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Translatable isotopes of finite groups

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Abstract. We prove the main result, that if $(Q, *)$ is a k-translatable isotope of a finite group $(Q, \oplus)$ of order n then $(Q, \oplus)$ is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n. Given a k-translatable ordering of a left cancellative groupoid Q of order n, we determine all k-translatable orderings of Q. We also prove that a left-cancellative, k-translatable groupoid Q is translatable for a single value of k. Finally, we prove that a left (or right) linear isotope of $\mathbb{Z}_n$ is linear and we give examples of k-translatable isotopes of $\mathbb{Z}_4$ that are neither left nor right linear.

1. Introduction

We assume that all sets considered in this note are finite and have form $Q = \{1, 2, \ldots, n\}$ with the *natural ordering* $1, 2, \ldots, n$.

A groupoid $(Q, *)$ of order n is called *k-translatable*, where $1 \leqslant k < n$, if its Cayley table is obtained by the following rule: If the first row of the Cayley table is $a_1, a_2, \ldots, a_n$, then the q-th row is obtained from the $(q-1)$-st row by taking the last k entries in the $(q-1)-$st row and inserting them as the first k entries of the q-th row and by taking the first $n-k$ entries of the $(q-1)$-st row and inserting them as the last $n-k$ entries of the q-th row, where $q \in \{2, 3, \ldots, n\}$. Then the (ordered) sequence $a_1, a_2, \ldots, a_n$ is called a *k-translatable sequence* of $(Q, *)$ with respect to the ordering $1, 2, \ldots, n$. A groupoid of order n is called *translatable* if it has a k-translatable sequence for some $k \in \{1, 2, \ldots, n-1\}$. A quasigroup of order n may be k-translatable only for k relatively prime to n. A group of order n is translatable if and only if it is cyclic. It is $(n-1)$-translatable.

It is important to note that a k-translatable sequence depends on the ordering of the elements in the Cayley table. A groupoid may be k-translatable for one ordering but not for another (see Example 2.4 below). Unless otherwise stated we will assume that the ordering of the Cayley table is $1, 2, \ldots, n$ and the first row of the table is $a_1, a_2, \ldots, a_n$.

The concept of translatability was first explored in [1] and [2]. It arose through the examination of the fine structure of quadratical quasigroups. Translatability determines the structure of certain types of quasigroups [3]. The question of when quadratical quasigroups, which are idempotent, are translatable was answered in [4] and [5]. There it was proved that a naturally ordered groupoid $(Q, *)$ is

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idempotent and k-translatable if and only if for all $i, j \in Q$ there exist $a, b \in \mathbb{Z}_n$ such that $i*j = (ai+bj)(\mathrm{mod}\, n)$, where $(a+b) = 1(\mathrm{mod}\, n)$ and $(a+bk) = 0(\mathrm{mod}\, n)$.

Now we are interested in the k-translatability of (α, β)-*isotopes* of a group $(Q, \oplus)$, i.e. quasigroups $(Q, *)$ with product $x * y = \alpha x \oplus \beta y$, where α, β are bijections of Q. We will prove our main result in Theorem 5.1, that if an isotope of a group $(Q, \oplus)$ is k-translatable then $(Q, \oplus)$ is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n. Then, for a given a bijection α of $\mathbb{Z}_n$, for particular values of k and n we will determine all possible bijections β for which $(Q, *)$ is k-translatable.

2. Preliminaries

For simplicity instead of $i \equiv j(\mathrm{mod}\, n)$ we will write $[i]_n = [j]_n$. Additionally, in calculations of modulo n, we assume that $0 = n$. Also the neutral element of a group $(Q, \oplus)$ will be denoted by 0. The inverse elements in $(Q, \oplus)$ and $\mathbb{Z}_n$ will be denoted by the same symbol; namely, as $-x$. The set $\{1, 2, \ldots, n\}$ will be denoted by $\overline{\{1, n\}}$. For $k \in \overline{\{1, n\}}$, $(k, n) = 1$ denotes that k and n are relatively prime.

With this convention a naturally ordered groupoid $(Q, *)$ is k-translatable if and only if $i * j = [i+1]_n * [j+k]_n$ for all $i, j \in Q$. Then $a_1, a_2, \ldots, a_n$, where $a_i = 1 * i$, is a k-translatable sequence.

