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SS-supplemented property in the lattices

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Abstract. Let L be a lattice with the greatest element 1. We introduce and investigate the latticial counterpart of the filter-theoretical concepts of ss-supplemented. The basic properties and possible structures of such filters are studied.

1. Introduction

Since Kasch and Mares [13] have defined the notions of perfect and semiperfect for modules, the notion of a supplemented module has been used extensively by many authors. For submodules U and V of a module M, V is said to be a supplement of U in M or U is said to have a supplement V in M if $U + V = M$ and $U \cap V \ll V$. The module M is called supplemented if every submodule of M has a supplement in M. In a series of papers, Zöschinger has obtained detailed information about supplemented and related modules [17]. Supplemented modules are also discussed in [14]. Recently, several authors have studied different generalizations of supplemented modules. Rad-supplemented modules have been studied in [15] and [3]. See [15]; these modules are called generalized supplemented modules. For submodules U and V of a module M, V is said to be a rad-supplement of U in M or U is said to have a rad-supplement V in M if $U + V = M$ and $U \cap V \subseteq \mathrm{rad}(V)$. M is called a rad-supplemented module if every submodule of M has a rad-supplement in M. We shall say that a module M is w-supplemented if every semisimple submodule of M has a supplement in M [1]. We say that V is an ss-supplement U in M if $M = U + V$ and $U \cap V \ll V$ and $V \cap U$ is semisimple. We call a module M is ss-supplemented if every submodule of M has an ss-supplement in M [12]. Recently, the study of the supplemented property in the rings, modules, and lattices has become quite popular (see for example [2, 3, 4, 10, 11, 12, 13]. Supplemented property (resp. w-supplemented property) in the lattices have already been investigated in [7] (resp. [6]). This paper is based on another variation of supplemented filters. In fact, in the present paper, we are interested in investigating strongly local filters and (amply) ss-supplemented filters in a distributive lattice with 1 to use other notions of ss-supplemented, and associate which exist in the literature as laid forth in [12] (see Sections 2, 3, 4).

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Let us briefly review some definitions and tools that will be used later [2]. By a lattice we mean a poset $(L, \leqslant)$ in which every couple elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$). A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A non-empty subset F of a lattice L is called a filter, if for $a \in F$, $b \in L$, $a \leqslant b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and $\{1\}$ is a filter of L). A proper filter P of L is said to be maximal if E is a filter in L with $P \subsetneqq E$, then $E = L$. If F is a filter of a lattice L, then the radical of F, denoted by $\mathrm{rad}(F)$, is the intersection of all maximal subfilters of F.

Let L be a lattice. If A is a subset of L, then the filter generated by A, denoted by $T(A)$, is the intersection of all filters that is containing A. A filter F is called finitely generated if there is a finite subset A of F such that $F = T(A)$. A subfilter G of a filter F of L is called *small* in F, written $G \ll F$, if, for every subfilter H of F, the equality $T(G \cup H) = F$ implies $H = F$ [7]. A subfilter G of F is called *essential* in F, written $G \trianglelefteq F$, if $G \cap H \neq \{1\}$ for any subfilter $H \neq \{1\}$ of F. Let G be a subfilter of a filter F of L. A subfilter H of F is called a *supplement* of G in F if $F = T(G \cup H)$ and H is minimal with respect to this property, or equivalently, $F = T(G \cup H)$ and $G \cap H \ll H$. H is said to be a supplement subfilter of F if H is a supplement of some subfilter of F. F is called a *supplemented* filter if every subfilter of F has a supplemented in F. A subfilter G of a filter F of L has *ample supplements* in F if, for every subfilter H of F with $F = T(H \cup G)$, there is a supplement H' of G with $H' \subseteq H$. If every subfilter of a filter F has ample supplements in F, then we call F *amply supplemented.* Let G, H be subfilters of a filter F of L. If $F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{rad}(H)$, then H is called a *rad-supplement* of G in F. If every subfilter of F has a rad-supplement in F, then F is called a *rad-supplemented filter*.

A lattice L is called *semisimple*, if for each proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$). In this case, we say that F is a *direct summand* of L, and we write $L = F \oplus G$. A filter F of L is called a *semisimple* filter, if every subfilter of F is a direct summand. A *simple* lattice (resp. filter), is a lattice (resp. filter) that has no filters besides the $\{1\}$ and itself. For a filter F, $\mathrm{Soc}(F) = T(\cup_{i \in \Lambda} F_i)$, where $\{F_i\}_{i \in \Lambda}$ is the set of all simple filters of L contained in F. In [17], Zhou and Zhang generalized the concept of socle a module M to that of $\mathrm{Soc}_g(M)$ by considering of all simple submodules of M that are small in M in place of the class of all simple submodules of M, that is, $\mathrm{Soc}_g(M) = \sum\{N \ll M : N \text{ is simple}\}$. For a filter F, we define $\mathrm{Soc}_g(F) = T(\cup_{i \in \Lambda} F_i)$, where $\{F_i\}_{i \in \Lambda}$ is the set of all simple filters of L contained in F and $F_i \ll F$ for each $i \in \Lambda$. Clearly, $\mathrm{Soc}_g(F) \subseteq \mathrm{Soc}(F)$ and $\mathrm{Soc}_g(F) \subseteq \mathrm{rad}(F)$. Let F be a filter of a lattice L. F is called hollow if $F \neq \{1\}$ and every proper subfilter G of F is small in F. F is called local if it has exactly one maximal subfilter that contains all proper subfilters.

Proposition 1.1. (cf. [9], [8]) *A non-empty subset F of a lattice L is a filter if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.*

Proposition 1.2. (cf. [6]) *Let F be a filter of a distributive lattice L with* 1.

(1) *If $A \ll F$ and $C \subseteq A$, then $C \ll F$.*

(2) *If A, B are subfilters of F with $A \ll B$, then A is a small subfilter in subfilters of F that contains the subfilter of B. In particular, $A \ll F$.*

(3) $\mathrm{rad}(F) = T(\cup_{G \ll F} G)$.

(4) *Every finitely generated subfilter of* $\mathrm{rad}(F)$ *is small in* $\mathrm{rad}(F)$.

(5) $x \in \mathrm{rad}(F)$ *if and only if* $T(\{x\}) \ll F$.

