpha * e_\beta) \overset{(8)}{=} (e_\beta * (e_\alpha * e_\beta)) * (e_\alpha * e_\beta) \overset{(14)}{=} (e_\beta * e_\alpha) * (e_\alpha * e_\beta) \overset{(12)}{=} e_\alpha * ((e_\beta * e_\alpha) * e_\beta) \overset{(14)}{=} e_\alpha * (e_\alpha * e_\beta) \overset{(12)}{=} e_\alpha * ((e_\beta * e_\alpha) * e_\alpha) \overset{(14)}{=} e_\alpha * (e_\alpha * (e_\alpha * e_\beta)) = e_\alpha * e_{\alpha\beta}$.

Then,

$$(15)\quad e_{\alpha\beta} * e_\alpha \overset{(14)}{=} e_{\alpha\beta} * (e_\alpha * e_{\alpha\beta}) = e_{\alpha\beta} * e_{\alpha\beta} = e_{\alpha\beta} = e_\alpha * (e_\alpha * e_\beta).$$

Since we have proved above that $e_{\alpha\beta} = (e_\beta * e_\alpha) * (e_\alpha * e_\beta)$, it follows from (8) that $(e_\alpha * e_\beta) * e_{\alpha\beta} = e_\beta * e_\alpha$. So, $e_\beta * e_\alpha = (e_\alpha * e_\beta) * e_{\alpha\beta} \overset{(14)}{=} e_\beta * (e_{\alpha\beta} * e_\alpha) \overset{(15)}{=} e_\beta * e_{\alpha\beta} \overset{(15)}{=} e_\beta * (e_\alpha * (e_\alpha * e_\beta)) \overset{(14)}{=} e_\beta * ((e_\beta * e_\alpha) * e_\alpha) \overset{(12)}{=} (e_\alpha * e_\beta) * (e_\alpha * e_\beta)$, which means that $(e_\alpha * e_\beta) * (e_\alpha * e_\beta) = (e_\alpha * e_\beta) * e_{\alpha\beta}$. Since $(DW_{\alpha\beta}, *|_{DW_{\alpha\beta}}, e_{\alpha\beta})$ is a quasigroup, $e_{\beta\alpha} = e_{\alpha\beta} = e_\alpha * e_\beta = e_\beta * e_\alpha$. Also, $(e_\alpha * e_\beta) * e_\gamma = e_{\alpha\beta} * e_\gamma = e_{(\alpha\beta)\gamma} = e_{\alpha(\beta\gamma)} = e_\alpha * (e_\beta * e_\gamma)$. Finally, the mapping $\Psi\colon (\{e_\alpha \mid \alpha \in V\}, *) \to V$ defined as $\Psi(e_\alpha) = \alpha$ satisfies $\Psi(e_\alpha * e_\beta) = \Psi(e_{\alpha\beta}) = \alpha\beta = \Psi(\alpha)\Psi(\beta)$ and so, since it is clearly one-one and onto V, Ψ is an isomorphism.

$(iii) \Rightarrow (i)$: This is obvious. □

3. Semigroup semilattices of groups

We have seen that Ward quasigroups, Ward dual quasigroups and double Ward quasigroups are in one-to-one correspondence with groups. In this section, we extend these results to semigroups that are semilattices of groups. Note that in semigroup theory a *semilattice*, a *union of groups* and a *semilattice of groups* are, by definition, semigroups. However, the definition of a semilattice (or idempotent groupoid) $(S,\cdot)$ of groupoids $(S_\alpha, *_{|S_\alpha})$ $(\alpha \in T)$ results in structures that are not necessarily associative, even when the S_α $(\alpha \in T)$ are all groups. Therefore, we use the terms *semigroup semilattice*, *semigroup union of groups* and *semigroup semilattice of groups*, terms that are redundant for semigroup theorists. The idea is a straightforward one. We simply "extend" the binary product that gives the bijection between groups and Ward quasigroups, for example, to the semigroup semilattice of groups and to the resultant structure(s). So, we are working with structures that result from defining binary operations on a semigroup semilattice of groups $(S,\cdot)$ as follows: $x * y = x \cdot y^{-1}$ (called *right division*), $x * y = x^{-1} \cdot y$ (called *left division*) and $x * y = x^{-1} \cdot y^{-1}$ (called *double division*). This is possible because a semigroup semilattice of groups is an inverse semigroup; that is, each element $x \in S$ has a unique inverse x^{-1} that is the inverse of the element x in the group to which it belongs [3, Theorem 4.11].

On the resultant structures $(S,*)$ we define binary operations as follows, respectively:

$$x_\alpha \otimes y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta),$$
$$x_\alpha \otimes y_\beta = (x_\alpha * e_{\alpha\beta}) * y_\beta,$$
$$x_\alpha \otimes y_\beta = (e_{\alpha\beta} * x_\alpha) * (e_{\alpha\beta} * y_\beta).$$

These structures $(S,\otimes)$ turn out to be semigroup semilattices of groups. In each of these three cases, the mappings $(S,\cdot) \to (S,*)$ and $(S,*) \to (S,\otimes)$ are inverse mappings. Hence, we find three different collections of structures, each of which is in one-to-one correspondence with the collection **SLG** of all semigroup semilattices of groups.