We will need the following results proven in our previous publications.

Lemma 2.1. (cf. [4, Lemma 9.1]) *The quasigroup $(\mathbb{Z}_n, *)$ with the operation $i * j = [ai + c + bj]_n$, where $a, b, c \in \mathbb{Z}_n$ and $(a, n) = (b, n) = 1$ is k-translatable if and only if $[a + kb]_n = 0$.*

Lemma 2.2. (cf. [2, Lemma 2.5]) *Let $a_1, a_2, \ldots, a_n$ be the first row of the Cayley table of a quasigroup $(Q, *)$ of order n. Then $(Q, *)$ is k-translatable if and only if for all $i, j \in Q$ the following (equivalent) conditions are satisfied.*

(*i*) $i * j = a_{[k-ki+j]_n}$,

(*ii*) $i * j = [i+1]_n * [j+k]_n$,

(*iii*) $i * [j-k]_n = [i+1]_n * j$.

Lemma 2.3. (cf. [2, Lemma 2.7]) *If a quasigroup $(Q, *)$ of order n is k-translatable with respect to the ordering $a_1, a_2, \ldots, a_n$ then it is k-translatable with respect to the ordering $a_n, a_1, a_2, \ldots, a_{n-1}$.*

Example 2.4. Consider the following tables:

$*$	1	2	3	4
1	1	2	3	4
2	2	3	4	1
3	3	4	1	2
4	4	1	2	3

$*$	4	1	2	3
4	3	4	1	2
1	4	1	2	3
2	1	2	3	4
3	2	3	4	1

$*$	1	3	4	2
1	1	3	4	2
3	3	1	2	4
4	4	2	3	1
2	2	4	1	3

These tables define the same quasigroup isomorphic to the additive group $\mathbb{Z}_4$. The first table shows that with respect to the natural ordering this quasigroup is 3-translatable. The second table is an example of Lemma 2.3. The third table shows that in another ordering this quasigroup is not translatable.

Lemma 2.5. *Let $(Q,*)$ be a k-translatable groupoid with respect to the natural ordering $1,2,\dots,n$, with k-translatable sequence $a_1,a_2,\dots,a_n$. Then $(Q,*)$ is k-translatable with respect to the ordering $n,n-1,\dots,2,1$, with k-translatable sequence $a_k,a_{k-1},\dots,a_1,a_n,a_{n-1},\dots,a_{k+1}$.*

Proof. The ordering $n,n-1,n-2,\dots,2,1$ can be expressed as $1',2',3',\dots,n'$, where $i'=[1-i]_n$. Then, by Lemma 2.2(*ii*) we have $i'*j'=[1-i]_n*[1-j]_n=[(1-i)-1]_n*[(1-j)-k]_n=[-i]_n*[1-(j+k)]_n=(i+1)'*(j+k)'$. So, $1',2',\dots,n'$ is a k-translatable ordering on $(Q,*)$. Since $n*j=a_{k-kn+j}=a_{k+j}$, this ordering has the k-translatable sequence $a_k,a_{k-1},\dots,a_1,a_n,a_{n-1},\dots,a_{k+1}$. □

Lemma 2.6. *Let $(Q,*)$ be a k-translatable groupoid with respect to the natural ordering with k-translatable sequence $a_1,a_2,\dots,a_n$ and suppose that $(s,n)=1$. Then $(Q,*)$ is k-translatable with respect to the ordering $1,[1+s]_n,[1+2s]_n,\dots,[1+(n-1)s]_n$ with k-translatable sequence $a_1,a_{1+s},a_{1+2s},\dots,a_{1+(n-1)s}$.*

Proof. Since $(s,n)=1$, we can introduce the new ordering $1',2',\dots,n'$ where $i'=[1+(i-1)s]_n$. Then, using Lemma 2.2(*ii*), we obtain $i'*j'=[1+(i-1)s]_n*[1+(j-1)s]_n=[(1+is)-s]_n*[(1+js)-s]_n=[1+is]_n*[(1+js)-s+ks]_n=[1+is]_n*[1+((j+k)-1)s]_n=(i+1)'*(j+k)'$. So, $1',2',\dots,n'$ is a k-translatable ordering on $(Q,*)$. Since $1'*j'=1*[1+(j-1)s]_n=a_{[1+(j-1)s]_n}$ the corresponding k-translatable sequence for this order is $a_1,a_{1+s},a_{1+2s},\dots,a_{1+(n-1)s}$. □

3. Translatable left cancellative groupoids

A groupoid $(Q,*)$ is *left cancellative* if for all $a,b,c\in Q$ $a*b=a*c$ implies $b=c$.

