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Some structures of Hom-Poisson color algebras

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Abstract. In many previous papers, the authors used an algebra endomorphism to twist the original algebraic structures in order to produce the corresponding Hom-algebraic structures. In this work, we use a bijective linear map, an element of centroid, an averaging operator, a Rota-Baxter operator or a multiplier to produce a Hom-Poisson color algebra from a given one.

1. Introduction

Poisson algebras are algebras which has simultaneously a Lie and a commutative associative algebra structures satisfying the Leibniz identity. They naturally appear in very different forms and contexts. Many examples coming from geometry and mathematical physics lead to a certain type of Poisson structures. These are always a key element coming along with interesting problems in the fields of classical/quantum mechanics, differential geometry and algebraic geometry.

The first motivation to study nonassociative Hom-algebras comes from quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras [1, 5, 6, 8, 9]. Hom-Lie algebras were first introduced by Hartwig, Larsson and Silvestrov in order to describe q-deformations of Witt and Virasoro algebras using σ-derivations [7]. The corresponding associative type objects, called Hom-associative algebras were introduced by Makhlouf and Silvestrov in [10]. Next, generalizations of Hom-type algebras are introduced and discussed in the framework of color algebras. In particular, Hom-associative color algebras [11] has been introduced as a generalization of both Hom-associative algebras and associative color algebras. Furthermore, relying on the well-known relationship between (Hom-)associative and (Hom-)Lie algebras, Hom-Lie color algebras were also introduced in [11] as a natural generalization of Hom-Lie algebras and as a special case of quasi-hom-Lie algebras. It is proved that the commutator of any Hom-associative color algebras gives rise to Hom-Lie color algebras and a way to obtain Hom-Lie color algebras from classical Lie color algebras along with even color algebra endomorphisms is presented. Also, we have introduced a multiplier σ on an abelian group and constructions of new Hom-Lie color algebras from given ones by the σ-twists are obtained. Furthermore, Hom-Poisson color algebras are

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introduced in [3] as the color version of Hom-Poisson algebras [4]. Some constructions of Hom-Poisson color algebras from Hom-associative color algebras which twisting map is an averaging operator or from a given Hom-Poisson color algebra together with an averaging operator or from a Hom-post-Poisson color algebra are given in [4]. In particular, it is shown that any Hom-pre-Poisson color algebra leads to a Hom-Poisson color algebra. Moreover, in [2] is obtained a description of Hom-Poisson color algebras by using only one operation of its two binary operations via the polarisation-depolarisation process.

The goal of this paper is to give a continuation of constructions of Hom-Poisson color algebras [4]. While many authors working on Hom-algebras use a morphism of Hom-algebras to build another one, we ask ourselves if there are others kinds of twists which are not morphisms such that we can get Hom-algebraic structures from others one. To give a positive answer to the above question, we organize this paper as follows. In Section 2, we recall some basic definitions about Rota-Baxter Hom-associative color algebras and Rota-Baxter Hom-Lie color algebras as well as averaging operators and centroids. In Section 3, we give the main results of the paper. The proceeding is by twisting the original multiplications of Hom-Poisson color algebras by a bijective linear map, an element of centroid, an averaging operator, a Rota-Baxter operator or a multiplier.

Throughout this paper, all graded vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

2. Definitions

In this section, we recall some relevant definitions about G-graded vetor space and color Hom-algebras. In particular, we recall the notion of a color Hom-associative algebra as well as the one of a color Hom-Lie algebra. Some examples are given and some results are also proved.

First, let recall that if G is an abelian group, a vector space L is said to be *G-graded* if, there exists a family $(L_a)_{a\in G}$ of vector subspaces of L such that $L = \oplus_{a\in G} L_a$. An element $u \in L$ is said to be *homogeneous of degree* $a \in G$ if $u \in L_a$. The set of all homogeneous elements in L is denoted by $\mathcal{H}(L)$.

Definition 2.1. Let G be an abelian group. A map $\varepsilon : G \times G \to \mathbb{K}^*$ is called a *skew-symmetric bicharacter* on G if the following identities hold:

(1) $\varepsilon(a, b)\varepsilon(b, a) = 1$,

(2) $\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)$,

(3) $\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)$,

for all $a, b, c \in G$.

Remark 2.2. (1) Observe that $\varepsilon(a, 0) = \varepsilon(0, a) = 1,\ \varepsilon(a, a) = \pm 1$ for all $a \in G$.

(2) If x and y are two homogeneous elements of degree a and b respectively and ε is a bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$. Also unless stated, in the sequel all the graded space are over the same abelian group G and the bicharacter will be the same for all the structures.

Example 2.3. For $G = \mathbb{Z}_2^n = \{(\alpha_1, \dots, \alpha_n) | \alpha_i \in \mathbb{Z}_2\}$,

$$\varepsilon((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) := (-1)^{\alpha_1\beta_1 + \dots + \alpha_n\beta_n}$$

is a skew-symmetric bicharacter.

Definition 2.4. Let G be an abelian group. A *bicharacter* on G is a map $\delta : G \times G \to \mathbb{K}^*$ defined by

$$\delta(x, y) := \sigma(x, y)\sigma(y, x)^{-1} \text{ for all } x, y, z \in G$$

where $\sigma : G \times G \to \mathbb{K}^*$ is any mapping such that

$$\sigma(x, y + z)\sigma(y, z) = \sigma(x, y)\sigma(x + y, z), \text{ for all } x, y, z \in G.$$

In this case, σ is called a *multiplier* on G and δ the bicharacter *associated with* σ.

Example 2.5. If we define the mapping $\sigma : G \times G \to \mathbb{R}^*$ by

$$\sigma((i_1, i_2), (j_1, j_2)) := (-1)^{i_1 j_2}, \text{ for all } i_k, j_k \in \mathbb{Z}_2,\ k = 1, 2,$$

it is easy to verify that σ is a multiplier on G and

$$\delta((i_1, i_2), (j_1, j_2)) := (-1)^{i_1 j_2 - i_2 j_1}, \text{for all } i_k, j_k \in \mathbb{Z}_2,\ i = 1, 2.$$

is a bicharacter on G.

Definition 2.6. A color Hom-algebra *is a quadruple* $(A, \mu, \varepsilon, \alpha)$ *in which*

(1) A *is a* G*-graded vector space i.e.,* $A = \bigoplus_{a \in G} A_a$*,*

(2) $\mu : A \times A \to A$ *is an even bilinear map i.e.,* $\mu(A_a, A_b) \subset A_{a+b}$*, for all* $a, b \in G$*,*

(3) $\alpha : A \to A$ *is an even linear map i.e.,* $\alpha(A_a) \subset A_a$ *for all* $a \in G$*,*

(4) $\varepsilon : G \times G \to \mathbb{K}^*$ *is a bicharacter.*

Definition 2.7. A *Hom-associative color algebra* is a color Hom-algebra $(A, \mu, \varepsilon, \alpha)$ satisfying the Hom-associativity condition:

$$as_\mu(x, y, z) := \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = 0,$$

for all $x, y, z \in \mathcal{H}(A)$.