Lemma 3.1. (cf. [3, Theorem 4.11]) *A semigroup $(S,\cdot)$ is a semigroup semilattice V of groups $(G_\alpha, \cdot_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$ if and only if $(S,\cdot)$ is a semigroup union of groups and has commuting idempotents if and only if $(S,\cdot)$ is an inverse semigroup that is a semigroup union of groups if and only if $(S,\cdot)$ is a semigroup and a semigroup semilattice $V \cong E(S,\cdot)$ of groups.*

Note that the following identity holds in inverse semigroups:

(16) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.

If $(S,\cdot)$ is a semigroup and a semilattice V of groups then it follows from Lemma 3.1 that

(17) $e_\alpha \cdot e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta \cdot e_\alpha$

for all $\alpha, \beta \in V$.

Lemma 3.2. *Suppose that $(S,\cdot)$ is a semigroup semilattice V of groups (G_α, e_α) $(\alpha \in V)$ and that $x_\alpha * y_\beta = x_\alpha \cdot y^{-1}$ for all $x_\alpha \in G_\alpha$, $y_\beta \in G_\beta$ and $\alpha, \beta \in V$. Then*

(18) *$(S,*)$ is an inverse groupoid with $x_\alpha^{-1} = e_\alpha * x_\alpha$ $(\alpha \in V)$,*

(19) $E(S,*) \cong E(S,\cdot) \cong V$,

(20) $(S,*)$ *is a semigroup semilattice* V *of Ward quasigroups* $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$,

(21) $(x_\alpha * y_\beta) * (z_\sigma * w_\gamma) = [x_\alpha * (w_\gamma^{-1} * y_\beta^{-1})] * z_\sigma$,

(22) $x_\alpha * (e_{\alpha\beta} * y_\beta) = x_\alpha * y_\beta^{-1}$,

(23) $x_\alpha * y_\beta = (y_\beta * x_\alpha)^{-1}$.

Proof. (18): It is straightforward to calculate that x_α^{-1}, the inverse of x_α in the group to which it belongs, is the unique inverse of x_α in $(S,*)$. That is, $x_\alpha^{-1} = e_\alpha * x_\alpha$.

(19): $x_\alpha = x_\alpha * x_\alpha$ if and only if $x_\alpha = x_\alpha \cdot x_\alpha^{-1} = e_\alpha$, the identity of the group to which x_α belongs. Then, $e_\alpha * e_\beta = e_\alpha \cdot e_\beta^{-1} = e_\alpha \cdot e_\beta \overset{(17)}{=} e_{\alpha\beta}$. Since, by Lemma 3.1, in $(S,\cdot)$ we have $E(S,\cdot) \cong V$, $E(S,*) \cong E(S,\cdot) \cong V$.

(20): Since $x_\alpha * y_\beta = x_\alpha \cdot y_\beta^{-1} \in G_\alpha \cdot G_\beta \subseteq G_{\alpha\beta}$. Since $x_\alpha * y_\alpha = x_\alpha \cdot y_\alpha^{-1}$ in each $(G_\alpha, *_{|G_\alpha}, e_\alpha)$, by fact $(F1)$, $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ is a Ward quasigroup for all $\alpha \in V$. By definition then, $(S,*)$ is a semigroup semilattice V of Ward quasigroups $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$.

(21): Using the facts that $x_\alpha * y_\beta = x_\alpha \cdot y_\beta^{-1}$ and $(x_\alpha \cdot y_\beta)^{-1} \overset{(16)}{=} y_\beta^{-1} \cdot x_\alpha^{-1}$ it is straightforward to calculate that $(x_\alpha * y_\beta) * (z_\sigma * w_\gamma) = x_\alpha \cdot y_\beta^{-1} \cdot w_\gamma \cdot z_\sigma^{-1} = [x_\alpha * (w_\gamma^{-1} * y_\beta^{-1})] * z_\sigma$.

(22): $x_\alpha * (e_{\alpha\beta} * y_\beta) = x_\alpha \cdot (e_{\alpha\beta} \cdot y_\beta^{-1})^{-1} \overset{(16)}{=} x_\alpha \cdot (y_\beta \cdot e_{\alpha\beta}^{-1}) = (x_\alpha \cdot y_\beta) \cdot e_{\alpha\beta} = x_\alpha \cdot y_\beta = x_\alpha * y_\beta^{-1}$.

(23): $(x_\alpha * y_\beta)^{-1} = (x_\alpha \cdot y_\beta^{-1})^{-1} \overset{(16)}{=} y_\beta \cdot x_\alpha^{-1} = y_\beta * x_\alpha$. □

Definition 3.3. If $(S,\cdot)$ is a semigroup semilattice V of groups (G_α, e_α) $(\alpha \in V)$ and $x * y = x \cdot y^{-1}$, then we denote $(S,*)$ by $SLWQ(S,\cdot)$. We define **SLWQ** as the collection of all semigroup semilattices V of Ward quasigroups $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$ that satisfy (21). In particular, $SLWQ(S,\cdot) \in$ **SLWQ**.

Note once again that a semigroup semilattice of inverse groupoids is an inverse groupoid. So, conditions (21), (22) and (23) have a clear meaning.

Lemma 3.4. *Suppose that* $(S,*)$ *is a semigroup semilattice* V *of Ward quasigroups* $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ $(\alpha \in V)$ *and satisfies* (21). *Define* $x_\alpha \cdot y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta)$. *Then* $(S,\cdot)$ *is a semigroup and a semigroup semilattice* V *of groups* $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ $(\alpha \in V)$ *with* $V \cong E(S,\cdot) \cong E(S,*)$.