Note that if $a_1,a_2,\dots,a_n$ is a k-translatable sequence of a left cancellative groupoid Q then for all $i\in\overline{\{1,n\}}$, $a_i=a_j$ if and only if $i=j$.

Definition 3.1. Let $Q=\{1,2,\dots,n\}$ be a groupoid of order n, with $a_1,a_2,\dots,a_n$ an ordering of Q. For $i\in\overline{\{1,n\}}$ we define the set A_i as the set consisting of the sequence $a_i,a_{i+1},\dots,a_n,a_1,a_2,\dots,a_{i-1}$ and B_j as the set consisting of the sequence $a_i,a_{i-1},\dots,a_1,a_n,a_{n-1},\dots,a_{i+1}$. Then we call $\bigcup(A_i\cup B_i)$, $i\in\overline{\{1,n\}}$, the *set of cyclic versions of the ordering* $a_1,a_2,\dots,a_n$.

Note that by Lemmas 2.3 and 2.5, a cyclic version of a k-translatable ordering is k-translatable.

Henceforth, $-j'$ will denote $-(j')$ and not $(-j)'$. Similarly $[x]'_n$ denotes $([x]_n)'$.

Theorem 3.2. *Let a left cancellative groupoid $(Q,*)$ be k-translatable with respect to the natural ordering, with k-translatable sequence $a_1, a_2, \ldots, a_n$. Then an ordering is k-translatable on $(Q,*)$ if and only if it is a cyclic version of the ordering $1, [1+s]_n, [1+2s]_n, \ldots, [1+(n-1)s]_n$ for some $s \in \overline{\{1,n\}}$, where $(s,n)=1$.*

Proof. $(\Leftarrow)$. This follows from Lemma 2.6 and the fact that a cyclic version of a k-translatable ordering is k-translatable.

$(\Rightarrow)$. By Lemma 2.2(ii) we can choose a k-translatable ordering $1', 2', \ldots, n'$ on $(Q,*)$, with $1'=1$ and with k-translatable sequence $a_1, a_2, \ldots, a_n$ say. Then, by Lemma 2.6(i), the first two rows of the multiplication table are as follows, with all subscripts of the entries being calculated modulo n.

	1	$2'$	$\ldots$	$(-k)'$	$(1-k)'$	$\ldots$	$(n-1)'$	n'
1	a_1	$a_{2'}$	$\ldots$	$a_{(-k)'}$	$a_{(1-k)'}$	$\ldots$	$a_{(n-1)'}$	$a_{n'}$
$2'$	$a_{k-k2'+1}$	$a_{k-k2'+2'}$	$\ldots$	$a_{k-k2'+(-k)'}$	$a_{k-k2'+(1-k)'}$	$\ldots$	$a_{k-k2'+(n-1)'}$	$a_{k-k2'+n'}$

Then, since the groupoid $(Q,*)$ is left cancellative and k-translatable, modulo n we have $k-k2' = (1-k)'-1 = (2-k)'-2' = \ldots = (n-1)'-(k-1)' = n'-k' = 1-(k+1)' = 2'-(k+2)' = \ldots = (-1-k)'-(n-1)' = (-k)'-n'$, which implies the following n identities:

$$
\begin{array}{cl}
(1) & (1-k)'-1 = (2-k)'-2' \\
(2) & (2-k)'-2' = (3-k)'-3' \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(k) & n'-k' = 1-(k+1)' \\
(k+1) & 1-(k+1)' = 2'-(k+2)' \\
(k+2) & 2'-(k+2)' = 3'-(k+3)' \\
\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-1) & (-1-k)'-(n-1)' = (-k)'-n' \\
(n) & (-k)'-n' = (1-k)'-1.
\end{array}
$$

We note that in any one of these n identities

(A) If j' is the first term on the left-hand side of the identity then $(j+1)'$ is the first term on the right-hand side of that identity.