If, in addition, μ satisfies $\mu = \varepsilon(\cdot, \cdot)\mu^{op}$ i.e., $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$ for all $x, y \in \mathcal{H}(A)$ (ε-commutativity), the Hom-associative color algebra $(A, \mu, \varepsilon, \alpha)$ is said to be a *ε-commutative Hom-associative color algebra.*

Whenever, $\alpha = Id_A$ we recover associative color algebra.

Proposition 2.8. *Let (A, μ, ε) be an associative color algebra and $\alpha : A \to A$ be an even linear map such that $(A, \mu, \varepsilon, \alpha)$ is a Hom-associative color algebra. Then, for any fixed element $\xi \in A$, the quadruple $(A, \mu_\xi, \varepsilon, \alpha)$ is a Hom-associative color algebra with*

$$\mu_\xi(x, y) := x\xi y,$$

for all $x, y \in \mathcal{H}(A)$.

Proof. For any $x, y, z \in \mathcal{H}(A)$ we have

$$\begin{aligned} as_{\mu_\xi}(x, y, z) &= \mu_\xi(\mu_\xi(x, y), \alpha(z)) - \mu_\xi(\alpha(x), \mu_\xi(y, z)) \\ &= (x\xi y)\xi\alpha(z) - \alpha(x)\xi(y\xi z) \\ &= x(\xi y\xi)\alpha(z) - \alpha(x)(\xi y\xi)z \text{ (associativity)} \\ &= (x\xi y\xi)\alpha(z) - (x\xi y\xi)\alpha(z) \text{ (Hom-associativity)} \\ &= 0. \end{aligned}$$

□

Now, we recall the definition of Hom-Lie color algebra.

Definition 2.9. A *Hom-Lie color algebra* is a color Hom-algebra $(A, [\ ,\ ], \varepsilon, \alpha)$ satisfying

(1) $[x, y] = -\varepsilon(x, y)[y, x]$ (ε-skew-symmetry),

(2) $\varepsilon(z, x)[\alpha(x), [y, z]] + \varepsilon(x, y)[\alpha(y), [z, x]] + \varepsilon(y, z)[\alpha(z), [x, y]] = 0$ (color Hom-Jacobi identity)

for any $x, y, z \in \mathcal{H}(A)$.

Example 2.10. It is clear that Lie color algebras are examples of Hom-Lie color algebras by setting $\alpha = id$. If, in addition, $\varepsilon(x, y) = 1$ (resp. $\varepsilon(x, y) = (-1)^{|x||y|}$) then, the Hom-Lie color algebra is a classical Lie algebra (resp. Lie superalgebra). Moreover, Hom-Lie algebras (resp. Hom-Lie superalgebras) are also obtained when $\varepsilon(x, y) = 1$ (resp. $\varepsilon(x, y) = (-1)^{|x||y|}$). See [11] for other examples as Hom-Lie color $sl(2, \mathbb{K})$, Heisenberg Hom-Lie color algebra and Hom-Lie color algebra of Witt type.

Definition 2.11. i) A *Rota-Baxter Hom-associative color algebra of weight $\lambda \in \mathbb{K}$* is a Hom-associative color algebra $(A, \cdot, \varepsilon, \alpha)$ together with an even linear map $R : A \to A$ that satisfies the identities

$$R \circ \alpha = \alpha \circ R, \tag{1}$$

$$R(x) \cdot R(y) = R\Big(R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y\Big), \tag{2}$$

for all $x, y \in \mathcal{H}(A)$.

ii) A *Rota-Baxter Hom-Lie color algebra of weight* $\lambda \in \mathbb{K}$ is a Hom-Lie color algebra $(L, [\ ,\ ], \varepsilon, \alpha)$ together with an even linear map $R : L \to L$ that satisfies the identities

$$\begin{aligned} R \circ \alpha &= \alpha \circ R, \\ [R(x), R(y)] &= R\Big([R(x), y] + [x, R(y)] + \lambda[x, y]\Big), \end{aligned} \tag{3}$$

for all $x, y \in \mathcal{H}(L)$.

Example 2.12. Consider the abelian multiplicative group $G = \{-1, +1\}$ and the G-graded 2-dimensional vector space $A = A_{(-1)} \oplus A_{(1)} = \langle e_2 \rangle \oplus \langle e_1 \rangle$. Then the quintuple $(A, \cdot, \varepsilon, \alpha, R)$ is a Rota-Baxter Hom-associative color algebra of weight λ with

- the multiplication: $e_1 \cdot e_1 := -e_1, \quad e_1 \cdot e_2 := e_2, \quad e_2 \cdot e_1 := e_2, \quad e_2 \cdot e_2 := e_1,$
- the bicharacter: $\varepsilon(i, j) := (-1)^{(i-1)(j-1)/4}$,
- the even linear map $\alpha : A \to A$ defined by $\alpha(e_1) := e_1, \quad \alpha(e_2) := -e_2,$
- the Rota-Baxter operator $R : A \to A$ given by $R(e_1) := -\lambda e_1$, $R(e_2) := -\lambda e_2$.

Definition 2.13. Let $k \geqslant 0$ be an integer.

i) An α^k*-averaging operator* over a Hom-associative color algebra $(A, \mu, \varepsilon, \alpha)$, is an even linear map $\beta : A \to A$ such that

$$\alpha \circ \beta = \beta \circ \alpha, \tag{4}$$
$$\beta(\mu(\beta(x), \alpha^k(y)) = \mu(\beta(x), \beta(y)) = \beta(\mu(\alpha^k(x), \beta(y))), \tag{5}$$

for all $x, y \in \mathcal{H}(A)$.

ii) An α^k*-averaging operator* over a Hom-Lie color algebra $(L, [,], \varepsilon, \alpha)$, is an even linear map $\beta : L \to L$ such that

$$\begin{aligned} \alpha \circ \beta &= \beta \circ \alpha, \\ [\beta(x), \beta(y)] &= \beta([\beta(x), \alpha^k(y)]), \end{aligned} \tag{6}$$

for all $x, y \in \mathcal{H}(L)$.

Definition 2.14. Let $k \geqslant 0$ be an integer.

An element of α^k*-centroid* of a Hom-associative color algebra $(A, \cdot, \varepsilon, \alpha)$, is an even linear map $\beta : A \to A$ such that

$$\beta \circ \alpha = \alpha \circ \beta, \tag{7}$$
$$\beta(x \cdot y) = \beta(x) \cdot \alpha^k(y) = \alpha^k(x) \cdot \beta(y), \tag{8}$$

for all $x, y \in \mathcal{H}(A)$.