Proof. As previously noted in the proof of $(F12)$, since each $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ is an inverse groupoid, with $x_\alpha^{-1} = e_\alpha * x_\alpha$and since a semigroup semilattice of inverse groupoids is an inverse groupoid, $(S, *)$ is an inverse groupoid.

We prove that (21) implies (22). We have

$$x_\alpha*(e_{\alpha\beta}*y_\beta) = (x_\alpha*e_\alpha)*(e_{\alpha\beta}*y_\beta) \overset{(21)}{=} [x_\alpha*(y_\beta^{-1}*e_\alpha^{-1})]*e_{\alpha\beta} = [x_\alpha*(y_\beta^{-1}*e_\alpha)] =$$
$$(x_\alpha * e_\alpha) * (y_\beta^{-1} * e_\alpha) \overset{(21)}{=} [x_\alpha * (e_\alpha * e_\alpha)] * y_\beta^{-1} = (x_\alpha * e_\alpha) * y_\beta^{-1} = x_\alpha * y_\beta^{-1},$$

so, (22) is valid.

Next, we prove that (21) implies (23). Since we have $x_\alpha * y_\beta = x_\alpha * (y_\beta^{-1})^{-1} \overset{(22)}{=} x_\alpha * (e_{\alpha\beta} * y_\beta^{-1})$, then $x_\alpha * y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta^{-1}) \overset{(5)}{=} [e_\alpha * (e_\alpha * x_\alpha)] * (e_{\alpha\beta} * y_\beta^{-1}) \overset{(21)}{=} [e_\alpha*(y_\beta*x_\alpha)]*e_{\alpha\beta} = [e_\alpha*(y_\beta*x_\alpha)]*(e_{\alpha\beta}*e_{\alpha\beta}) \overset{(21)}{=} (e_\alpha*[e_{\alpha\beta}*(y_\beta*x_\alpha)^{-1}])*e_{\alpha\beta} = e_\alpha * (y_\beta * x_\alpha) = (e_\alpha * e_\alpha) * [(y_\beta * x_\alpha) * e_{\alpha\beta}] \overset{(21)}{=} [e_\alpha * (e_{\alpha\beta} * e_\alpha)] * (y_\beta * x_\alpha) \overset{(F12)}{=} e_{\alpha\beta} * (y_\beta * x_\alpha) = (y_\beta * x_\alpha)^{-1}$, so, (23) is valid.

Now $x_\alpha = x_\alpha \cdot x_\alpha$ if and only if $x_\alpha = x_\alpha * (e_\alpha * x_\alpha) = x_\alpha * e_\alpha$ if and only if $e_\alpha = e_\alpha*x_\alpha = x_\alpha*x_\alpha$ if and only if $x_\alpha = e_\alpha$. Also, $e_\alpha*e_\beta = [e_\alpha*(e_\alpha*e_\alpha)]*e_\beta) \overset{(21)}{=} e_\alpha * (e_\beta * e_\alpha)$. Then, $e_\alpha \cdot e_\beta = e_\alpha * (e_{\alpha\beta} * e_\beta) = (e_\alpha * e_\alpha) * (e_{\alpha\beta} * e_\beta) \overset{(21)}{=} [e_\alpha * (e_\beta * e_\alpha)] * e_{\alpha\beta} = e_\alpha * (e_\beta * e_\alpha) \overset{(21)}{=} e_\alpha * e_\beta$. So, the operations $\cdot$ and $*$ coincide on $E(S, *)$. Thus, $E(S, \cdot) \cong E(S, *)$. Using $(F12)$, $E(S, \cdot) \cong E(S, *) \cong V$ is a semigroup semilattice. Since, for each $(W_\alpha, *_{|W_\alpha}, e_\alpha)$, $x_\alpha \cdot y_\alpha = x_\alpha * (e_\alpha * y_\alpha)$, by $(F4)$, each $(W_\alpha, \cdot_{|W_\alpha}, e_\alpha)$ is a group. Since $x_\alpha \cdot y_\beta = x_\alpha*(e_{\alpha\beta} * y_\beta) \in W_{\alpha\beta}$, $W_\alpha \cdot W_\beta \subseteq W_{\alpha\beta}$, and so $(S, \cdot)$ is a semigroup semilattice V of groups. So, we only need to prove that $(S, \cdot)$ is a semigroup.

We have $(x_\alpha \cdot y_\beta) \cdot z_\gamma = [x_\alpha*(e_{\alpha\beta}*y_\beta)]*(e_{\alpha\beta\gamma}*z_\gamma) \overset{(22)}{=} (x_\alpha*y_\beta^{-1})*(e_{\alpha\beta\gamma}*z_\gamma) \overset{(21)}{=} [x_\alpha * (z_\gamma^{-1} * y_\beta)] * e_{\alpha\beta\gamma} = [x_\alpha * (z_\gamma^{-1} * y_\beta)] \overset{(23)}{=} x_\alpha * (y_\beta * z_\gamma^{-1})^{-1} \overset{(22)}{=} x_\alpha * [e_{\alpha\beta\gamma} * (y_\beta * z_\gamma^{-1})] \overset{(22)}{=} x_\alpha * [e_{\alpha\beta\gamma} * [y_\beta * (e_{\beta\gamma} * z_\gamma)]) = x_\alpha \cdot (y_\beta \cdot z_\gamma)$. □

Corollary 3.5. *Let $(S, *)$ be a semigroup semilattice V of Ward quasigroups $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ $(\alpha \in V)$. If $(S, *)$ satisfies* (21)*, then*