(6) *If $F_1, F_2, \ldots, F_n$ are small subfilters of F, then $T(F_1 \cup F_2 \cup \cdots \cup F_n)$ is also small in F.*

Lemma 1.3. (cf. [6])

(1) *$T(A) = \{x \in L : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x$ for some $a_i \in A$ $(1 \leqslant i \leqslant n)\}$ an arbitrary non-empty subset A of L. Moreover, if F is a filter and A is a subset of L with $A \subseteq F$, then $T(A) \subseteq F$, $T(F) = F$ and $T(T(A)) = T(A)$.*

(2) *$T(T(A \cup B) \cup C) \subseteq T(A \cup T(B \cup C))$ for subfilters A, B, C of a filter F of L. In particular, $F = T(T(C \cup B) \cup A) = T(T(A \cup C) \cup B)$ for all $F = T(T(A \cup B) \cup C)$.*

(3) *(Modular law) $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$ for filters F_1, F_2, F_3 of L such that $F_2 \subseteq F_1$.*

Proposition 1.4. (cf. [11])

(a) *Let G be a semisimple subfilter of a filter F of L such that $G \subseteq \mathrm{rad}(F)$. Then $G \ll F$.*

(b) *Let H and G be subfilters of a filter F of L . Then the following hold:*

(1) *If H is semisimple, then $\frac{T(H \cup G)}{G}$ is a semisimple subfilter in $\frac{F}{G}$.*

(2) *If* $\mathrm{Soc}(F) = \cap_{K \trianglelefteq F} K$.

(3) $\mathrm{Soc}(G) = G \cap \mathrm{Soc}(F)$.

(c) *Let U, V be subfilters of a filter F of L such that V is a direct summand of F with $U \subseteq V$. Then $U \ll F$ if and only if $U \ll V$.*

2. Strongly Local Filters

Throughout this paper, we shall assume unless otherwise stated, that L is a distributive lattice with 1. In this section we collect some properties concerning strongly local filters of L. Our starting point is the following lemma.

Lemma 2.5. *Let F be a filter of L. Then the following hold:*
(1) *If E is a simple subfilter of F, then $E = T(\{a\})$ for some $1 \neq a \in E$.*
(2) *If $f_1, f_2, \ldots, f_n \in F$, then $T(T(\{f_1\}) \cup \ldots \cup T(\{f_n\})) = T(\{f_1, \ldots, f_n\})$.*
(3) *If F is semisimple, then F is a direct sum of a finite family of simple subfilters if and only if F is finitely generated.*

Proof. (1). Since E is simple, there is an element $1 \neq a \in E$ such that $T(\{a\}) \neq \{1\}$ is a subfilter of E; hence $E = T(\{a\})$.

(2). Since the inclusion $A = T(\{f_1, \ldots, f_n\}) \subseteq T(T(\{f_1\}) \cup \ldots \cup T(\{f_n\})) = B$ is clear we will prove the reverse inclusion. Let $x \in B$. Then $a_1 \wedge a_2 \wedge \cdots \wedge a_n \leqslant x$ for some $a_i \in T(\{f_i\})$ $(1 \leqslant i \leqslant n)$. By assumption, there exist $s_1, s_2, \ldots, s_n \in L$ such that $a_i = f_i \vee s_i$ $(1 \leqslant i \leqslant n)$. Then $(f_1 \vee s_1) \wedge \ldots \wedge (f_n \vee s_n) \leqslant x$. Since for each i, $f_i \leqslant f_i \vee s_i$ and $f_i \in A$, we get $f_i \vee s_i \in A$ $(1 \leqslant i \leqslant n)$; so $x \in A$, and so we have equality.

(3). Let $F = F_1 \oplus \cdots \oplus F_n$, where for each i $(1 \leqslant i \leqslant n)$, F_i is a simple subfilter of F, so by (1), $F_i = T(\{f_i\})$ for some $1 \neq f_i \in F_i$. Then by (2), $F = T(T(\{f_1\}) \cup \cdots \cup T(\{f_n\})) = T(\{f_1, \ldots, f_n\})$. Thus F is finitely generated. Conversely, suppose that $F = T(A)$, where $A = \{e_1, \ldots, e_m\}$. As F is semisimple, we can write $F = T(\cup_{i \in I} F_i)$, where for each $i \in I$, F_i is simple. We can now pick out a finite collection $i_1, i_2, \ldots, i_r$ of elements of I such that $e_i \in T(F_{i_1} \cup \cdots \cup F_{i_r})$ for $1 \leqslant i \leqslant m$. But then $F = T(F_{i_1} \cup \cdots \cup F_{i_r})$, that is, $F = F_{i_1} \oplus \cdots \oplus F_{i_r}$. □

Proposition 2.6. *If F is a filter of L, then $\mathrm{Soc}_g(F) = \mathrm{rad}(F) \cap \mathrm{Soc}(F)$.*

Proof. It suffices to show that $\mathrm{rad}(F) \cap \mathrm{Soc}(F) \subseteq \mathrm{Soc}_g(F)$. Let $a \in \mathrm{rad}(F) \cap \mathrm{Soc}(F)$. Then $T(\{a\})$ is semisimple and so there exist simple subfilters F_i of F such that $T(\{a\}) = F_1 \oplus \cdots \oplus F_n$ by Lemma 2.5 (3). By Proposition 1.2 (5), $T(\{a\}) \ll \mathrm{rad}(F)$; hence it is small in F by Proposition 1.2 (2). Since for each i, $F_i \subseteq T(\{a\})$, we get $F_i \ll F$ by Proposition 1.2 (1). Thus $a \in T(\{a\}) \subseteq \mathrm{Soc}_g(F)$, and so we have equality. □

A filter F is called *indecomposable* if $F \neq \{1\}$ and $F = T(G \cup H)$ with $G \cap H = \{1\}$, then either $G = \{1\}$ or $H = \{1\}$.

Lemma 2.7. *Let F be an indecomposable filter of L. Then F is either simple or* $\mathrm{Soc}(F) \subseteq \mathrm{rad}(F)$.

Proof. If F is simple, we are done. Thus we may assume that F is not simple. It suffices to show that $\mathrm{Soc}(F) \ll F$ by Proposition 1.2 (3). Let $F = T(K \cup \mathrm{Soc}(F))$ for some subfilter K of F. By assumption, there is a semisimple subfilter H of $\mathrm{Soc}(F)$ such that $\mathrm{Soc}(F) = (\mathrm{Soc}(F) \cap K) \oplus H$, and so by Lemma 1.3 (2), $F = T(K \cup T(H \cup (\mathrm{Soc}(F) \cap K))) = T(K \cup H)$and $K \cap H = H \cap (\mathrm{Soc}(F) \cap K) = \{1\}$. Since F is indecomposable and not simple, we get $H = \{1\}$; hence $F = K$. Thus $\mathrm{Soc}(F) \ll F$, as required. □

By [6, Remark 2.19 (2)], every local filter is hollow and by [6, Remark 2.19 (1)], every hollow filter is indecomposable. Using Proposition 2.6 and Lemma 2.7 we have the following Corollary:

Corollary 2.8. *Let F be a local filter of L such that it is not simple. Then* $\mathrm{Soc}_g(F) = \mathrm{Soc}(F)$.

Definition 2.9. A filter F of L is called strongly local if it is local and $\mathrm{rad}(F)$ is semisimple. A filter F of L is called radical if F has no maximal subfilters, that is, $F = \mathrm{rad}(F)$.