In the case of a Hom-Lie color algebra $(L, [\ ,\ ], \varepsilon, \alpha)$, *an element of* α^k*-centroid* is an even linear map $\beta : L \to L$ such that

$$\begin{aligned} \beta \circ \alpha &= \alpha \circ \beta, \\ \beta([x, y]) &= [\beta(x), \alpha^k(y)], \end{aligned} \tag{9}$$

for all $x, y \in \mathcal{H}(L)$.

Observe that $\beta([x, y]) = [\alpha^k(x), \beta(y)]$ thanks to the ε-skew-symmetry.

3. Hom-Poisson Color Algebras

This section is devoted to various constructions of Hom-Poisson color algebras. It contains relevant results of this paper. In the most proofs, we don't establish the ε-skew-symmetry condition as well as the color Hom-Jacobi identity.

Definition 3.1. *A Hom-Poisson color algebra* consists of a G-graded vector space A, a multiplication $\mu : A \times A \to A$, an even bilinear bracket $\{\ ,\}: A \times A \to A$ and an even linear map $\alpha : A \to A$ such that :

(1) $(A, \mu, \varepsilon, \alpha)$ is a Hom-associative color algebra,

(2) $(A, \{\ ,\}, \varepsilon, \alpha)$ is a Hom-Lie color algebra,

(3) the Hom-Leibniz color identity

$$\{\alpha(x), \mu(y, z)\} = \mu(\{x, y\}, \alpha(z)) + \varepsilon(x, y)\mu(\alpha(y), \{x, z\}),$$

is satisfied for any $x, y, z \in \mathcal{H}(A)$.

A Hom-Poisson color algebra $(A, \mu, \{\ ,\}, \varepsilon, \alpha)$ in which μ is ε-commutative is said to be a *commutative Hom-Poisson color algebra.*

Example 3.2. Let $A = A_{(0)} \oplus A_{(1)} = \langle e_1, e_2\rangle \oplus \langle e_3\rangle$ be a 3-dimensional graded vector space and $\cdot : A \times A \to A$ and $[\ ,\ ] : A \times A \to A$ the multiplications defined by

$$\begin{aligned} &e_1 \cdot e_1 := e_1, e_1 \cdot e_2 := e_2, e_1 \cdot e_3 := ae_3, e_2 \cdot e_1 := e_2, \\ &e_2 \cdot e_1 := \frac{1}{a} e_2, e_2 \cdot e_3 := e_3, e_3 \cdot e_1 := ae_3, [e_2, e_3] := e_3 \end{aligned}$$

and the omitted products being zero. Then, the quintuple $(A, \cdot, [\ ,\ ], \varepsilon, \alpha)$ is a Hom-Poisson color algebra with

$$\alpha(e_1) := e_1, \quad \alpha(e_2) := e_2, \quad \alpha(e_3) := ae_3,$$

and any bicharacter ε.

Theorem 3.3. *Let* $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ *be a Hom-Poisson color algebra and a map* $\sigma : G \times G \to \mathbb{K}^*$ *be a symmetric multiplier on* G *i.e.,*

(1) $\sigma(x,y)=\sigma(y,x), \forall x,y\in G$,

(2) $\sigma(x,y)\sigma(z,x+y)$ *is invariant under cyclic permutation of* $x,y,z\in G$.

Then, $P^\sigma=(P,\cdot^\sigma,[\ ,\ ]^\sigma,\varepsilon,\alpha)$ *is also a Hom-Poisson color algebra with*

$$x\cdot^\sigma y:=\sigma(x,y)x\cdot y \quad \text{and} \quad [x,y]^\sigma:=\sigma(x,y)[x,y],$$

for any $x,y\in\mathcal{H}(P)$.

Proof. For any homogeneous elements $x,y,z\in P$,

$$\begin{aligned}
as_{\cdot^\sigma}(x,y,z) &= (x\cdot^\sigma y)\cdot^\sigma \alpha(z)-\alpha(x)\cdot^\sigma(y\cdot^\sigma z)\\
&=\sigma(x,y)\sigma(x+y,z)(x\cdot y)\cdot z-\sigma(x,y+z)\sigma(y,z)\alpha(x)\ \cdot(y\cdot z)\\
&=\sigma(x,y)\sigma(x+y,z)as_{\cdot}(x,y,z)\\
&=0.
\end{aligned}$$

Thus the Hom-associativity condition holds in P^σ. Next, the color Hom-Jacobi identity follows from [11]. Finally, for verifying the Hom- Leibniz color identity consider any homogeneous elements $x,y,z\in P$,

$$\begin{aligned}
&[\alpha(x),y\cdot^\sigma z]^\sigma\\
&=[\alpha(x),\sigma(y,z)y\cdot z]^\sigma\\
&=\sigma(y,z)\sigma(x,y+z)[\alpha(x),y\cdot z]\\
&=\sigma(y,z)\sigma(x,y+z)[x,y]\cdot\alpha(z)+\sigma(y,z)\sigma(x,y+z)\varepsilon(x,y)\alpha(y)\cdot[x,z]\\
&=\sigma(x,y)\sigma(z,x+y)[x,y]\cdot\alpha(z)+\sigma(z,x)\sigma(y,z+x)\varepsilon(x,y)\alpha(y)\cdot[x,z]\\
&=\sigma(z,x+y)[x,y]\cdot^\sigma\alpha(z)+\sigma(y,x+z)\varepsilon(x,y)\alpha(y)\cdot^\sigma[x,z]\\
&=[x,y]^\sigma\cdot^\sigma\alpha(z)+\varepsilon(x,y)\alpha(y)\cdot^\sigma[x,z]^\sigma.
\end{aligned}$$

□

The following theorem can be proved as the previous one.

Theorem 3.4. *Let* $(P,\cdot,[\ ,\ ],\varepsilon,\alpha)$ *be a Hom-Poisson color algebra and a map* $\delta: G\times G\to\mathbb{K}^*$ *be the bicharacter associated with the multiplier* σ *on* G*. Then,* $(P,\cdot^\sigma,[\ ,\ ]^\sigma,\varepsilon\delta,\alpha)$ *is also a Hom-Poisson color algebra with*

$$x\cdot^\sigma y:=\sigma(x,y)x\cdot y,\ [x,y]^\sigma:=\sigma(x,y)[x,y] \text{ and } \varepsilon\delta(x,y):=\varepsilon(x,y)\sigma(x,y)\sigma(y,x)^{-1},$$

for any $x,y\in\mathcal{H}(P)$*. Moreover, an endomorphism of* $(P,\cdot,[\ ,\ ],\varepsilon,\alpha)$ *is also an endomorphism of* $(P,\cdot^\sigma,[\ ,\ ]^\sigma,\varepsilon\delta,\alpha)$.