(*i*) *it satisfies* (22) *and* (23)*,*

(*ii*) *there exists $(S, \cdot) \in$ **SLG** such that $x * y = x \cdot y^{-1}$ for all $x, y \in S$.*

Proof. Part (i) was proved in Lemma 3.4. For part (ii), let $(S, \cdot)$ be as in our Lemma 3.4. Then, as proved in Lemma 3.4, $(S, \cdot) \in$ **SLG**. Also, $x_\alpha \cdot y_\beta^{-1} = (x_\alpha*e_\alpha)*(e_{\alpha\beta}*y_\beta^{-1}) \overset{(21)}{=} [x_\alpha*(y_\beta*e_\alpha)]*e_{\alpha\beta} = [x_\alpha*(y_\beta*e_\alpha)] = [(x_\alpha*e_\alpha)*(y_\beta*e_\alpha)] \overset{(21)}{=} (x_\alpha * e_\alpha) * y_\beta = x_\alpha * y_\beta$. □

Definition 3.6. Let $(S,*)$ be a semigroup semilattice V of Ward quasigroups $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ $(\alpha \in V)$ that satisfies (21), and if we define $x_\alpha \cdot y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta)$, then we denote the semigroup semilattice V of groups $(S,\cdot)$ as $SLG(S.*)$.

Theorem 3.7. *For all $(S,*) \in$ **SLWQ**, $SLWQ(SLG(S,*)) = (S,*)$ and for all $(S,\cdot) \in$ **SLG**, $SLG(SLWQ(S,\cdot)) = (S,\cdot)$.*

Proof. In $SLG(S,*)$ the product is $x_\alpha \cdot y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta)$. The product $\otimes$ in $SLWQ(SLG(S,*))$ is $x_\alpha \otimes y_\beta = x_\alpha \cdot y_\beta^{-1}$. So, $x_\alpha * y_\beta = (x_\alpha * y_\beta) * (e_{\alpha\beta} * e_{\alpha\beta}) \overset{(22)}{=} (x_\alpha * (e_{\alpha\beta}^{-1} * y_\beta^{-1})) * e_{\alpha\beta} = x_\alpha * (e_{\alpha\beta} * y_\beta^{-1}) = x_\alpha \cdot y_\beta^{-1} = x_\alpha \otimes y_\beta$ and consequently, $SLWQ(SLG(S,*)) = (S,*)$.

In $SLWQ(S,\cdot)$ the product is $x*y = x\cdot y^{-1}$. The product $\oplus$ in $SLG(SLWQ(S,\cdot))$ is $x_\alpha \oplus y_\beta = x_\alpha * (e_{\alpha\beta} * y_\beta) = x_\alpha \cdot (e_{\alpha\beta} \cdot y_\beta^{-1})^{-1} \overset{(12)}{=} x_\alpha \cdot y_\beta \cdot e_{\alpha\beta}^{-1} = x_\alpha \cdot y_\beta$. Hence, $SLG(SLWQ(S,\cdot)) = (S,\cdot)$. □

The second part of the following Corollary can be viewed as an "extension" of $(F1)$.

Corollary 3.8. *There is a one-to-one correspondence between semigroup semilattices of groups* **SLG** *and groupoids $(S,*)$ that are semigroup semilattices V of Ward quasigroups $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ and that satisfy (21). Also, $(S,*) \in$* **SLWQ** *if and only if there exists $(S,\cdot) \in$* **SLG** *such that $x * y = x \cdot y^{-1}$ for all $x, y \in S$.*

Corollary 3.9. *There is a one-to-one correspondence between semigroup semilattices of abelian groups and groupoids $(S,*)$ that are semigroup semilattices V of medial Ward quasigroups $(W_\alpha, *_{|W_\alpha}, e_\alpha)$ $(\alpha \in V)$ and that satisfy (21).*

Proof. The proof here follows that of Lemmas 3.2, 3.4 and Theorem 3.7, using the additional fact that a groupoid is a medial Ward quasigroup if and only if it is induced by an abelian group. □

Lemma 3.10. *Let $(S,\cdot)$ be a semigroup semilattice V of groups $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$ and let $x * y = x^{-1} \cdot y$. Then*

(24) $(S,*)$ *an inverse groupoid with* $x_\alpha^{-1} = x_\alpha * e_\alpha$, $(\alpha \in V)$,

(25) $E(S,*) \cong E(S,\cdot) \cong V$,

(26) $(S,*)$ *is a semigroup semilattice V of Ward dual quasigroups* $(S_\alpha, *_{|S_\alpha}, e_\alpha)$ $(\alpha \in V)$,

(27) $(x_\alpha * y_\beta) * (z_\gamma * w_\sigma) = y_\beta * ((z_\gamma^{-1} * x_\alpha^{-1}) * w_\sigma)$,

(28) $(x_\alpha * e_{\alpha\beta}) * y_\beta = x_\alpha^{-1} * y_\beta$,

(29) $x_\alpha * y_\beta = (y_\beta * x_\alpha)^{-1}$.

Proof. Note that it follows from Lemma 3.1 that $\mathbf{SLG} = \overline{\mathbf{SLG}}$. Since $x \bar{*} y = y * x = y^{-1} \cdot x = x \bar{\cdot} y^{-1}$ and $(S, \bar{\cdot}) \in \mathbf{SLG}$, $(S, \bar{*})$ satisfies (18) to (23). Hence, $(S, *)$ satisfies (24) to (29). □

Definition 3.11. $SLWD(S, \cdot)$ will denote $(S, *)$ in Lemma 3.10 above. We denote the collection of all groupoids $(S, *)$ that are semigroup semilattices V of Ward dual quasigroups $(WD_\alpha, *_{|_{WD_\alpha}}, e_\alpha)$ $(\alpha \in V)$ that satisfy (27) as **SLWDQ**. So, $SLWD(S, \cdot) = (S, *) \in \mathbf{SLWDQ}$.