(B) If $-(j')$ is the second term on the left-hand side of the identity then $-(j+1)'$ is the first term on the right-hand side of that identity.

(C) If j' is the first term on the left (right)-hand side of the identity the second term on the left (right)-hand side of the identity is $-(j+k)'$.

It follows that for all $j = 1, 2, \ldots, n$,

(D) $j'-(j+k)' = (j+1)'-(j+1+k)'$.

Now $n'-1 \overset{(k)}{=} k'-(k+1)'$. But (D) implies $k'-(2k)'=(k+1)'-(2k+1)'$. So, $k'-(k+1)'=(2k)'-(2k+1)'$ and $n'-1=k'-(k+1)'=(2k)'-(2k+1)'$. Continuing in this manner we get $n'-1=k'-(k+1)'=(2k)'-(2k+1)'=(3k)'-(3k+1)'=\ldots=(-2k)'-(-2k+1)'=(-k)'-(1-k)'$.

Since $(k,n)=1$, the elements $k',(2k)',\ldots,(-2k)',(-k)'$ are all different. Therefore $n'-1=1-2'=2'-3'=\ldots=(n-1)'-n'$ and this implies $j'=(j+1)'+n'-1$. Hence, $j'=1+(1-j)(n'-1)$, $(n'-1,n)=1$ and $1',2',3',\ldots,n'$ is the order $1,1-(n'-1),1-2(n'-1),\ldots,1-(n-1)(n'-1)$, a cyclic version of which returns us to the original k-translatable ordering, as required. □

Theorem 3.3. *If a left cancellative groupoid $(Q,*)$ is k-translatable then it is k-translatable for a single value of k.*

Proof. Suppose that $1,2,3,\ldots,n$ is a k-translatable ordering on $(Q,*)$, with k-translatable sequence $a_1,a_2,a_3,\ldots,a_n$ and that $1',2',3',\ldots,n'$ is a k^*-translatable ordering on $(Q,*)$, with the k^*-translatable sequence $b_1,b_2,b_3,\ldots,b_n$. By Lemma 2.5, there is a k^*-translatable ordering $1'',2'',3'',\ldots,n''$ with a k^*-translatable sequence $c_1,c_2,c_3,\ldots,c_n$ and with $1''=1$. Then, $1*j''=a_{k-k+j''}=c_{k^*-k^*+j''}$. Therefore, $a_j=c_j$ for all $j\in\overline{\{1,n\}}$. Then, $2*n=a_{[k-2k+n]_n}=c_{[k^*-2k^*+n]_n}=a_{[k^*-2k^*+n]_n}$ and, since $(Q,*)$ is left cancellative, $-k=-(k^*)$ and $k=k^*$, completing the proof. □

Note that the condition of left cancellation is necessary in the previous theorem. For example, a constant groupoid of order $n>1$ is k-translatable for all $k=1,2,\ldots,n-1$. Similarly, the groupoid $(Q,*)$ of order $2m$, with $x*y=1$ for all odd y and $x*y=2$ for all even y, is $2k$-translatable for every $k=1,\ldots,m-1$.

4. Translatable T-quasigroups

A quasigroup $(Q,*)$ is called a *T-quasigroup* if there exist an abelian group $(Q,\oplus)$ and its automorphisms φ,ψ such that $x*y=\varphi(x)\oplus\psi(y)\oplus c$ for all $x,y\in Q$ and some fixed $c\in Q$. Obviously, each T-quasigroup induced by $(Q,\oplus)$ is (α,β)-isotope of $(Q,\oplus)$.

By the Toyoda theorem (cf. for example [6] or [7]) a quasigroup $(Q,*)$ is medial if and only if it is a T-quasigroup with $\varphi\psi=\psi\varphi$.

Theorem 4.1. *A translatable T-quasigroup $(Q,*)$ of order n is isomorphic to a translatable medial quasigroup induced by the group $\mathbb{Z}_n$.*

Proof. Let $(Q,*)$ be a finite quasigroup of order n induced by the group $(Q,+)$, Then $x*y=\varphi(x)+\psi(y)+c$ for some fixed $c\in Q$ and automorphisms φ,ψ of $(Q,+)$. Denote the k-translatable ordering of Q by $1,2,3,\ldots,n$. By Lemma 2.2(ii), $(Q,*)$ is k-translatable $(1\leqslant k<n)$ with respect to the ordering $1,2,\ldots,n$

if and only if $\varphi(i)+\psi(j)+c = i*j = [i+1]_n*[j+k]_n = \varphi([i+1]_n)+\psi([j+k]_n)+c$, i.e. if and only if $\varphi(i)+\psi(j) = \varphi([i+1]_n)+\psi([j+k]_n)$ for all $i,j \in \{1,2,\dots,n\}$.