Assume that F is a filter of L and let $P(F)$ be the filter generated by $\cup_{G\subseteq F} G$, where for each subfilter G of F, $G = \mathrm{rad}(G)$, that is, $P(F) = T(\cup_{G\subseteq F} G)$, where $G = \mathrm{rad}(G)$. It is easy to see that $P(F) \subseteq \mathrm{rad}(F)$.

Lemma 2.10. *If F is a filter of L, then $P(F)$ is the largest radical subfilter of F.*

Proof. It suffices to show that $P(F) \subseteq \mathrm{rad}(P(F))$. If $x \in P(F)$, then there exist radical subfilters $G_1, ..., G_n$ of F and $g_1 \in G_1,.., g_n \in G_n$ such that $g_1 \wedge \cdots \wedge g_n \leqslant x$. Since $g_1 \in G_1 = \mathrm{rad}(G_1), \ldots, g_n \in G_n = \mathrm{rad}(G_n)$, by Proposition 1.2 (5) we have $T(\{g_i\}) \ll G_i$, for each $1 \leqslant i \leqslant n$. By Proposition 1.2 (2), $T(\{g_i\}) \ll P(F)$, for each $1 \leqslant i \leqslant n$. Therefore $g_i \in \mathrm{rad}(P(F))$, for each $1 \leqslant i \leqslant n$. This implies that $x \in \mathrm{rad}(P(F))$. □

Proposition 2.11. *If a filter F of L is strongly local, then F is reduced (that is, $P(F) = \{1\}$).*

Proof. Since $\mathrm{rad}(F)$ is semisimple and $P(F) \subseteq \mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$, we get $P(F)$ is semisimple and so $P(F) = \mathrm{rad}(P(F)) = \{1\}$ by [6, Proposition 2.16 (2)] and Lemma 2.10, as required. □

Example 2.12. The collection of ideals of Z, the ring of integers, form a lattice under set inclusion which we shall denote by $L(Z)$ with respect to the following definitions: $mZ \vee nZ = (m,n)Z$ and $mZ \wedge nZ = [m,n]Z$ for all ideals mZ and nZ of Z, where (m,n) and $[m,n]$ are greatest common divisor and least common multiple of m,n, respectively. Note that $L(Z)$ is a distributive complete lattice with least element the zero ideal and the greatest element Z. Then by [7, Proposition 2.1 (iii) and Theorem 3.1 (ii)], every simple filter of $L(Z)$ is of the form $F = \{Z, pZ\}$ for some prime number p. Let $\mathbf{P}$ be the set of all prime numbers. For each $p \in \mathbf{P}$, set $F_p = \{Z, pZ\}$. Then $\{F_p\}_{p\in\mathbf{P}}$ is the set of all simple filters of $L(Z)$. Moreover, by [7, Lemma 3.1], $\mathbf{m} = L(Z) \setminus \{0\}$ is the only maximal filter of $L(Z)$; so $L(Z)$ is a local filter of $L(Z)$ (so it is hollow). If G is a proper subfilter of $L(Z)$ with $G \neq \mathrm{rad}(G)$, then G has a maximal subfilter, say H. There exists $x \in G$ such that $x \notin H$; hence $T(H \cup T(\{x\})) = G$. By [6, Remark 2.19 (4)], G has a supplement K in $L(Z)$; so by Lemma 1.3,

$$L(Z) = T(T(H \cup T(\{x\})) \cup K) = T(H \cup T(K \cup T(\{x\})));$$

hence $L(Z) = H$ which is impossible since $T(K \cup T(\{x\})) \ll L(Z)$. Thus $P(L(Z)) = \mathbf{m} \neq \{1\}$. If $L(Z) = T(\cup_{p\in \mathbf{P}} F_p)$, then $\{0\} = p_{i_1}Z \wedge \cdots \wedge p_{i_k}Z = p_{i_1}\cdots p_{i_k}Z$, a contradiction. So $L(Z)$ is not semisimple. Similarly, $\mathrm{rad}(L(Z)) = \mathbf{m}$ is not semisimple. Therefore the condition "strongly" in the Proposition 2.11 is necessary.

3. SS-supplemented Filters

In this section, the basic properties and possible structures of ss-supplemented filters are investigated. Our starting point is the following lemma.

Lemma 3.1. *Let G and H be subfilters of a filter F of L. If G is a maximal subfilter of F, then H is a supplement of G in F if and only if $F = T(G \cup H)$ and H is local.*

Proof. Let H be a supplement of G in F. By [6, Theorem 2.9 (4)], H is cyclic, and $G \cap H = \mathrm{rad}(H)$ is a (the unique) maximal subfilter of H; so H is local. Conversely, let H be local (so it is hollow) and $F = T(G \cup H)$. If $H \cap G = H$, then $F = G$ which is impossible. Thus $H \cap G \neq H$. Now H is hollow gives $H \cap G \ll H$. Thus H is a supplement of G in F. □

Definition 3.2. Let G be any subfilter of a filter F of L. We say that H is an *ss-supplement* G in F if $F = T(G \cup H)$ and $G \cap H \ll H$ and $G \cap H$ is semisimple. We call a filter F *ss-supplemented* if every its subfilter has an ss-supplement in F.

A subfilter G of F has *ample ss-supplements* in F if every subfilter K of F such that $F = T(K \cup G)$ contains an ss-supplement of G in F. We call a filter F *amply ss-supplemented* if every subfilter of F has ample ss-supplements in F.

We next give two other characterizations of ss-supplements filters.

Proposition 3.3. *Let G, H be subfilters of a filter F of L. Then the following statements are equivalent:*

(1) *$F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{Soc}_g(H)$;*
(2) *$F = T(G \cup H)$ and $G \cap H \subseteq \mathrm{rad}(H)$ and $G \cap H$ is semisimple;*
(3) *$F = T(G \cup H)$ and $G \cap H \ll H$ and $G \cap H$ is semisimple.*

Proof. (1) ⇒ (2). By (1) and Proposition 2.6, $G \cap H$ is semisimple and $G \cap H \subseteq \mathrm{rad}(H) \cap \mathrm{Soc}(H) \subseteq \mathrm{rad}(H)$.

(2) ⇒ (3). It is clear by (2) and Proposition 1.4 (a).