Theorem 3.5. *Let* $(P',\cdot',[\ ,\ ]',\varepsilon,\alpha')$ *be a Hom-Poisson color algebra and* P *a graded vector space with an even bilinear map "$\cdot$", a* ε*-skew-symmetric even bilinear bracket "$[,]$" and an even linear map* α*. Let* $f:P\to P'$ *be an even bijective linear map such that* $f\circ\alpha=\alpha'\circ f$,

$$f(x\cdot y)=f(x)\cdot' f(y) \quad \text{and} \quad f([x,y])=[f(x),f(y)]', \forall x,y\in\mathcal{H}(P).$$

Then $(P,\cdot,[\ ,\ ],\varepsilon,\alpha)$ *is a Hom-Poisson color algebra.*

Proof. First, we obtain for all $x, y, z \in \mathcal{H}(P)$,

$$\begin{aligned}
&(x \cdot y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) \\
&= f^{-1}\Big((f(x) \cdot' f(y)) \cdot' f(\alpha(z))\Big) - f^{-1}\Big(f(\alpha(x)) \cdot' (f(y) \cdot' f(z))\Big) \\
&= f^{-1}\Big((f(x) \cdot' f(y)) \cdot' \alpha'(f(z) - \alpha'(f(x)) \cdot' (f(y) \cdot' f(z))\Big).
\end{aligned}$$

Thus, the Hom-associativity identity follows from the one in P'. Similarly, we get the color Hom-Jacobi identity. Finally, for any $x, y, z \in \mathcal{H}(P)$, the Hom-Leibniz color identity is proved as follows

$$\begin{aligned}
[\alpha(x), y \cdot z] &= f^{-1}[f(\alpha(x)), f(y \cdot z)]' \\
&= f^{-1}[f(\alpha(x)), f\Big(f^{-1}\Big(f(y) \cdot' f(z)\Big)\Big)]' \\
&= f^{-1}[\alpha'(f(x)), f(y) \cdot' f(z)]' \\
&= f^{-1}\Big([f(x), f(y)]' \cdot' \alpha'(f(z)) + \varepsilon(x, y)\alpha'(f(y)) \cdot' [f(x) \cdot' f(z)]'\Big) \\
&= f^{-1}\Big(f\Big(f^{-1}[f(x), f(y)]'\Big) \cdot' \alpha'(f(z))\Big) \\
&\quad + \varepsilon(x, y) f^{-1}\Big(\alpha'(f(y)) \cdot' f\Big(f^{-1}[f(x) \cdot' f(z)]'\Big)\Big) \\
&= f^{-1}\Big(f([x, y]) \cdot' f(\alpha(z))\Big) + \varepsilon(x, y) f^{-1}\Big(f(\alpha(y)) \cdot' f([x \cdot z])\Big) \\
&= [x, y] \cdot \alpha(z) + \varepsilon(x, y)\alpha(y) \cdot [x \cdot z].
\end{aligned}$$

□

Definition 3.6. Let $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ be a Hom-Poisson color algebra. An even linear map $\beta : P \to P$ is said to be

(1) *an element of* α^k*-centroid* of P if (7), (8) and (9) hold.

(2) *an* α^k*-averaging operator* of P if (4), (5) and (6) hold.

(3) *a Rota-Baxter operator* over P if (1), (2) and (3) hold.

Example 3.7. The even linear map $R : P \to P$ defined on the Hom-Poisson color algebra of Example 3.2, by

$$R(e_1) := -\lambda e_1, \quad R(e_2) := -\lambda e_2, \quad R(e_3) := -\lambda e_3,$$

is a Rota-Baxter operator of weight λ on P.

Example 3.8. If $(A, \mu, \varepsilon, \alpha, R)$ is a Rota-Baxter Hom-associative color algebra, then

$$(A, \mu, \{\ ,\ \} := \mu - \varepsilon(\cdot\ ,\cdot)\mu^{op}, \varepsilon, \alpha, R),$$

is a Rota-Baxter Hom-Poisson color algebra.

Now, we have the following result whose proof is similar to the one of the previous.

Theorem 3.9. *Let $P^\bullet := (P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ be a Hom-Poisson color algebra and $\beta : P \to P$ be an element of α^0-centroid of P. If we define the multiplications $* : P \times P \to P$ and $\{\ ,\} : P \times P \to P$ by*

$$x * y := x \cdot y \quad \textit{and} \quad \{x, y\} := [\beta(x), y], \forall x, y \in \mathcal{H}(P), \tag{10}$$

then $P^* := (P, *, \{\ ,\}, \varepsilon, \alpha)$ is also a Hom-Poisson color algebra. Moreover, the map $\beta : (P, *, \{\ ,\}, \varepsilon, \alpha) \longrightarrow (P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ becomes a morphism of Hom-Poisson color algebras.

Proof. It is clear that the Hom-associativity identity in P^* follows from the one in $P^\bullet$. Next, the color Hom-Jacobi identity is proved as follows

$$\begin{aligned}
&\varepsilon(z, x)\{\alpha(x), \{y, z\}\} + \varepsilon(x, y)\{\alpha(y), \{z, x\}\} + \varepsilon(y, z)\{\alpha(z), \{x, y\}\}\\
&= \varepsilon(z, x)[\beta(\alpha(x)), [\beta(y), z]] + \varepsilon(x, y)[\beta(\alpha(y)), [\beta(z), x]] + \varepsilon(y, z)[\beta(\alpha(z)), [\beta(x), y]]\\
&= \varepsilon(z, x)[\beta(\alpha(x)), \beta([y, z])] + \varepsilon(x, y)[\beta(\alpha(y)), \beta([z, x])] + \varepsilon(y, z)[\beta(\alpha(z)), \beta([x, y])]\\
&= \beta^2(\varepsilon(z, x)[\alpha(x), [y, z]] + \varepsilon(x, y)[\alpha(y), [z, x]] + \varepsilon(y, z)[\alpha(z), [x, y]])\\
&= \beta^2(0) = 0.
\end{aligned}$$

In order to prove the Hom-Leibniz color identity we consider $x, y, z \in \mathcal{H}(P)$. Then

$$\begin{aligned}
\{\alpha(x), y * z\} &= [\beta(\alpha(x)), y \cdot z] = [\alpha(\beta(x)), y \cdot z]\\
&= [\beta(x), y] \cdot \alpha(z) + \varepsilon(x, y)\alpha(y)) \cdot [\beta(x), z]\\
&= \{x, y\} * \alpha(z) + \varepsilon(x, y)\alpha(y) * \{x, y\}.
\end{aligned}$$

□

Theorem 3.10. *Let $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ be a Hom-Poisson color algebra and $\beta : P \to P$ be an α^0-averaging operator. Then with the products defined as*

$$x * y := \beta(x) \cdot \beta(y) \quad \textit{and} \quad \{x, y\} := [\beta(x), \beta(y)], \forall x, y \in \mathcal{H}(P), \tag{11}$$

$(P, *, \{\ ,\}, \varepsilon, \alpha)$ is a Hom-Poisson color algebra.