Lemma 3.12. *Suppose that $(S, *)$ is a semigroup semilattice V of Ward dual quasigroups $(WD_\alpha, *_{|_{WD_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and satisfies (27). Then $(S, \cdot)$ with the operation $x_\alpha \cdot y_\beta = (x_\alpha * e_{\alpha\beta}) * y_\beta$, is a semigroup and a semigroup semilattice V of groups.*

Proof. It is clear that $(S, \bar{*})$ is a semigroup semilattice V of Ward quasigroups $(WD_\alpha, *_{|_{WD_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and satisfies (21). Also, $x_\alpha \bar{\cdot} y_\beta = (y_\beta * e_{\alpha\beta}) * x_\alpha = x_\alpha \bar{*} (e_{\alpha\beta} \bar{*} y_\beta)$. By Lemma 3.4, $(S, \bar{\cdot})$ is a semigroup and a semigroup semilattice V of groups with $V \cong E(S, \bar{\cdot}) \cong E(S, \bar{*})$. Hence, $(S, \cdot)$ is a semigroup and a semigroup semilattice V of groups with $V \cong E(S, \cdot) \cong E(S, *)$. □

Definition 3.13. $SLG(S, *)$ will denote $(S, \cdot)$ in Lemma 3.12 above.

Theorem 3.14. *For all $(S, \cdot) \in \mathbf{SLG}$, $SLG(SLWD(S, \cdot)) = (S, \cdot)$ and for all $(S, *) \in \mathbf{SLWDQ}$, $SLWD(SLG(S, *)) = (S, *)$.*

Proof. Observe that the product in $SLWD(S, \cdot)$ is $x * y = x^{-1} \cdot y$. The product in $SLG(SLWD(S, \cdot))$ is

$$x_\alpha \otimes y_\beta = (x_\alpha * e_{\alpha\beta}) * y_\beta = (x_\alpha^{-1} \cdot e_{\alpha\beta}) * y_\beta \overset{(12)}{=} e_{\alpha\beta} \cdot x_\alpha \cdot y_\beta = x_\alpha \cdot y_\beta.$$

Hence, $SLG(SLWD(S, \cdot)) = (S, \cdot)$.

The product in $SLG(S, *)$ is $x_\alpha \cdot y_\beta = (x_\alpha * e_{\alpha\beta}) * y_\beta$. Hence, the product in $SLWD(SLG(S, *))$ is $x_\alpha \oplus y_\beta = x^{-1} \cdot y = ((x_\alpha * e_\alpha) * e_{\alpha\beta}) * y_\beta \overset{(27)}{=} x_\alpha * y_\beta$ and so $SLWD(SLG(S, *)) = (S, *)$. □

Corollary 3.15. *There is a one-to-one correspondence between semigroup semilattices of groups* **SLG** *and groupoids $(S, *)$ that are semigroup semilattices V of Ward dual quasigroups $(WD_\alpha, *_{|_{WD_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and that satisfy (24), (26) and (27). Also, $(S, *) \in$* **SLWDQ** *if and only if there exists $(S, \cdot) \in$* **SLG** *such that $x * y = x^{-1} \cdot y$ for all $x, y \in S$.*

Corollary 3.16. *There is a one-to-one correspondence between semigroup semilattices of abelian groups and groupoids $(S, *)$ that are semigroup semilattices V of unipotent, left-unital right modular quasigroups $(Q_\alpha, *_{|_\alpha}, e_\alpha)$ satisfying (27).*

Proof. A Ward dual quasigroup is a unipotent, left-unital right modular quasigroup if and only if it is medial if and only if it is induced by an abelian group. Using this fact, the proof of Corollary 3.16 exactly follows those of Lemmas 3.10, 3.12 and Theorem 3.14. □

Lemma 3.17. *Let* $(S,\cdot)$ *be a semigroup semilattice* V *of groups* $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$ *such that* $x_\alpha * y_\beta = x^{-1} \cdot y_\beta^{-1}$. *Then*

(30) $(\{e_\alpha \,|\, \alpha \in V\}, *) \cong (\{e_\alpha \,|\, \alpha \in V\}, \cdot) \cong V$ *is a semigroup semilattice,*

(31) $(S,*)$ *is a semigroup semilattice* V *of double Ward quasigroups* $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ $(\alpha \in V)$,

(32) $(S,*)$ *satisfies the identity*

$$(e_{\alpha\beta\gamma}*((e_{\alpha\beta}*x_\alpha)*(e_{\alpha\beta}*y_\beta)))*(e_{\alpha\beta\gamma}*z_\gamma) = (e_{\alpha\beta\gamma}*x_\alpha)*(e_{\alpha\beta\gamma}*((e_{\beta\gamma}*y_\beta)*(e_{\beta\gamma}*z_\gamma))),$$

(33) $(S,*)$ *satisfies the identity* $x_\alpha * y_\beta = (e_{\alpha\beta} * (e_\alpha * x_\alpha)) * (e_{\alpha\beta} * (e_\beta * y_\beta))$.