(3) ⇒ (1). It is clear by (3) and Proposition 2.6. □

Analogous to that Lemma 3.1 we have the following proposition:

Proposition 3.4. *Let G and H be subfilters of a filter F of L. If G is a maximal subfilter of F, then H is a ss-supplement of G in F if and only if $F = T(G \cup H)$ and H is strongly local.*

Proof. Let H be an ss-supplement of G in F. By [6, Theorem 2.9 (4)], H is local with the unique maximal subfilter $G \cap H = \mathrm{rad}(H)$; so H is strongly local since $G \cap H$ is semisimple. Conversely, since H is local and $F = T(G \cup H)$, we can write $G \cap H \subseteq \mathrm{rad}(H)$. Now $\mathrm{rad}(H)$ is semisimple gives $G \cap H$ is semisimple. Hence, H is an ss-supplement of G in F. □

Lemma 3.5. *Let G be a subfilter of a ss-supplemented filter F of L. If $G \ll F$, then $G \subseteq \mathrm{Soc}_g(F)$. In particular, if* $\mathrm{rad}(F) \ll F$, *then* $\mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$.

Proof. Let H be an ss-supplement of G in F. Then $F = T(G \cup H)$ and $G \ll F$ gives $H = F$ and $G = G \cap H$ is semisimple; so $G \subseteq \mathrm{rad}(F) \cap \mathrm{Soc}(F) = \mathrm{Soc}_g(F)$ by Proposition 2.6. The in particular statement is clear. □

Let F be a local filter of L (so it is hollow). It is easy to see that F has no supplement subfilter except for $\{1\}$ and F. Thus every local filter is amply supplemented. Analogous to that we have:

Proposition 3.6. *Every strongly local filter of L is amply ss-supplemented.*

Proof. Let F be a strongly local filter (so $\mathrm{rad}(F)$ is semisimple). Then F is local and so it is amply supplemented. If G is a proper subfilter of F, then $F = T(F \cup G)$ and $G = G \cap F \ll F$; so $G \subseteq \mathrm{rad}(F)$; hence G is semisimple. Thus F is amply ss-supplemented. □

Proposition 3.7. *If F is a hollow filter of L, then F is* (*amply*) *ss-supplemented if and only if it is strongly local.*

Proof. Assume that F is ss-supplemented and let $x \in \mathrm{rad}(F)$. By Proposition 1.2 (5), $T(\{x\}) \ll \mathrm{rad}(F)$, and so it is small in F by Proposition 1.2 (2). As F is ss-supplemented, it follows from Lemma 3.5 that $x \in T(\{x\}) \subseteq \mathrm{Soc}_g(F) = \mathrm{rad}(F) \cap \mathrm{Soc}(F)$; hence $x \in \mathrm{Soc}(F)$, and so $\mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$. Suppose that $F = \mathrm{rad}(F)$. Then $F = \mathrm{rad}(F) = \mathrm{Soc}(F)$, and so F is semisimple. Thus $F = \{1\}$ by [6, Proposition 2.16 (2)]. This is a contradiction because F is hollow. So we may assume that $F \neq \mathrm{rad}(F)$, that is, F is local by [6, Theorem 2.21]. Hence F is strongly local. The other implication follows from Proposition 3.6. □

The following example shows in general a (amply) supplemented filter need not be (amply) ss-supplemented.

Example 3.8. Assume that R is a local Dedekind domain with unique maximal ideal $P = Rp$ and let $E = E(R/P)$, the R-injective hull of R/P. For each positive integer n, set $A_n = (0 :_E P^n)$. Then by [9, Lemma 2.6], every non-zero proper submodule of E is equal to A_m for some m with a strictly increasing sequence of submodules $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$. The collection of submodules of E form a complete lattice which is a chain under set inclusion which we shall denote by $L(E)$ with respect to the following definitions: $A_n \vee A_m = A_n + A_m$ and $A_n \wedge A_m = A_n \cap A_m$ for all submodules A_n and A_m of E. Then by [7, Example 2.3

(b)], every proper filter of $L(E)$ is of the form $[A_n, E]$ for some n. Clearly, $L(E)$ is a hollow filter which is not local. As hollow filters are (amply) supplemented, $L(E)$ is (amply) supplemented. However, $L(E)$ is not (amply) ss-supplemented filter by Proposition 3.7.

Theorem 3.9. *If F is a filter of L with* $\mathrm{rad}(F) \ll F$*, then the following statements are equivalent:*

(1) *F is ss-supplemented;*
(2) *F is supplemented and* $\mathrm{rad}(F)$ *has an ss-supplement in F;*
(3) *F is supplemented and* $\mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$*.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (3). Since $\mathrm{rad}(F) \ll F$ and $\mathrm{rad}(F)$ has ss-supplement in F, we get F is a supplement of $\mathrm{rad}(F)$; hence $\mathrm{rad}(F) = \mathrm{rad}(F) \cap F$ is semisimple. Thus $\mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$.

(3) $\Rightarrow$ (1). Let G be a subfilter of F. By assumption, there exists a subfilter H of F such that $F = T(G \cup H)$ and $G \cap H \ll H$. Then $G \cap H \subseteq \mathrm{rad}(H) \subseteq \mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$; so $G \cap H$ is semisimple. It means that F is ss-supplemented. □

Corollary 3.10. *If F is a finitely generated filter of L, then F is ss-supplemented if and only if it is supplemented and* $\mathrm{rad}(F) \subseteq \mathrm{Soc}(F)$*.*

Proof. By Theorem 3.9, it suffices to show that $\mathrm{rad}(F) \ll F$. Assume that $F = T(A)$, where $A = \{a_1, \ldots, a_n\}$ and let $F = T(H \cup \mathrm{rad}(F))$ for some subfilter H of F. Since $\mathrm{rad}(F) = T(\cup_{G \ll F} G)$, there exists a finite subfilters $F_{i_1} \ll F$, $F_{i_2} \ll F$, $\ldots$, $F_{i_r} \ll F$ such that $a_i \in T(T(F_{i_1} \cup \cdots \cup F_{i_r}) \cup H)$ for $1 \leqslant i \leqslant r$ which implies that $F = T(T(F_{i_1} \cup \cdots \cup F_{i_r}) \cup H)$; hence $H = F$ by Proposition 1.2 (6).. □

Lemma 3.11. *If K and H are semisimple filters of L, then $T(K \cup H)$ is semisimple.*

Proof. Let G be a subfilter of $T(K \cup H)$. There exist a subfilter K' of K and a subfilter H' of H such that $K = (G \cap K) \oplus K'$ (so $K' \cap G = \{1\}$) and $H = (H \cap G) \oplus H'$ (so $H' \cap G = \{1\}$). If $x \in G \cap T(K' \cup H')$, then $a \wedge b \leqslant x$ for some $a \in K'$ and $b \in H'$; so $x = (x \vee a) \wedge (x \vee b) = 1$. Thus $G \cap T(K' \cup H') = \{1\}$. It enough to show that $T(H \cup K) = T(G \cup T(K' \cup H'))$. Since the inclusion $T(G \cup T(K' \cup H')) \subseteq T(K \cup H)$ is clear, we will prove the reverse inclusion. Let $z \in T(K \cup H)$. Then $c \wedge d \leqslant z$ for some $c \in K = T((G \cap K) \cup K')$ and $d \in H = T((H \cap G) \cup H')$. It follows that there are elements $c_1 \in G \cap K$, $c_2 \in K'$, $d_1 \in G \cap H$ and $d_2 \in H'$ such that $c_1 \wedge c_2 \leqslant c$ and $d_1 \wedge d_2 \leqslant d$; hence $(c_1 \wedge d_1) \wedge (c_2 \wedge d_2) \leqslant z$, where $c_1 \wedge d_1 \in G$ and $c_2 \wedge d_2 \in T(H' \cup K')$ which implies that $z \in T(G \cup T(K' \cup H'))$, and so we have equality. Thus $T(K \cup H) = G \oplus T(K' \cup H')$. □