Proof. First, the ε-skew-symmetry is obvious to obtain. Next, let $x, y, z \in \mathcal{H}(P)$, then

$$\begin{aligned}
&(x * y) * \alpha(z) - \alpha(x) * (y * z)\\
&= \beta(\beta(x) \cdot \beta(y)) \cdot \beta(\alpha(z)) - \beta(\alpha(x)) \cdot \beta(\beta(y) \cdot \beta(z))\\
&= \beta\Big((\beta(x) \cdot \beta(y)) \cdot \alpha(\beta(z)) - \alpha(\beta(x)) \cdot (\beta(y) \cdot \beta(z))\Big)\\
&= \beta(0) = 0,
\end{aligned}$$

which is the Hom-associativity. Similarly, we get the color Hom-Jacobi identity as follows

$$\begin{aligned}
&\varepsilon(z,x)\{\alpha(x),\{y,z\}\}+\varepsilon(x,y)\{\alpha(y),\{z,x\}\}+\varepsilon(y,z)\{\alpha(z),\{x,y\}\}\\
=&\ \varepsilon(z,x)[\beta(\alpha(x)),\beta([\beta(y),\beta(z)])]+\varepsilon(x,y)[\beta(\alpha(y)),\beta([\beta(z),\beta(x)])]\\
&+\varepsilon(y,z)[\beta(\alpha(z)),\beta([\beta(x),\beta(y)])]\\
=&\ \beta\Big(\varepsilon(z,x)[\alpha(\beta(x)),[\beta(y),\beta(z)]]+\varepsilon(x,y)[\alpha(\beta(y)),[\beta(z),\beta(x)]]\\
&+\varepsilon(y,z)[\alpha(\beta(z)),[\beta(x),\beta(y)]]\Big)\\
=&\ \beta(0)=0.
\end{aligned}$$

Finally, let us prove the Hom-Leibniz color identity as follows

$$\begin{aligned}
\{\alpha(x),y*z\} &= [\beta(x),\beta(y*z)]\\
&= [\beta\alpha(x),\beta(\beta(y)\cdot\beta(z))]\\
&= [\alpha\beta(x),\beta^2(y)\cdot\beta(z)]\\
&= [\beta(x),\beta^2(y)]\cdot\alpha\beta(z)+\varepsilon(x,y)\alpha\beta^2(y)\cdot[\beta(x),\beta(z)]\\
&= \beta[\beta(x),\beta(y)]\cdot\beta\alpha(z)+\varepsilon(x,y)\alpha\beta^2(y)\cdot[\beta(x),\beta(z)]\\
&= \beta[\beta(x),\beta(y)]\cdot\beta\alpha(z)+\varepsilon(x,y)\beta^2\alpha(y)\cdot\beta([x,\beta(z)])\\
&= \beta[\beta(x),\beta(y)]\cdot\beta\alpha(z)+\varepsilon(x,y)\beta\Big(\beta\alpha(y)\cdot\beta([x,\beta(z)])\Big)\\
&= \beta[\beta(x),\beta(y)]\cdot\beta\alpha(z)+\varepsilon(x,y)\beta\Big(\beta\alpha(y)\cdot[\beta(x),\beta(z)])\Big)\\
&= \beta[\beta(x),\beta(y)]\cdot\beta\alpha(z)+\varepsilon(x,y)\beta\alpha(y)\cdot\beta[\beta(x),\beta(z)]\\
&= \{x,y\}*\alpha(z)+\varepsilon(x,y)\alpha(y)*\{x,z\}.
\end{aligned}$$
□

The following theorem is proved by a straighforward calculation.

Theorem 3.11. *Let $(P,\cdot,[\ ,\ ],\varepsilon)$ be a Poisson color algebra and $\beta:P\to P$ an α^0-averaging operator. Then with the products*

$$x*y:=\beta(x)\cdot y \quad and \quad \{x,y\}:=[\beta(x),y],\forall x,y\in\mathcal{H}(P), \tag{12}$$

$(P,*,\{\ ,\ \},\varepsilon,\beta)$ becomes a Hom-Poisson color algebra.

Theorem 3.12. *Let $(P,\cdot,[\ ,\ ],\varepsilon,\alpha)$ be a Hom-Poisson color algebra and $\beta:P\to P$ be an injective α^k-averaging operator. Then with the products*

$$x*y:=\beta(x)\cdot\alpha^k(y) \quad and \quad \{x,y\}:=[\beta(x),\alpha^k(y)],\forall x,y\in\mathcal{H}(P) \tag{13}$$

$(P,*,\{\ ,\ \},\varepsilon,\alpha)$ is a Hom-Poisson color algebra. Moreover, $\beta:(P,*,\{\ ,\ \},\varepsilon,\alpha)\to(P,\cdot,[\ ,\ ],\varepsilon,\alpha)$ is a morphism of Hom-Poisson color algebras.

Proof. Note that the ε-skew-symmetry is obvious to prove. Next, to prove the Hom-associativity, pick $x, y, z \in \mathcal{H}(P)$ then

$$\begin{aligned}
&\beta\Big((x * y) * \alpha(z) - \alpha(x) * (y * z)\Big)\\
&= \beta\Big(\beta(\beta(x) \cdot \alpha^k(y)) \cdot \alpha^{k+1}(z) - \beta\alpha(x) \cdot \alpha^k(\beta(y) \cdot \beta(z))\Big)\\
&= \beta\Big((\beta(x) \cdot \alpha^k(y)\Big) \cdot \alpha\beta(z) - \alpha\beta(x) \cdot \beta\Big(\beta(y) \cdot \alpha^k(z)\Big)\\
&= (\beta(x) \cdot \beta(y)) \cdot \alpha(\beta(z)) - \alpha(\beta(x)) \cdot (\beta(y) \cdot \beta(z))\\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\beta\Big(\varepsilon(z, x)\{\alpha(x), \{y, z\}\} + \varepsilon(x, y)\{\alpha(y), \{z, x\}\} + \varepsilon(y, z)\{\alpha(z), \{x, y\}\}\Big)\\
&= \beta\Big(\varepsilon(z, x)[\beta\alpha(x), \alpha^k([\beta(y), \alpha^k(z)])] + \varepsilon(x, y)[\beta\alpha(y), \alpha^k([\beta(z), \alpha^k(x)])]\\
&\quad + \varepsilon(y, z)[\beta\alpha(z), \alpha^k([\beta(x), \alpha^k(y)])]\Big) = \varepsilon(z, x)[\beta\alpha(x), \beta([\beta(y), \alpha^k(z)])]\\
&\quad + \varepsilon(x, y)[\beta\alpha(y), \beta([\beta(z), \alpha^k(x)])] + \varepsilon(y, z)[\beta\alpha(z), \beta([\beta(x), \alpha^k(y)])]\\
&= \varepsilon(z, x)[\alpha(\beta(x)), [\beta(y), \beta(z)]] + \varepsilon(x, y)[\alpha(\beta(y)), [\beta(z), \beta(x)]]\\
&\quad + \varepsilon(y, z)[\alpha(\beta(z)), [\beta(x), \beta(y)]]\\
&= \beta(0) = 0,
\end{aligned}$$