By Lemma 2.3, we can choose the ordering such that the group element in the n-th position in this ordering is 0, the identity element of $(Q,+)$. We define $t_i = \bar{i}-\bar{1}$, where $\bar{i}$ is the group element of $(Q,+)$ located in the i^{th} position of the ordering $1,2,\dots,n$. Note that $t_1 = 0$ and $t_n = -\bar{1}$. Then, $\varphi(i)+\psi(j) = \varphi([i+1]_n)+\psi([j+k]_n) \Leftrightarrow \varphi(\bar{i})+\psi(\bar{j}) = \varphi(\overline{[i+1]_n})+\psi(\overline{[j+k]_n}) \Leftrightarrow \psi(\bar{j}-\overline{[j+k]_n}) = \varphi(\overline{[i+1]_n}-\bar{i}) \Leftrightarrow \psi((\bar{1}+t_j)-(\bar{1}+t_{[j+k]_n})) = \varphi((\bar{1}+t_{[i+1]_n})-(\bar{1}-t_i)) \Leftrightarrow \psi(t_j-t_{[j+k]_n}) = \varphi(t_{[i+1]_n}-t_i)$ for all $i,j \in \overline{\{1,n\}}$.

For $j=1$ and $i \in \overline{\{1,n\}}$, $\psi(-t_{[1+k]_n}) = \varphi(t_{[i+1]_n}-t_i)$. So, $\psi(-t_{[1+k]_n}) = \varphi(t_{[s+1]_n}-t_s)$ for all $s \in \overline{\{1,n\}}$. Hence, $t_n-t_{n-1} = t_{n-1}-t_{n-2} = \ldots = t_2-t_1 = t_1-t_n = 0-(-\bar{1}) = \bar{1}$. Thus, $t_2 = \bar{1}$, $t_i = (i-1)\bar{1}$ and $\bar{i} = i\bar{1}$. This means that $\bar{1}$ generates the group $(Q,+)$ and so $(Q,+)$ is a cyclic group isomorphic to $\mathbb{Z}_n$. Hence, by Lemma 2.1, $(Q,*)$ is isomorphic to a translatable medial quasigroup $i\diamond j = [ai+bj+c]_n$, where $(a,n)=1=(b,n)$ and $[a+bk]_n = 0$. □

Corollary 4.2. *A medial quasigroup of order n is translatable if and only if it is induced by a group isomorphic to the additive group $\mathbb{Z}_n$.*

Proof. The necessity follows from Theorem 4.1. To prove the sufficiency observe that a medial quasigroup of order n induced by the group $\mathbb{Z}_n$ has the form $x*y = [ax+by+c]_n$, where $a,b,c \in \mathbb{Z}_n$ and $(a,n) = (b,n) = 1$. By Lemma 2.1 this quasigroup is k-translatable if and only if $[a+bk]_n = 0$. This equation is always uniquely solvable with $k = [-a\bar{b}\,]_n$, where $[b\bar{b}\,]_n = 1$. □

5. Translatability of isotopes of a finite group

Theorem 5.1. *If an (α,β)-isotope $(Q,*)$ of a group $(Q,\oplus)$ of order n is k-translatable then there is an ordering $1,2,\dots,n$ on Q such that for some $s \in \overline{\{1,n\}}$ and all $i,j \in \overline{\{1,n\}}$*

(i) $\alpha n = 0 = \beta s$,

(ii) $\alpha[i+1]_n = \alpha i \oplus \alpha 1$,

(iii) $\alpha i = \underbrace{\alpha 1 \oplus \alpha 1 \oplus \ldots \oplus \alpha 1}_{i\ times} = i(\alpha 1)$,

(iv) $(Q,\oplus)$ *is isomorphic to the group* $\mathbb{Z}_n$,

(v) $\beta[j+k]_n = \beta j - \alpha 1$ *and* $\beta[s+jk]_n = j(-\alpha 1)$.