Proposition 3.12. *Let F_1 and G be subfilters of a filter F of L with F_1 ss-supplemented. If there is a ss-supplement for $T(F_1 \cup G)$ in F, then the same is true for G.*

Proof. Let X be an ss-supplement of $T(F_1 \cup G)$ in F and Y is an ss-supplement $T(X \cup G) \cap F_1$ in F_1. Then by an argument like that in [6, Proposition 2.10], we get $F = T(G \cup T(X \cup Y))$ and $T(X \cup Y) \cap G \ll T(X \cup Y)$. Moreover, $A = X \cap T(Y \cup G)$ is semisimple as a subfilter of the semisimple filter $X \cap T(F_1 \cup G)$. Also, $Y \cap (F_1 \cap T(X \cup G)) = Y \cap T(X \cup G) = B$ is semisimple; so $T(A \cup B)$ is semisimple by Lemma 3.11. Since $T(A \cup B) = G \cap T(X \cup Y)$, we get $T(X \cup Y)$ is an ss-supplement of G in F. □

Theorem 3.13. *Let F_1 and F_2 be subfilters of F such that $F = T(F_1 \cup F_2)$. If F_1 and F_2 are ss-supplemented, then F is ss-supplemented.*

Proof. Let G be a subfilter of F. The subfilter $\{1\}$ is ss-supplement of $F = T(F_1 \cup T(F_2 \cup G))$ in F. Since F_1 is ss-supplemented, $T(F_2 \cup G)$ has an ss-supplement in F by Proposition 3.12. Again applying Proposition 3.12, G has an ss-supplement in F. This completes the proof. □

Corollary 3.14. *If $F_1, \dots, F_n$ are ss-supplemented filters of L, then $T(U_{i=1}^n F_i)$ is an ss-supplemented filter.*

Proof. Apply Theorem 3.13. □

Lemma 3.15. *Let F be a filter of L. If every subfilter of F is ss-supplemented, then F is amply ss-supplemented.*

Proof. Let G and H be subfilters of F such that $F = T(G \cup H)$. By the assumption, $H = T((G \cap H) \cup H')$, $(G \cap H) \cap H' = G \cap H' \ll H'$ and $G \cap H'$ is semisimple for some subfilter H' of H. Since $F = T(G \cup T((G \cap H) \cup H')) = T(G \cup H')$, we get G has ample ss-supplements in F. Thus F is amply ss-supplemented. □

Lemma 3.16. *Assume that F is a amply ss-supplemented filter of L and let H be an ss-supplement subfilter in F. Then H is amply ss-supplemented.*

Proof. Let H be an ss-supplement of a subfilter G of F. Let X and Y be subfilters of H such that $H = T(X \cup Y)$. Then

$$F = T(H \cup G) = T(G \cup T(X \cup Y)) = T(Y \cup T(G \cup X)).$$

As F is amply ss-supplemented, $T(X \cup G)$ has an ss-supplement $Y' \subseteq Y$ in F; so $F = T(Y' \cup T(X \cup G)) = T(G \cup T(X \cup Y'))$. Since $X \cup Y' \subseteq X \cup Y$, we obtain $T(X \cup Y') \subseteq T(X \cup Y) = H$. Then H is an ss-supplement of G in F gives $H = T(X \cup Y')$ by minimality of H. Moreover, $X \cap Y' \subseteq T(G \cup X) \cap Y' \ll Y'$, and so $X \cap Y' \ll Y'$ by Proposition 1.2 (1). As $T(G \cup X) \cap Y'$ is semisimple, $X \cap Y' \subseteq T(G \cup X) \cap Y'$ is semisimple. Thus H is amply ss-supplemented. □

The next theorem gives a more explicit description of amply ss-supplemented filters.

Theorem 3.17. *For a filter F of L, the following statements are equivalent:*
(1) *F is amply ss-supplemented;*
(2) *Every subfilter G of F is of the form $G = T(X \cup Y)$, where X is ss-supplemented and $Y \subseteq \mathrm{Soc}_g(F)$.*

Proof. (1) $\Rightarrow$ (2). Assume that F is amply ss-supplemented and let G be a subfilter of F. Since F is ss-supplemented, G has an ss-supplements H in F; so $F = T(H \cup G)$. By the assumption, there exists a subfilter X of G such that X is an ss-supplement of H in F; so $F = T(X \cup H)$. Set $Y = G \cap H$. Since H is an ss-supplement of G in F, we have $Y = G \cap H \subseteq \mathrm{Soc}_g(H) \subseteq \mathrm{Soc}(F)$ by Proposition 3.3. By the modular law, $G = G \cap T(X \cup H) = T(X \cup (G \cap H)) = T(X \cup Y)$, where $Y \subseteq \mathrm{Soc}_g(F)$ and X is ss-supplemented by Lemma 3.16.

(2) $\Rightarrow$ (1). By the assumption, if G is a subfilter of F, then $G = T(X \cup Y)$ with X is ss-supplemented and $Y \subseteq \mathrm{Soc}_g(F) \subseteq \mathrm{Soc}(F)$ (so Y is ss-supplemented). By Theorem 3.13, G is ss-supplemented. Therefore F is amply ss-supplemented by Lemma 3.15. □

Corollary 3.18. *For a filter F of L, the following statements are equivalent:*
(1) *F is amply ss-supplemented;*
(2) *Every subfilter of F is ss-supplemented;*
(3) *Every subfilter of F is amply ss-supplemented.*

Proof. Apply Theorem 3.17. □

4. SS-supplemented Quotient Filters

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(L, \leqslant)$, we define a relation on L, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on L, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by $\frac{L}{F}$. We set up a partial order $\leqslant_Q$ on $\frac{L}{F}$ as follows: for each $a \wedge F, b \wedge F \in \frac{L}{F}$, we write $a \wedge F \leqslant_Q b \wedge F$ if and only if $a \leqslant b$. It is straightforward to check that $(\frac{L}{F}, \leqslant_Q)$ is a poset. The following notation below will be kept in this section: Let $a \wedge F, b \wedge F \in \frac{L}{F}$ and set $X = \{a \wedge F, b \wedge F\}$. By definition of $\leqslant_Q$, $(a \vee b) \wedge F$ is an upper bound for the set X. If $c \wedge F$ is any upper bound of X, then we can easily show that $(a \vee b) \wedge F \leqslant_Q c \wedge F$. Thus $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$. Similarly, $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$. Thus $(\frac{L}{F}, \leqslant_Q)$ is a lattice.