which is the color Hom-Jacobi identity. Finally, let us prove the Hom-Leibniz color identity as follows

$$\begin{aligned}
\beta(\{\alpha(x), y * z\}) &= [\beta\alpha(x), \alpha^k(y * z)] = [\beta\alpha(x), \alpha^k(\beta(y) \cdot \alpha^k(z))]\\
&= [\beta\alpha(x), \beta(\beta(y) \cdot \alpha^k(z))] = [\alpha\beta(x), \beta(y) \cdot \beta(z)]\\
&= [\beta(x), \beta(y)] \cdot \alpha\beta(z) + \varepsilon(x, y)\alpha\beta(y) \cdot [\beta(x), \beta(z))]\\
&= \beta[\beta(x), \alpha^k(y)] \cdot \alpha\beta(z) + \varepsilon(x, y)\alpha\beta(y) \cdot \beta[\beta(x), \alpha^k(z))]\\
&= \beta\Big(\beta[\beta(x), \alpha^k(y)] \cdot \alpha^{k+1}(z) + \varepsilon(x, y)\beta\alpha(y) \cdot \alpha^k[\beta(x), \alpha^k(z))]\Big)\\
&= \beta\Big(\beta\{x, y\} \cdot \alpha^{k+1}(z) + \varepsilon(x, y)\beta\alpha(y) \cdot \alpha^k\{x, z\}\Big)\\
&= \beta\Big(\{x, y\} * \alpha(z) + \varepsilon(x, y)\alpha(y) * \{x, z\}\Big).
\end{aligned}$$

□

Theorem 3.13. *Let $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ be a Hom-Poisson color algebra and the map $R : P \to P$ be a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on P. Then P is a Hom-Poisson color algebra with the multiplications:*

$$\begin{aligned}
x * y &:= R(x) \cdot y + x \cdot R(y) + \lambda x \cdot y,\\
\{x, y\} &:= [R(x), y] + [x, R(y)] + \lambda[x, y],
\end{aligned}$$

for all $x, y \in \mathcal{H}(P)$. Moreover, R is a morphism of Hom-Poisson color algebra $(P, *, \{\ ,\ \}, \varepsilon, \alpha)$ onto $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$.

Proof. First, let $x, y, z \in \mathcal{H}(P)$, then

$$\begin{aligned}
&(x*y)*\alpha(z) = R(x*y)\cdot\alpha(z) - (x*y)\cdot R\alpha(z) + \lambda(x*y)\cdot\alpha(z)\\
&= (R(x)\cdot R(y))\cdot\alpha(z) + (R(x)\cdot y)\alpha R(z) + (x\cdot R(y))\cdot\alpha R(z) + \lambda(x*y)\cdot\alpha R(z)\\
&+\lambda(R(x)\cdot y)\cdot\alpha(z) + \lambda(x\cdot R(y))\cdot\alpha(z) + \lambda(x\cdot y)\cdot\alpha(z)\\
&= \alpha R(x)\cdot(R(y)\cdot z) + \alpha R(x)\cdot(y\cdot R(z)) + \alpha(x)\cdot(R(y)\cdot R(z)) + \lambda\alpha(x)\cdot(y\cdot R(z))\\
&+\lambda\alpha R(x)\cdot(y\cdot z) + \lambda\alpha(x)\cdot(R(y)\cdot z) + \lambda(x\cdot y)\cdot\alpha(z),
\end{aligned}$$

and also,

$$\begin{aligned}
&\alpha(x)*(y*z) = R\alpha(x)\cdot(y*z) + \alpha(x)\cdot(y*z) + \lambda\alpha(x)\cdot(y*z)\\
&= \alpha R(x)\cdot(R(y)\cdot z) + \alpha R(x)\cdot(y\cdot R(z)) + \alpha(x)\cdot(R(y)\cdot R(z)) + \lambda\alpha(x)\cdot(y\cdot R(z))\\
&+\lambda\alpha R(x)\cdot(y\cdot z) + \lambda\alpha(x)\cdot(R(y)\cdot z) + \lambda(x\cdot y)\cdot\alpha(z)\\
&\ \text{(using (1), (2) and rearranging terms)}
\end{aligned}$$

therefore, the Hom-associativity holds. Next, we get by using Equation (3) that

$$\begin{aligned}
&\varepsilon(z,x)\{\alpha(x),\{y,z\}\} = \varepsilon(z,x)\Big([R\alpha(x),\{y,z\}] + [\alpha(x), R(\{y,z\})] + \lambda[\alpha(x),\{y,z\}]\Big)\\
&= \varepsilon(z,x)\Big([\alpha R(x),[R(y),z]] + [\alpha R(x),[y,R(z)]] + \lambda[\alpha R(x),[y,z]] + [\alpha(x),[R(y),R(z)]]\\
&\lambda[\alpha(x),[R(y),z]] + \lambda[\alpha(x),[y,R(z)]] + \lambda^2[\alpha(x),[y,z]]\Big)
\end{aligned}$$

and similarly, after rearranging terms

$$\begin{aligned}
&\varepsilon(x,y)\{\alpha(y),\{z,x\}\}\\
&= \varepsilon(x,y)\Big([\alpha R(y),[z,R(x)]] + [\alpha(y),[R(z),R(x)]] + \lambda[\alpha(y),[z,R(x)]]\\
&+[\alpha R(y),[R(z),x]] + \lambda[\alpha R(y),[z,x]] + \lambda[\alpha(y),[R(z),x]] + \lambda^2[\alpha(y),[z,x]]\Big),\\
&\varepsilon(y,z)\{\alpha(z),\{x,y\}\}\\
&= \varepsilon(y,z)\Big([\alpha R(z),[x,R(y)]] + [\alpha(z),[R(x),R(y)]] + \lambda[\alpha(z),[x,R(y)]]\\
&+[\alpha R(z),[R(x),y]] + \lambda[\alpha R(z),[x,y]] + \lambda[\alpha(z),[R(x),y]] + \lambda^2[\alpha(z),[x,y]]\Big).
\end{aligned}$$

Then adding memberwise these two previous equalities, we observe that the color Hom-Jacobi identity in $(P, *, \{\ ,\ \}, \varepsilon, \alpha)$ follows from the one in $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$.