Proof. From Lemma 2.3, there is a k-translatable ordering $1,2,\dots,n$ on Q such that $\alpha n = 0$ and, since β is a bijection, $\beta s = 0$ for some $s \in Q$.

Then, using k-translatability and Lemma 2.2(ii), $0 = n * s = 1 * [s+k]_n = \alpha 1 \oplus \beta[s+k]_n$. Hence, $\beta[s+k]_n = -\alpha 1$.

Thus, $\alpha i = \alpha i \oplus 0 = i * s = [i+1]_n * [s+k]_n = \alpha[i+1]_n \oplus \beta[s+k]_n = \alpha[i+1]_n - \alpha 1$, which implies

$$\alpha[i+1]_n = \alpha i \oplus \alpha 1. \tag{1}$$

Then, by induction on i, it is easy to prove that for all $i \in \overline{\{1,n\}}$, $\alpha i = \alpha 1 \oplus \alpha 1 \oplus \ldots \oplus \alpha 1$, (with i number of summands). Consequently, $\alpha i \oplus \alpha j = \alpha[i+j]_n$. We then define a bijection $\varphi : Q \to \mathbb{Z}_n$ as $\varphi \alpha i = i$ and so, we have $\varphi(\alpha i \oplus \alpha j) = \varphi(\alpha[i+j]_n) = [i+j]_n = [\varphi \alpha i + \varphi \beta j]_n$. Hence, φ is an isomorphism.

Finally, $\beta j = 0 \oplus \beta j = n * j = 1 * [j+k]_n = \alpha 1 \oplus \beta[j+k]_n$ and, since the groups $(Q, \oplus)$ and $(\mathbb{Z}, +)$ are isomorphic, the operation $\oplus$ is commutative, for all $j \in \overline{\{1,n\}}$ we have $\beta[j+k]_n = \beta j - \alpha 1$. By induction on j it is then easy to prove that for all $j \in \overline{\{1,n\}}$, $\beta[s+jk]_n = -\alpha 1 - \alpha 1 - \ldots - \alpha 1$ (j times). $\square$

Proposition 5.2. *If an (α,β)-isotope $(Q,*)$ of the commutative group $(Q,\oplus)$ satisfies (ii) and (v) of Theorem 5.1, then it is k-translatable.*

Proof. $[i+1]_n * [j+k]_n = \alpha[i+1]_n \oplus \beta[j+k]_n \overset{(ii),(v)}{=} \alpha i \oplus \alpha 1 \oplus \beta j - \alpha 1 = \alpha i \oplus \beta j = i * j$, for all $i, j \in \overline{\{1,n\}}$. By Lemma 2.2($ii$), $(Q,*)$ is k-translatable. $\square$

The following Corollary follows readily from Theorem 5.1 and Proposition 5.2. The proof is omitted.

Corollary 5.3. *The quasigroup $(\mathbb{Z}_n, *)$ with $i * j = [\alpha i + \beta j]_n$, where α, β are bijections of $\mathbb{Z}_n$ is k-translatable for some k if and only if there is an ordering $1', 2', \ldots, n'$ of $\mathbb{Z}_n$ such that for some $s \in \overline{\{1,n\}}$ and all $i \in \overline{\{1,n\}}$*

(i) $\alpha n' = 0 = \beta s'$,

(ii) $\alpha([i+1]'_n) = [\alpha i' + \alpha 1']_n$,

(iii) $\alpha i' = [i(\alpha 1')]_n$ *for* $i \in \overline{\{1,n\}}$,

(iv) $\beta([i+k]'_n) = \beta i' - \alpha 1'$ *and* $\beta([s+ik]'_n) = [i(-\alpha 1')]_n$,

(v) $(\alpha 1', n) = 1$.