Remark 4.1. Let G be a subfilter of a filter F of L.
(1). If $a \in F$, then $a \wedge F = F$. By the definition of $\leqslant_Q$, it is easy to see that $1 \wedge F = F$ is the greatest element of $\frac{L}{F}$.
(2). If $a \in F$, then $a \wedge F = b \wedge F$ (for every $b \in L$) if and only if $b \in F$. In particular, $c \wedge F = F$ if and only if $c \in F$. Also, if $a \in F$, then $a \wedge F = F = 1 \wedge F$.
(3). By the definition $\leqslant_Q$, we can easily show that if L is distributive, then $\frac{L}{F}$ is

distributive.
(4). $\frac{F}{G} = \{a \wedge G : a \in F\}$ is a filter of $\frac{L}{G}$.
(5). If K is a filter of $\frac{L}{G}$, then $K = \frac{F}{G}$ for some filter F of L.
(6). If H is a filter of L such that $G \subseteq H$ and $\frac{F}{G} = \frac{H}{G}$, then $F = H$.
(7). If H and V are filters of L containing G, then $\frac{F}{G} \cap \frac{H}{G} = \frac{V}{G}$ if and only if $V = H \cap F$.
(8). If H is a filter of L containing G, then $\frac{T(F \cup H)}{G} = T(\frac{H}{G} \cup \frac{F}{G})$.

Proposition 4.2. *Every quotient filter of a strongly local filter of L is strongly local.*

Proof. Let G be a subfilter of a strongly local filter F of L. Clearly, if H is a subfilter of F with $G \subseteq H$, then H is a maximal subfilter of F if and only if $\frac{H}{G}$ is a maximal subfilter of $\frac{F}{G}$; so the quotient filter $\frac{F}{G}$ is local. By assumption, $\mathrm{rad}(\frac{F}{G}) = \frac{\mathrm{rad}(F)}{G} \subseteq \frac{\mathrm{Soc}(F)}{G} = \frac{\cap_{K \trianglelefteq F} K}{G} \subseteq \cap_{\frac{K}{G} \trianglelefteq \frac{F}{G}} \frac{K}{G} \subseteq \mathrm{Soc}(\frac{F}{G})$; so $\mathrm{rad}(\frac{F}{G})$ is semisimple. Thus $\frac{F}{G}$ is strongly local. □

Lemma 4.3. *Let G, H, K be filters of L such that $H \ll K$. Then $\frac{T(H \cup G)}{G} \ll \frac{T(K \cup G)}{G}$.*

Proof. Let $\frac{T(K \cup G)}{G} = T(\frac{U}{G} \cup \frac{T(H \cup G)}{G}) = \frac{T(U \cup T(H \cup G))}{G}$ for some subfilter $\frac{U}{G}$ of $\frac{T(K \cup G)}{G}$ (so $U \subseteq T(K \cup G)$); hence $T(K \cup G) = T(U \cup H)$. As $K = K \cap T(U \cup H) = T(H \cup (U \cap K))$, we get $U \cap K = K$ since $H \ll K$. It follows that $T(K \cup G) \subseteq U$, and so $\frac{T(K \cup G)}{G} = \frac{U}{G}$. □

Theorem 4.4. *If F is an ss-supplemented filter, then every quotient filter of F is ss-supplemented.*

Proof. Assume that F is an ss-supplemented filter and let $\frac{F}{G}$ be a quotient filter of F. Let $\frac{H}{G}$ be a subfilter of $\frac{F}{G}$. By the assumption, there exists a subfilter K of F such that $F = T(K \cup H)$, $K \cap H \ll H$ and $H \cap K$ is semisimple. Then $\frac{F}{G} = T(\frac{H}{G} \cup \frac{T(K \cup G)}{G})$ and

$$\frac{H}{G} \cap \frac{T(K \cup G)}{G} = \frac{H \cap T(K \cup G)}{G} = \frac{T((H \cap K) \cup G)}{G} \ll \frac{T(K \cup G)}{G}$$

by Lemma 4.3 and Lemma 1.3. Since $H \cap K$ is semisimple, it follows from Proposition 1.4 that $\frac{H}{G} \cap \frac{T(K \cup G)}{G} = \frac{T((H \cap K) \cup G)}{G}$ is semisimple; so $\frac{T(K \cup G)}{G}$ is an ss-supplement of $\frac{H}{G}$ in $\frac{F}{G}$. This completes the proof. □

Corollary 4.5. *If F is an amply ss-supplemented filter of L, then every quotient filter of F is amply ss-supplemented.*

Proof. Let $\frac{V}{X}$ be a subfilter of $\frac{F}{X}$ such that $\frac{F}{X} = T(\frac{V}{X} \cup \frac{U}{X})$ for some subfilter $\frac{U}{X}$ of $\frac{F}{X}$; so $F = T(V \cup U)$. Since F is amply ss-supplemented, there is a subfilter $H \subseteq U$ such that H is a ss-supplement of V in F. By a similar argument like that in Theorem 4.4, $\frac{T(H\cup X)}{X} \subseteq \frac{U}{X}$ is a ss-supplement $\frac{V}{X}$ in $\frac{F}{X}$. Thus $\frac{F}{X}$ is amply ss-supplemented. □

Lemma 4.6. *Let G and H be subfilters of a filter F of L such that $F = T(G \cup H)$. If K is a proper subfilter of F such that $G \subsetneqq K$, then $K \cap H$ is a proper subfilter of H.*

Proof. If $H \subseteq K$, then $F = T(G \cup H)$ gives $F = K$, a contradiction. Thus $H \nsubseteq K$ and $K \cap H \neq H$. By the relations, $K = K \cap T(G \cup H) = T(G \cup (H \cap K))$ and $K \neq G$, we obtain $K \cap H \neq \{1\}$. Therefore, $K \cap H$ is a proper subfilter of H. □

Lemma 4.7. *Let G and H be proper subfilters of a filter F of L. If $F = T(G \cup H)$ and H is simple, then G is a maximal subfilter of F.*

Proof. If K is a subfilter of F such that $G \subsetneqq K \subsetneqq F$, then $K \cap H$ is a proper subfilter of H by Lemma 4.6 which is impossible since H is simple. This completes the proof. □

Proposition 4.8. *Let G and H be subfilters of a filter F of L. Assume H to be a supplement of G in F. Then the following hold:*

(1). *If K is a maximal subfilter of H, then $T(G \cup K)$ is a maximal subfilter of F. In this case, $K = T(G \cup K) \cap H$.*

(2). *If $\mathrm{rad}(F) \ll F$, then G is contained in a maximal subfilter of F.*

Proof. (1). Since K is a maximal subfilter of H, we find $K \neq H$. Since H is a supplement of G in F, we get $F \neq T(G \cup K)$. As $G \cap H \ll H$ and K is a maximal subfilter of H, we conclude that $H \cap G \subseteq K$; hence $K = T(K \cup (G \cap H)) = H \cap T(G \cup K)$. Since $\frac{H}{K}$ is simple and $\frac{F}{K} = T(\frac{H}{K} \cup \frac{T(G\cup K)}{K})$, we conclude that $\frac{T(G\cup K)}{K}$ is a maximal filter of $\frac{F}{K}$ by Lemma 4.7 which implies that $T(G \cup K)$ is a maximal subfilter of F which contains G.