Finally, let us prove the Hom-Leibniz color identity as follows

$$\begin{aligned}
&\{\alpha(x), y * z\}\\
=&\{\alpha(x), R(y)z + yR(z) + \lambda yz\}\\
=&[R\alpha(x), R(y)z + yR(z) + \lambda yz] + [\alpha(x), R(R(y)z + yR(z) + \lambda yz)]\\
&+\lambda[\alpha(x), R(y)z + yR(z) + \lambda yz]\\
=&[R\alpha(x), R(y)z] + [R\alpha(x), yR(z)] + \lambda[R\alpha(x), yz] + [\alpha(x), R(y)R(z)]\\
&+\lambda[\alpha(x), R(y)z] + \lambda[\alpha(x), yR(z)] + \lambda^2[\alpha(x), yz].
\end{aligned}$$

By the Hom-Leibniz the color identity in $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$,

$$\begin{aligned}
&\{\alpha(x), y * z\}\\
=&[R(x), R(y)]\alpha(z) + \varepsilon(x, y)\alpha(R(y))[R(x), z] + [R(x), y]\alpha(R(z))\\
&+\varepsilon(x, y)\alpha(y) \cdot [R(x), R(z)] + \lambda[R(x), y]\alpha(z) + \lambda\varepsilon(x, y)\alpha(y) \cdot [R(x), z]\\
&+[x, R(y)]\alpha(R(z)) + \varepsilon(x, y)\alpha(R(y))[x, R(z)] + \lambda[x, R(y)]\alpha(z)\\
&+\varepsilon(x, y)\lambda\alpha R(y)[x, z] + \lambda[x, y]\alpha(R(z)) + \lambda\varepsilon(x, y)\alpha(y)[x, R(z)]\\
&+\lambda^2[x, y]\alpha(z) + \lambda^2\varepsilon(x, y)\alpha(y)[x, z].
\end{aligned}$$

By reorganizing the terms, we have

$$\begin{aligned}
&\{\alpha(x), y * z\}\\
=&[R(x), R(y)]\alpha(z) + [R(x), y]\alpha(R(z)) + \lambda[x, y]\alpha(R(z)) + [x, R(y)]\alpha(R(z))\\
&+\lambda[R(x), y]\alpha(z) + \lambda[x, R(y)]\alpha(z) + \lambda^2[x, y]\alpha(z)\\
&+\varepsilon(x, y)\alpha(R(y))[R(x), z] + \varepsilon(x, y)\alpha(R(y))[x, R(z)] + \varepsilon(x, y)\lambda\alpha R(y)[x, z]\\
&+\varepsilon(x, y)\alpha(y)[R(x), R(z)] + \lambda\varepsilon(x, y)\alpha(y)[R(x), z] + \lambda\varepsilon(x, y)\alpha(y)[x, R(z)]\\
&+\lambda^2\varepsilon(x, y)\alpha(y)[x, z]\\
=&R\Big([R(x), y]\alpha(z) + [x, R(y)] + \lambda[x, y]\Big)\alpha(z)\\
&+\Big([R(x), y] + [x, R(y)] + \lambda[x, y]\Big)\alpha(R(z))\\
&+\lambda\Big([R(x), y] + [x, R(y)] + \lambda[x, y]\Big)\alpha(z)\\
&+\varepsilon(x, y)\alpha(R(y))\Big([R(x), z] + [x, R(z)] + \lambda[x, z]\Big) + \varepsilon(x, y)\alpha(y)[R(x), R(z)]\\
&+\lambda\varepsilon(x, y)\alpha(y)\Big([R(x), z] + [x, R(z)] + \lambda[x, z]\Big).\\
=&\Big([R(x), y] + [x, R(y)] + \lambda[x, y]\Big) * \alpha(z)\\
&+\varepsilon(x, y)\alpha(y) * \Big([R(x), z] + [x, R(z)] + \lambda[x, z]\Big).\\
=&\{x, y\} * \alpha(z) + \varepsilon(x, y)\alpha(y) * \{x, z\}.
\end{aligned}$$

□

The following result can be proved easily.

Theorem 3.14. *Let $(P, \cdot, [\ ,\ ], \varepsilon, \alpha)$ be a Hom-Poisson color algebra over a field $\mathbb{K}$ and suppose $\hat{\mathbb{K}}$ an extension of $\mathbb{K}$. Then, the graded $\mathbb{K}$-vector space*

$$\hat{\mathbb{K}} \otimes P = \sum_{g \in G} (\mathbb{K} \otimes P)_g = \sum_{g \in G} \mathbb{K} \otimes P_g$$

is a Hom-Poisson color algebra with

(1) *the associative product $(\xi \otimes x) \cdot' (\eta \otimes y) := \xi\eta \otimes (x \cdot y)$,*

(2) *the bracket $[\xi \otimes x, \eta \otimes y]' := \xi\eta \otimes [x, y]$,*

(3) *the even linear map $\alpha'(\xi \otimes x) := \xi \otimes \alpha(x)$,*

(4) *the bicharacter $\varepsilon(\xi + x, \eta + y) := \varepsilon(x, y)$,*

for all $\xi, \eta \in \hat{\mathbb{K}}$ and $x, y \in \mathcal{H}(P)$.

Theorem 3.15. *Let $(A, \cdot, \varepsilon, \alpha_A)$ be a commutative Hom-associative color algebra and $(P, *, [\ ,\ ], \varepsilon, \alpha_P)$ be a Hom-Poisson color algebra. Then the tensor product $A \otimes P$ endowed with the even linear map $\alpha = \alpha_A \otimes \alpha_P : A \otimes P \to A \otimes P$, the even bilinear maps $\diamond : (A \otimes P) \times (A \otimes P) \to A \otimes P$ and $\{\ ,\ \} : (A \otimes P) \times (A \otimes P) \to A \otimes P$ defined, for any $a, b \in \mathcal{H}(A)$, $x, y \in \mathcal{H}(P)$, by*

(1) $\alpha(a \otimes x) := \alpha_A(a) \otimes \alpha_P(x)$,

(2) $(a \otimes x) \diamond (b \otimes y) := \varepsilon(x, b)(a \cdot b) \otimes (x * y)$,

(3) $\{a \otimes x, b \otimes y\} := \varepsilon(x, b)(a \cdot b) \otimes [x, y]$,

is a Hom-Poisson color algebra.