Proof. (30): For any $e_\alpha, e_\beta \in V$, $e_\alpha * e_\beta = e_\alpha^{-1} \cdot e_\beta^{-1} = e_\alpha \cdot e_\beta$. Then, $e_\alpha * e_\beta = e_\alpha \cdot e_\beta = e_\beta \cdot e_\alpha = e_{\alpha\beta}$, by Lemma 3.1. Hence, $(e_\alpha * e_\beta) * e_\gamma = e_{\alpha\beta} * e_\gamma = e_{(\alpha\beta)\gamma} = e_{\alpha(\beta\gamma)} = e_\alpha * (e_\beta * e_\gamma)$. By Lemma 3.1, $(\{e_\alpha \,|\, \alpha \in V\}, *) \cong V$ is a semigroup semilattice and so (30) is valid.

(31): Each $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ has product $x_\alpha * y_\alpha = x_\alpha^{-1} \cdot y_\alpha^{-1}$ and therefore, by $(F3)$, each $(G_\alpha, *_{|G_\alpha}, e_\alpha)$ is a double Ward quasigroup. Since $x_\alpha * y_\beta = x_\alpha^{-1} \cdot y_\beta^{-1} \in G_{\alpha\beta}$, (31) is valid.

(32): We have

$$\begin{aligned}
&(e_{\alpha\beta\gamma} * ((e_{\alpha\beta} * x_\alpha) * (e_{\alpha\beta} * y_\beta))) * (e_{\alpha\beta\gamma} * z_\gamma) = \\
&(e_{\alpha\beta\gamma} * ((e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1}) * (e_{\alpha\beta}^{-1} \cdot y_\beta^{-1}))) * (e_{\alpha\beta\gamma}^{-1} \cdot z_\gamma^{-1}) = \\
&(e_{\alpha\beta\gamma} * (x_\alpha \cdot y_\beta)) * (e_{\alpha\beta\gamma}^{-1} \cdot z_\gamma^{-1}) = (e_{\alpha\beta\gamma}^{-1} \cdot y_\beta^{-1} \cdot x_\alpha^{-1})^{-1} \cdot z_\gamma \cdot e_{\alpha\beta\gamma} = \\
&(x_\alpha \cdot y_\beta) \cdot (e_{\alpha\beta\gamma} \cdot (z_\gamma \cdot e_{\alpha\beta\gamma})) = (x_\alpha \cdot y_\beta) \cdot (z_\gamma \cdot e_{\alpha\beta\gamma}) = \\
&(x_\alpha \cdot y_\beta \cdot z_\gamma) \cdot e_{\alpha\beta\gamma} = x_\alpha \cdot y_\beta \cdot z_\gamma.
\end{aligned}$$

Also,

$$\begin{aligned}
&(e_{\alpha\beta\gamma} * x_\alpha) * (e_{\alpha\beta\gamma} * ((e_{\beta\gamma} * y_\beta) * (e_{\beta\gamma} * z_\gamma))) = \\
&(e_{\alpha\beta\gamma}^{-1} \cdot x_\alpha^{-1})^{-1} * (e_{\alpha\beta\gamma}^{-1} \cdot ((e_{\beta\gamma}^{-1} \cdot y_\beta^{-1})^{-1})^{-1})^{-1} = \\
&(x_\alpha \cdot e_{\alpha\beta\gamma}) \cdot e_{\alpha\beta\gamma} \cdot (y_\beta \cdot e_{\beta\gamma}) \cdot (z_\gamma \cdot e_{\beta\gamma}) = \\
&x_\alpha \cdot (e_\alpha \cdot e_{\beta\gamma}) \cdot (y_\beta \cdot (e_{\beta\gamma} \cdot (z_\gamma \cdot e_{\beta\gamma}))) = \\
&(x_\alpha \cdot e_\alpha) \cdot (e_{\beta\gamma} \cdot ((y_\beta \cdot z_\gamma) \cdot e_{\beta\gamma})) = x_\alpha \cdot y_\beta \cdot z_\gamma.
\end{aligned}$$

This proves (32).

(33): By the definition of the operation $*$,

$$(e_{\alpha\beta} * (e_\alpha * x_\alpha)) * (e_{\alpha\beta} * (e_\beta * y_\beta)) =$$
$$(e_{\alpha\beta}^{-1} \cdot (e_\alpha^{-1} \cdot x_\alpha^{-1})^{-1}) * (e_{\alpha\beta}^{-1} \cdot (e_\beta^{-1} \cdot y_\beta^{-1})^{-1}) =$$
$$e_\alpha \cdot x_\alpha^{-1} \cdot (e_{\alpha\beta} \cdot (e_\beta \cdot y_\beta^{-1} \cdot e_{\alpha\beta})) = e_\alpha \cdot x_\alpha^{-1} \cdot e_\beta \cdot y_\beta^{-1} \cdot e_{\alpha\beta} =$$
$$e_\alpha \cdot x_\alpha^{-1} \cdot e_\beta \cdot y_\beta^{-1} = x_\alpha^{-1} \cdot y_\beta^{-1} = x_\alpha * y_\beta.$$

This proves (33) and completes the proof of Lemma 3.17. □

Definition 3.18. $SLDWQ(S, \cdot)$ denotes $(S, *)$ of Lemma 3.17. The collection of all semilattices of double Ward quasigroups that satisfy (30)− (34) is denoted by **SLDWQ**.