(2). If $G \subseteq \mathrm{rad}(F)$, then the assertion is clear. If $G \nsubseteq \mathrm{rad}(F)$, then by [6, Theorem 2.9 (3)], $\mathrm{rad}(H) = H \cap \mathrm{rad}(F) \neq H$, i.e. there is a maximal subfilter K of H. Now the assertion follows from (1). □

Definition 4.9. Let F be a filter of L. F is called the *internal direct sum* of the set $\{F_i : i \in I\}$ of subfilters of F: $F = \oplus_{i\in I} F_i$ if and only if $F = T(\cup_{i\in I} F_i)$ and for each $j \in I$, $F_j \cap T(\cup_{i\in I_{i\neq j}} F_i) = \{1\}$.

Lemma 4.10. *If $\{F_i\}_{i\in I}$ is an indexed set of subfilters of a filter F of L with $F = \oplus_{i\in I} F_i$, then $\mathrm{rad}(F) = \oplus_{i\in I}\mathrm{rad}(F_i)$ and $\mathrm{Soc}(F) = \oplus_{i\in I}\mathrm{Soc}(F_i)$.*

Proof. By the assumption, for each $i \in I$, $\mathrm{rad}(F_i) = F_i \cap \mathrm{rad}(F)$ by [6, Theorem 2.9 (3)]. It suffices to show that $\mathrm{rad}(F) \subseteq \oplus_{i\in I}\mathrm{rad}(F_i)$. Let $x \in \mathrm{rad}(F)$. Then $(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}) \leqslant x$, where $x_{i_1} \in F_{i_1} \subseteq \mathrm{rad}(F), \ldots, x_{i_k} \in F_{i_k} \subseteq \mathrm{rad}(F)$. Therefore, $F = \oplus_{i\in I}F_i$ gives there exist subfilters $F_{t_1}, \cdots, F_{t_s}$ of F such that $x_{i_1} \in F_{t_1} \cap \mathrm{rad}(F) = \mathrm{rad}(F_{t_1}), \ldots, x_{i_k} \in F_{t_s} \cap \mathrm{rad}(F) = \mathrm{rad}(F_{t_s})$; so $x \in T(\mathrm{rad}(F_{t_1}) \cup \cdots \cup \mathrm{rad}(F_{t_s})) \subseteq \oplus_{i\in I}\mathrm{rad}(F_i)$, and so we have equality. Since the inclusion $\oplus_{i\in I}\mathrm{Soc}(F_i) \subseteq \mathrm{Soc}(F)$ is clear, we will prove the reverse inclusion. Let $z \in \mathrm{Soc}(F)$. Then

$$z = (z \vee a_1) \wedge \cdots \wedge (z \vee a_n)$$

for some $a_1 \in F_{j_1} \subseteq F, \ldots, a_n \in F_{j_n} \subseteq F$; hence $z \vee a_1 \in F_{j_1} \cap \mathrm{Soc}(F) = \mathrm{Soc}(F_{j_1}), \ldots, z \vee a_n \in F_{j_n} \cap \mathrm{Soc}(F) = \mathrm{Soc}(F_{j_1})$. It follows that $z \in T(\mathrm{Soc}(F_{j_1}) \cup \cdots \cup \mathrm{Soc}(F_{j_n})) \subseteq \oplus_{i\in I}\mathrm{Soc}(F_i)$. This completes the proof. □

Let L, L' be two lattice. Then a lattice homomorphism $f : L \to L'$ is a map from L to L' satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$. A bijective lattice homomorphism f is called a lattice isomorphism (in this case we write $L \cong L'$).

Lemma 4.11. *If A and B are filters of L, then $\frac{T(A\cup B)}{A} \cong \frac{B}{A\cap B}$.*

Proof. Define $f : \frac{B}{A\cap B} \to \frac{T(A\cup B)}{A}$ by $f(b \wedge (A \cap B)) = b \wedge A$. It is clear that f is well-defined. We will show f is one-to-one: Let $f(b_1 \wedge (A \cap B)) = f(b_2 \wedge (A \cap B))$, where $b_1, b_2 \in B$. Then $b_1 \wedge A = b_2 \wedge A$; and so $b_1 \wedge c_1 = b_2 \wedge c_2$ for some $c_1, c_2 \in A$. Hence

$$(b_1 \wedge c_1) \vee (b_1 \wedge b_2) = (b_2 \wedge c_2) \vee (b_2 \wedge b_1)$$

The left side is equal to $[b_1 \vee (b_1 \wedge b_2)] \wedge [c_1 \vee (b_1 \wedge b_2)] = b_1 \wedge [c_1 \vee (b_1 \wedge b_2)]$. Similarly, the right side is equal to $b_1 \wedge [c_1 \vee (b_1 \wedge b_2)]$. Thus $b_1 \wedge (A \cap B) = b_2 \wedge (A \cap B)$. We claim f is surjective: Let $z \wedge A \in \frac{T(A\cup B)}{A}$, where $z \in T(A \cup B)$. Hence there exist $a \in A, b \in B$ such that $a \wedge b \leqslant z$. Thus $(z \vee b) \wedge a = (z \wedge a) \vee (b \wedge a) = z \wedge a$. Therefore $(z \vee b) \wedge A = z \wedge A$. Thus $f((z \vee b) \wedge (A \cap B)) = (z \vee b) \wedge A = z \wedge A$ and $(z \vee b) \wedge (A \cap B) \in \frac{B}{A\cap B}$. Now, we show that f is a lattice homomorphism. Let $b_1 \wedge (A \cap B), b_2 \wedge (A \cap B) \in \frac{B}{A\cap B}$. Then $f((b_1 \wedge (A \cap B)) \wedge_Q (b_2 \wedge (A \cap B))) = f((b_1 \wedge b_2) \wedge (A \cap B)) = (b_1 \wedge b_2) \wedge A = (b_1 \wedge A) \wedge_Q (b_2 \wedge A) = f(b_1 \wedge (A \cap B)) \wedge_Q f(b_2 \wedge (A \cap B))$.