Corollary 5.4. *For a given ordering on $\mathbb{Z}_n$ and any $k, t \in \overline{\{1,n\}}$ such that $(k,n) = (t,n) = 1$ there are bijections α_t and β_s ($s \in \overline{\{1,n\}}$) on $\mathbb{Z}_n$ such that the quasigroup $(\mathbb{Z}_n, *_s)$ defined by $i *_s j = [\alpha_t i + \beta_s y]_n$ is k-translatable with respect to this ordering.*

Proof. Suppose that $1', 2', \ldots, n'$ is a fixed ordering on $\mathbb{Z}_n$ and that $k, t \in \overline{\{1,n\}}$ be such that $(k,n) = (t,n) = 1$. Then, we define the bijection α_t on $\mathbb{Z}_n$ by putting $\alpha_t i' = [it]_n$ for any $i \in \overline{\{1,n\}}$. It is easy to see that $\alpha_t[i+t]'_n = [\alpha_t i' + t]_n$ for any $i \in \overline{\{1,n\}}$. Now for any $s \in \overline{\{1,n\}}$ we define the bijection β_s by putting $\beta_s[s+ik]_n = [-it]_n$ for any $i \in \overline{\{1,n\}}$. Since $(k,n) = 1$, we have $\{1, 2, \ldots, n\} =$

$\{[s+k]_n, [s+2k]_n, \ldots, [s+nk]_n = s\}$. It follows that $\beta_s([i+k]'_n) = [\beta_s i' - t]_n$ for any $i \in \overline{\{1,n\}}$. Then $[i+1]'_n *_s [j+k]'_n = [\alpha_t([i+1]'_n) + \beta_s([j+k]'_n)]_n = [\alpha_t i' + t + \beta_s j' - t]_n = [\alpha_t i' + \beta_s j']_n = i' *_s j'$. So, by Lemma 2.2(*ii*). $(\mathbb{Z}_n, *_s)$ is k-translatable with respect to this ordering. □

Note that, as a result of Theorem 5.1 and Corollary 5.4, a finite group of order n is isomorphic to $\mathbb{Z}_n$ if and only if it has a k-translatable isotope for some $k \in \overline{\{1, n-1\}}$. In fact, a finite group of order n either has no k-translatable isotope or it has k-translatable isotopes for all values of $k \in \overline{\{1, n-1\}}$.

Example 5.5. Let $n = 8$. Then $(t, 8) = 1$ for $t \in \{1, 3, 5, 7\}$. Then for $t = 5$, $s = 1$, $k = 3$ and the given ordering $4, 6, 1, 3, 2, 8, 5, 7$ we see that $\alpha_5 = (1, 7, 8, 6, 2)(3, 4, 5)$ and $\beta_1 = (1, 2, 4, 8, 5, 6)(3)(7)$. The Cayley table of $i' *_1 j' = [\alpha_5 i' + \beta_1 j']_8$ follows.

$*_1$	4	6	1	3	2	8	5	7
$1' = 4$	5	6	7	8	1	2	3	4
$2' = 6$	2	3	4	5	6	7	8	1
$3' = 1$	7	8	1	2	3	4	5	6
$4' = 3$	4	5	6	7	8	1	2	3
$5' = 2$	1	2	3	4	5	6	7	8
$6' = 8$	6	7	8	1	2	3	4	5
$7' = 5$	3	4	5	6	7	8	1	2
$8' = 7$	8	1	2	3	4	5	6	7

Example 5.6. For $t = 5$ we want to determine all the k-translatable quasigroups $(\mathbb{Z}_8.*)$ of the form $i * j = [\alpha i + \beta j]_8$, where α is an automorphism of the group $\mathbb{Z}_8$. Such automorphisms are of the form $\alpha i = [mi]_8$, where $m \in \{1, 3, 5, 7\}$. Then $\alpha_5 1' = 5$, $\alpha_5 2' = 2$, $\alpha_5 3' = 7$, $\alpha_5 4' = 4$, $\alpha_5 5' = 1$, $\alpha_5 6' = 6$, $\alpha_5 7' = 3$, $\alpha_5 8' = 8$.

Now let $\alpha = \alpha_5$ be an automorphism of $\mathbb{Z}_8$. If $\alpha i = 1i = i$, then $i' = 5i$. If $\alpha i = 3i$, then $i' = 7i$. If $\alpha i = 5i$, then $i' = i$. If $\alpha i = 7i$, then $i' = 3i$. These automorphisms, respectively, give the following orderings: $\alpha i = i$ gives the ordering $5, 2, 7, 4, 1, 6, 3, 8$; $\alpha i = 3i$ gives the ordering $7, 6, 5, 4, 3, 2, 1, 8$; $\alpha i = 5i$ gives $1, 2, 3, 4, 5, 6, 7, 8$; $\alpha i = 7x$ gives $3, 6, 1, 4, 7, 2, 5, 8$.