Lemma 3.19. *Suppose that $(S, *)$ is a semigroup semilattice V of double Ward quasigroups $(DW_\alpha, *_{|DW_\alpha}, e_\alpha)$ $(\alpha \in V)$, that $(\{e_\alpha \,|\, \alpha \in V\}, *) \cong V$ and that $(S, *)$ satisfies* (32) *and* (33)*. Define $x_\alpha \cdot y_\beta = (e_{\alpha\beta} * x_\alpha) * (e_{\alpha\beta} * y_\beta)$. Then*

(34) $(\{e_\alpha \,|\, \alpha \in V\}, \cdot) \cong (\{e_\alpha \,|\, \alpha \in V\}, *)$ *is a semigroup semilattice,*

(35) $(S, \cdot)$ *is a semigroup and a semigroup semilattice of groups* $(DW_\alpha, *_{|DW_\alpha}, e_\alpha)$ $(\alpha \in V)$,

(36) *for all $\alpha, \beta \in V$ and all $x_\alpha \in DW_\alpha$, $y_\beta \in DW_\beta$, $x_\alpha * y_\beta = x_\alpha^{-1} \cdot y_\beta^{-1}$.*

Proof. We have

$$e_\alpha \cdot e_\beta = (e_{\alpha\beta} * e_\alpha) * (e_{\alpha\beta} * e_\beta) = (e_{\alpha\beta} * (e_\alpha * e_\alpha)) * (e_{\alpha\beta} * (e_\beta * e_\beta)) \overset{(33)}{=} e_\alpha * e_\beta$$

and so, (34) is valid.

For each $(DW_\alpha, *_{|DW_\alpha}, e_\alpha)$ the product is $x_\alpha \cdot y_\beta = (e_\alpha * x_\alpha) * (e_\alpha * y_\alpha)$, by $(F6)$ each $(DW_\alpha, *_{|DW_\alpha}, e_\alpha)$ is a group. By (32), $(S, \cdot)$ is a semigroup and, by Lemma 3.1, (35) is valid. Finally, by $(F6)$, $x_\alpha^{-1} = e_\alpha * x_\alpha$ in $(S, \cdot)$. Then, by (33),

$$x_\alpha^{-1} \cdot y_\beta^{-1} = (e_\alpha * x_\alpha) \cdot (e_\beta * y_\beta) = (e_{\alpha\beta} * (e_\alpha * x_\alpha)) * (e_{\alpha\beta} * (e_\beta * y_\beta)) = x_\alpha * y_\beta,$$

which completes the proof. □

Definition 3.20. $SLG(S, *)$ denotes $(S, \cdot)$ of Lemma 3.19.

Theorem 3.21. *For all $(S, \cdot) \in$ **SLG**, $SLG(SLDWQ(S, \cdot)) = (S, \cdot)$ and for all $(S, *) \in$ **SLDWQ**, $SLDWQ(SLG(S, *)) = (S, *)$.*

Proof. The product in $SLDWQ(S, \cdot)$ is $x_\alpha * y_\beta = x_\alpha^{-1} \cdot y_\beta^{-1}$. So, the product in $SLG(SLDWQ(S, \cdot))$ is $x_\alpha \otimes y_\beta = (e_{\alpha\beta} * x_\alpha) * (e_{\alpha\beta} * y_\beta) = (e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1})^{-1} \cdot (e_{\alpha\beta}^{-1} \cdot y_\beta^{-1})^{-1}$ $= x_\alpha \cdot (e_{\alpha\beta} \cdot (y_\beta \cdot e_{\alpha\beta})) = (x_\alpha \cdot y_\beta) \cdot e_{\alpha\beta} = x_\alpha \cdot y_\beta$. Hence, $SLG(SLDWQ(S, \cdot)) = (S, \cdot)$.

The product in $SLG(S, *)$ is $x_\alpha \cdot y_\beta = (e_{\alpha\beta} * x_\alpha) * (e_{\alpha\beta} * y_\beta)$. The product in $SLDWQ(SLG(S, *))$ is $x_\alpha \oplus y_\beta = x_\alpha^{-1} \cdot y_\beta^{-1} = (e_{\alpha\beta} * (e_\alpha * x_\alpha)) * (e_{\alpha\beta} * (e_\beta * e y_\beta)) \overset{(33)}{=}$ $x_\alpha * y_\beta$. So, $SLDWQ(SLG(S, *)) = (S, *)$. □

Corollary 3.22. *There is a one-to-one correspondence between elements of* **SLG** *and* **SLDWQ**.

Note that since **SLG** is in one-one correspondence with **SLWQ**, **SLWDQ** and **SLDWQ**, **SLWQ** and **SLWDQ** are in one-one correspondence with each other, as are **SLDWQ** and **SLWQ**. The next results give the explicit forms of these bijective mappings.

Theorem 3.23. $\overline{\mathbf{SLG}} = \mathbf{SLG}$.

Proof. The dual groupoid of a semigroup union of groups with commuting idempotents is a semigroup union of groups with commuting idempotents. As previously noted, the required result then follows from Lemma 3.1. □

Theorem 3.24. . $\overline{\mathbf{SLWQ}} = \mathbf{SLWDQ}$.

Proof. If $(S,*) \in \mathbf{SLWQ}$, then $x * y = x \cdot y^{-1}$ for some $(S,\cdot) \in \mathbf{SLG}$. So, if $(T,\bar{\circ}) \in \overline{\mathbf{SLWQ}}$, then, using Theorem 3.23, $x \bar{\circ} y = x^{-1} \bar{\cdot} y$ for some $(T,\bar{\cdot})$ from **SLG**. As in the proof of Lemma 3.10, $(T,\bar{\circ})$ satisfies (24) – (27). Therefore, $(T,\bar{\circ}) \in \mathbf{SLWDQ}$. Hence, $\overline{\mathbf{SLWQ}} \subseteq \mathbf{SLWDQ}$.