Similarly, $f((b_1 \wedge (A \cap B)) \vee_Q (b_2 \wedge (A \cap B))) = f(b_1 \wedge (A \cap B)) \vee_Q f(b_2 \wedge (A \cap B))$. This completes the proof. □

Lemma 4.12. *Assume that $\{F_i\}_{i\in I}$ is an indexed set of subfilters of a filter F of L such that $F = \oplus_{i\in I}F_i$ and let G be a subfilter of F. Then $\frac{F}{G} = \oplus_{i\in I}\frac{T(F_i\cup G)}{G}$.*

Proof. For each $j \in I$, let $x \wedge G \in \frac{T(F_j\cup G)}{G} \cap T(\cup_{i\in I_{i\neq j}} \frac{T(F_i\cup G)}{G})$. Then $x \in T(F_j \cup G)$ gives there exist $f_j \in F_j$ and $g_j \in G$ such that $x \wedge G = ((x \vee f_j) \wedge (x \vee g_j)) \wedge G = (x \vee f_j) \wedge G$; so $x = f_j \vee x \in F_j$. Similarly, there are subfilters $F_{i_1}, \ldots, F_{i_s}$ such that $x \in T(\cup_{k=1_{k\neq j}}^{s} F_{i_k})$; hence $x = 1$. Thus $\frac{T(F_j\cup G)}{G} \cap T(\cup_{i\in I_{i\neq j}} \frac{T(F_i\cup G)}{G}) = \{1 \wedge G\}$.

It is enough to show that $\frac{F}{G} \subseteq \oplus_{i\in I}\frac{T(F_i\cup G)}{G}$. Let $y \wedge G \in \frac{F}{G}$. Then there exist $f_{i_1} \in F_{i_1}, \dots, f_{i_t} \in F_{i_t}$ such that $f_{i_1} \wedge \cdots \wedge f_{i_t} \leqslant y$; so $(f_{i_1} \wedge G) \wedge_Q \cdots \wedge_Q (f_{i_t} \wedge G) \leqslant y \wedge G$. It follows that $y \wedge G \in T(\frac{T(F_{i_1}\cup G)}{G} \cup \cdots \cup \frac{T(F_{i_t}\cup G}{G}) \subseteq T(\cup_{i\in I}\frac{T(F_i\cup G}{G}))$, as required. □

Remark 4.13. Let F be a filter of F.

(1). If G is a hollow subfilter of a filter F of L that is not small in F. Then there exists a proper subfilter K of F such that $F = T(G \cup K)$. Since G is hollow, we get $G \cap K \ll G$. Thus G is a supplement in F. Thus $\text{rad}(G) = G \cap \text{rad}(F)$ by [6, Theorem 2.9 (3)].

(2). If G is a direct summand of F such that $G \ll F$, then $G = \{1\}$.

(3). A filter F of L is said to be coatomic if every proper subfilter of F is contained in a maximal subfilter of F. It is easy to see that $\text{rad}(F) \ll F$.

Lemma 4.14. *Let $\{H_\alpha\}_{\alpha\in A}$ be an indexed set of simple subfilters of the filter F of a lattice L. If $F = T(\cup_{\alpha\in A}H_\alpha)$, then for each subfilter K of F there is a subset B of A such that $\{H_\alpha\}_{\alpha\in B}$ is independent and $F = K \oplus (T(\cup_{\alpha\in B}H_\alpha))$.*

Proof. Let K be a subfilter of F. Then there is a subset B of A maximal with respect to conditions that $\{H_\alpha\}_{\alpha\in B}$ is independent and $K \cap (T(\cup_{\alpha\in B}H_\alpha)) = \{1\}$. Let $M = T(K \cup (T(\cup_{\alpha\in B}H_\alpha)))$. For each $\alpha \in A$, we have either $H_\alpha \cap M = \{1\}$ or $H_\alpha \cap M = H_\alpha$. If $H_\alpha \cap M = \{1\}$, then we have a contradiction with the maximality of B. Thus $H_\alpha \subset M$ for each $\alpha \in A$, hence $F = K \oplus (T(\cup_{\alpha\in B}H_\alpha))$. □

Proposition 4.15. *Let $F = \oplus_{i\in I}F_i$ be a filter of L, where each F_i is a local filter. If $\text{rad}(F) \ll F$, then F is supplemented.*

Proof. By [6, Theorem 2.21] and Remark 4.13, for each $i \in I$, F_i is not small in F (so $\text{rad}(F_i) = F_i \cap \text{rad}(F) \neq F_i$) and $\frac{F_i}{\text{rad}(F_i)}$ is simple. Let U be a subfilter of F. By Lemma 4.11 and Lemma 4.12, we have $\bar{F} = \frac{F}{\text{rad}(F)} = \oplus_{i\in I}\frac{T(F_i\cup\text{rad}(F))}{\text{rad}(F)} \cong \oplus_{i\in I}\frac{F_i}{\text{rad}(F_i)}$ is a direct sum of simple filters, and so $\bar{F} = \bar{U} \oplus (\oplus_{i\in J}\frac{F_i}{\text{rad}(F_i)})$ for some $J \subseteq I$, where $\bar{U} = \frac{T(U\cup\text{rad}(F))}{\text{rad}(F)}$, by Lemma 4.15. Now we set $\bar{V} = \oplus_{i\in J}\frac{F_i}{\text{rad}(F_i)})$ (so $V = \oplus_{i\in J}F_i$). Since $\bar{F} = \bar{U} \oplus \bar{V}$, we get that $F = T(\text{rad}(F) \cup T(U \cup V))$ which implies $F = T(U \cup V)$ since $\text{rad}(F) \ll F$. Moreover, $\bar{U} \cap \bar{V} = \{\text{rad}(F)\}$ gives $U \cap V \subseteq \text{rad}(F)$; so $U \cap V \ll F$ by Proposition 1.2 (1). Since V is a direct summand of F, $U \cap V \ll V$ by Proposition 1.4 (c). Thus F is supplemented. □

Theorem 4.16. *Let $F = \oplus_{i\in I}F_i$ be a filter of L, where each F_i is a strongly local filter. Then F is ss-supplemented and coatomic.*

Proof. Since F_i is strongly local for every $i \in I$, it is local and $\text{rad}(F_i) \subseteq \text{Soc}(F_i)$ $(i \in I)$. It then follows from Lemma 4.10 that $\text{rad}(F) = \oplus_{i\in I}\text{rad}(F_i) \subseteq \oplus_{i\in I}\text{Soc}(F_i)$ $= \text{Soc}(F)$; hence $\text{rad}(F) \ll F$ by Proposition 1.4 (a). As strongly local filters are local, Proposition 4.16 gives F is supplemented. Therefore, F is ss-supplemented by Theorem 3.9. Let H be a proper subfilter of F. By Proposition 4.8 (2), H is contained in a maximal subfilter of F, that is, F is coatomic. □

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