Proof. First, let $a, b, c \in \mathcal{H}(A)$ and $x, y, z \in \mathcal{H}(P)$. By the Hom-associativity of $\cdot$ and $*$, we get:

$$\begin{aligned}
&\Big((a \otimes x) \diamond (b \otimes y)\Big) \diamond \alpha(c \otimes z) \\
&= \varepsilon(x, b)\varepsilon(x + y, c)(a \cdot b) \cdot \alpha_A(c) \otimes (x * y) * \alpha_P(z) \\
&= \varepsilon(x, b)\varepsilon(x, c)\varepsilon(y, c)(a \cdot b) \cdot \alpha_A(c) \otimes (x * y) * \alpha_P(z) \\
&= \alpha(a \otimes x) \diamond \Big((b \otimes y) \diamond (c \otimes z)\Big).
\end{aligned}$$

Hence the Hom-associativity of $\diamond$ holds. Next, we get

$$\begin{aligned}
&\varepsilon(c + z, a + x)\{\alpha(a \otimes x), \{b \otimes y, c \otimes z\}\} \\
&= \varepsilon(c, a)\varepsilon(x, b)\varepsilon(y, c)\varepsilon(z, a)\varepsilon(z, x)\Big(\alpha_A(a) \cdot (b \cdot c) \otimes [\alpha_P(x), [y, z]]\Big)
\end{aligned}$$

and similarly, by the ε-commutativity and the Hom-associativity of $\cdot$, we get

$$\begin{aligned}
&\varepsilon(a+x,b+y)\{\alpha(b\otimes y),\{c\otimes z,a\otimes x\}\}\\
&=\varepsilon(c,a)\varepsilon(x,b)\varepsilon(y,c)\varepsilon(z,a)\varepsilon(x,y)\Big(\alpha_A(a)\cdot(b\cdot c)\otimes[\alpha_P(y),[z,x]]\Big),\\
&\varepsilon(b+y,c+z)\{\alpha(c\otimes z),\{a\otimes x,b\otimes y\}\}\\
&=\varepsilon(c,a)\varepsilon(x,b)\varepsilon(y,c)\varepsilon(z,a)\varepsilon(y,z)\Big(\alpha_A(a)\cdot(b\cdot c)\otimes[\alpha_P(z),[x,y]]\Big).
\end{aligned}$$

Thus the color Hom-Jacobi identity in $(P,\diamond,\{\,,\},\varepsilon,\alpha)$ follows from the one in $(P,*,[\,,\,],\varepsilon,\alpha)$. Finally, let us prove the Hom-Leibniz color identity as follows

$$\begin{aligned}
&\{\alpha(a\otimes x),(b\otimes y)\diamond(c\otimes z)\}\\
&=\varepsilon(y,c)\{\alpha_A(a)\otimes\alpha_P(x),(b\cdot c)\otimes(y*z)\}\\
&=\varepsilon(y,c)\varepsilon(x,b+c)\alpha_A(a)\cdot(b\cdot c)\otimes[\alpha_P(x),y*z]\\
&=\varepsilon(y,c)\varepsilon(x,b+c)\alpha_A(a)\cdot(b\cdot c)\otimes([x,y]*\alpha_P(z)+\varepsilon(x,y)\alpha_P(y)*[x,z]\\
&=\varepsilon(y,c)\varepsilon(x,b)\varepsilon(x,c)\alpha_A(a)\cdot(b\cdot c)\otimes[x,y]*\alpha_P(z)+\\
&\quad\varepsilon(y,c)\varepsilon(x,b+c)\varepsilon(x,y)(a\cdot b)\cdot\alpha_A(c)\otimes\alpha_P(y)*[x,z]\\
&=\varepsilon(x,b)\varepsilon(x+y,c)(a\cdot b)\cdot\alpha_A(c)\otimes[x,y]*\alpha_P(z)+\\
&\quad\varepsilon(y,c)\varepsilon(x,b+c)\varepsilon(x,y)\varepsilon(a,b)(b\cdot a)\cdot\alpha_A(c)\otimes\alpha_P(y)*[x,z]\\
&=\varepsilon(x,b)(a\cdot b\otimes[x,y])\diamond(\alpha_A(c)\otimes\alpha_P(z))+\\
&\quad\varepsilon(y,c)\varepsilon(x,b)\varepsilon(x,y)\varepsilon(a,b)(\alpha_A(b)\otimes\alpha_P(y))\diamond(a\cdot c\otimes[x,z])\\
&=\{a\otimes x,b\otimes y\}\diamond\alpha(c\otimes z)+\varepsilon(a+x,b+y)\alpha(b\otimes y)\diamond\{a\otimes x,c\otimes z\}.\qquad\square
\end{aligned}$$

References

[1] **N. Aizawa and H. Sato**, *q-deformation of the Virasoro algebra with central extension,* Physics Letters B, Phys. Lett. B, **256** (1991), no. 1, 185-190.

[2] **S. Attan**, *Some characterizations of color Hom-Poisson algebras*, Hacettepe J. Math. Statist., **47** (2018), no. 6, 1552-1563.

[3] **I. Bakayoko**, *Modules over color Hom-Poisson algebras*, J. Gen. Lie Theory Appl., **8** (2014), no. 1, Art. ID ID 1000226, 7 pp.

[4] **I. Bakayoko and B.M. Touré**, *Constructing Hom-Poisson color algebras*, Intern. J. Algebra, **13** (2019), no. 1, 1-16

[5] **M. Chaichian, P. Kulish and J. Lukierski**, *q-Deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras,* Phys. Lett. B, **237** (1990), no. 3-4, 401-406.

[6] **T.L. Curtright and C.K. Zachos**, *Deforming maps for quantum algebras,* Phys. Lett. B, **243** (1990), no. 3, 237-244.

[7] **J.T. Hartwig, D. Larsson and S.D. Silvestrov**, *Deformations of Lie algebras using σ-derivations,* J. Algebra **295** (2006), 314-361.

[8] **N. Hu**, *q-Witt algebras, q-Lie algebras, q-holomorph structure and representations,* Algebra Colloq. **6**(1999), no. 1, 51-70.

[9] **C. Kassel**, *Cyclic homology of differential operators, the Virasoro algebra and a q-analogue,* Commun. Math. Phys. **146** (1992), 343-351.

[10] **A. Makhlouf and S. Silvestrov**, *Notes on formal deformations of Hom-Associative and Hom-Lie algebras,* Forum Math., **22** (2010), no. 4, 715-759.

[11] **L. Yuan**, *Hom-Lie color algebras*, Commun. Algebra, **40** (2012), 575-592.

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