If $(S,*) \in \mathbf{SLWDQ}$, then $x * y = x^{-1} \cdot y$ for some $(S,\cdot) \in \mathbf{SLG}$. So, using Theorem 3.23, $x \bar{*} y = x \bar{\cdot} y^{-1}$ for some $(S,\bar{\cdot}) \in \mathbf{SLG}$. Therefore, as in the proof of Lemma 3.2, $(S,\bar{*})$ satisfies (17). Hence, $(S,\bar{*}) \in \mathbf{SLWQ}$ and $(S,*) \in \overline{\mathbf{SLWQ}}$. So, $\mathbf{SLWDQ} \subseteq \overline{\mathbf{SLWQ}}$. □

Corollary 3.25. $(S,*) \in \mathbf{SLWDQ}$ *if and only if* $(S,*)$ *is a semilattice of Ward dual quasigroups and satisfies the identity* $(x*y)*(z*w) = y*((z^{-1}*x^{-1})*w)$.

Theorem 3.26. $\overline{\mathbf{SLDWQ}} = \mathbf{SLDWQ}$.

Proof. If $(S,*) \in \overline{\mathbf{SLDWQ}}$, then $x * y = y \bar{*} x$ for some $(S,\bar{*}) \in \mathbf{SLDWQ}$. So, $x * y = y \bar{*} x = x^{-1} \bar{\cdot} y^{-1}$ for some $(S,\bar{\cdot}) \in \mathbf{SLG}$. Therefore, by the proof of Lemma 3.17, $(S,*) \in \mathbf{SLDWQ}$. Hence, $\overline{\mathbf{SLDWQ}} \subseteq \mathbf{SLDWQ} \subseteq \overline{\mathbf{SLDWQ}}$. □

Theorem 3.27. **SLDWQ** *and* **SLWQ** *are in one-one correspondence.*

Proof. For $(S,*) \in \mathbf{SLDWQ}$ we define $SLWQ(S,*) = (S,\circ)$, where $x_\alpha \circ y_\beta = (e_{\alpha\beta} * x_\alpha) * y_\beta$. If $(S,\otimes) \in \mathbf{SLWQ}$, we define $SLDWQ(S,\otimes) = (S,\oplus)$, where $x_\alpha \oplus y_\beta = (e_{\alpha\beta} \otimes x_\alpha) \otimes y_\beta$. Note that, since $(S,*) \in \mathbf{SLDWQ}$, $x * y = x^{-1} \cdot y^{-1}$ for some $(S,\cdot) \in \mathbf{SLG}$. Therefore, $x_\alpha \circ y_\beta = (e_{\alpha\beta} * x_\alpha) * y_\beta = (e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1})^{-1} \cdot y_\beta^{-1} = x_\alpha \cdot e_{\alpha\beta} \cdot y_\beta^{-1} = x_\alpha \cdot e_\alpha \cdot e_\beta \cdot y_\beta^{-1} = x_\alpha \cdot y_\beta^{-1}$. By Lemma 3.2, $SLWQ(S,*) = (S,\circ)$ is in **SLWQ**. Therefore, $SLDWQ(S,\circ) = (S,\oplus)$, where $x_\alpha \oplus y_\beta = (e_{\alpha\beta} \circ x_\alpha) \circ y_\beta = (e_{\alpha\beta} \cdot x_\alpha^{-1}) \cdot y_\beta^{-1} = x_\alpha^{-1} \cdot y_\beta^{-1}$ and so $(S,\oplus) \in \mathbf{SLDWQ}$.

Then, the product in $SLDWQ(SLWQ(S,*))$ is $x_\alpha \oplus y_\beta = (e_{\alpha\beta} \circ x_\alpha) \circ y_\beta = (e_{\alpha\beta} * (e_{\alpha\beta} \circ x_\alpha)) * y_\beta = (e_{\alpha\beta}^{-1} \cdot (e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1})^{-1})^{-1} \cdot y_\beta^{-1} = ((e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1})^{-1})^{-1} \cdot y_\beta^{-1} = (e_{\alpha\beta}^{-1} \cdot x_\alpha^{-1}) \cdot y_\beta^{-1} = x_\alpha^{-1} \cdot y_\beta^{-1} = x_\alpha * y_\beta$. Therefore, $SLDWQ(SLWQ(S,*)) = (S,*)$.

Similarly, $SLWQ(SLDWQ(S,\otimes)) = (S,\otimes)$. □

Questions. Suppose that $(S,*)$ is a semigroup semilattice V of double Ward quasigroups $(DW_\alpha, *_{|_{DW_\alpha}}, e_\alpha)$ $(\alpha \in V)$ and that $(S,*)$ satisfies (32) and (33). Then, is $(\{e_\alpha \,|\, \alpha \in V\}, *) \cong V$.

1. Can groupoids in **SLDWQ** be described by a single identity, in place of (32) and (33)?

2. Is there a structure theorem for groupoids in **SLWQ**, **SLWDQ** and **SLDWQ** analogous to the structure theorem for semigroups that are semigroup semilattices of groups [3, Theorem 4.11]?

A remaining area for investigation is right, left and double division on completely simple semigroups, where x^{-1} is the inverse of x in the group to which it belongs.

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