By Corollary 5.4, for each $s \in \{1, 2, \ldots, 8\}$ and each $k \in \{1, 3, 5, 7\}$ we can calculate β_s. It turns out that β_s is an automorphism of $\mathbb{Z}_8$ if and only if $s = 8$ (as long as α is an automorphism of $\mathbb{Z}_8$). These calculations give: for $\alpha_5 i = i$ and $k = 1$, $\beta_8 i = 7i$; for $\alpha_5 i = 3i$ and $k = 1$, $\beta_8 i = 5i$; for $\alpha_5 i = 5x$ and $k = 1$, $\beta_8 i = 3i$ and for $\alpha_5 i = 7i$ and $k = 1$, $\beta_8 i = i$, which matches Lemma 2.1.

For $s \neq 8$, $i *_s j = [\alpha_t i + \beta_s j]_n$ is a 1-translatable, left linear quasigroup. For example, in the case when $\alpha_5 i = i$, $k = 1$, the ordering $5, 2, 7, 4, 1, 6, 3, 8$ and $s = 1$, $\beta_1 = (1, 4)(2, 3)(5, 8)(6, 7)$ is not an automorphism of $\mathbb{Z}_8$. This quasigroup has the following Cayley table that is clearly 1-translatable. It has a right neutral element; namely, 5, and it is unipotent.

$*_1$	5	2	7	4	1	6	3	8
5	5	8	3	6	1	4	7	2
2	2	5	8	3	6	1	4	7
7	7	2	5	8	3	6	1	4
4	4	7	2	5	8	3	6	1
1	1	4	7	2	5	8	3	6
6	6	1	4	7	2	5	8	3
3	3	6	1	4	7	2	5	8
8	8	3	6	1	4	7	2	5

An (α,β)-isotope $(Q,*)$ of the group $(Q,\oplus)$ is *left* (*right*) *linear* over $(Q,\oplus)$ if α (respectively, β) is an automorphism of $(Q,\oplus)$. If an (α,β)-isotope can be written as $x*y=\hat{\alpha}x\oplus c\oplus\hat{\beta}y$ for automorphisms $\hat{\alpha},\hat{\beta}$ of $(Q,\oplus)$ and some $c\in Q$, then the quasigroup $(Q,*)$ is called *linear* over $(Q,\oplus)$.

The following Theorem finds all k-translatable quasigroups that are left linear over $\mathbb{Z}_n$.

Theorem 5.7. *If an (α,β)-isotope $(\mathbb{Z}_n,*)$ of the group $\mathbb{Z}_n$ is left linear over $\mathbb{Z}_n$, then it is k-translatable if and only if there exist $m,s,t\in\overline{\{1,n\}}$ such that $(t,n)=1=(m,n)$ and $\beta_s j=[\overline{k}(st-mj)]_n$ for all $j\in\overline{\{1,n\}}$, where $\alpha i=[mi]_n$ and $[\overline{k}k]_n=1$.*

Proof. ($\Rightarrow$): Since α is an automorphism of the group $\mathbb{Z}_n$, $\alpha i=[mi]_n$ for some $(m,n)=1$. Using Corollary 5.3, there exists an ordering $1',2',\ldots,n'$ on $\mathbb{Z}_n$ and $s\in\overline{\{1,n\}}$ such that $\alpha n'=0=\beta s'$ and, for all $i\in\overline{\{1,n\}}$, $\alpha i'=[i(\alpha 1')]_n$, $(\alpha 1',n)=1$ and $\beta_s([s+ik]'_n)=-[i(\alpha 1')]_n$. Thus for $t=\alpha 1'$ we obtain $[mi']_n=\alpha i'=[i(\alpha 1')]_n=[it]_n$. Hence, for $(m,\overline{m})=1$ $i'=[\overline{m}ti]_n$ and $[s+ik]'_n=[\overline{m}t(s+ik)]_n=[\overline{m}ts+\overline{m}tki]_n$. Therefore, $-[it]_n=$ $beta_s([s+ik]'_n)=\beta_s[\overline{m}ts+\overline{m}tki]_n$. This for $i=[-\overline{k}s+\overline{k}\overline{t}mj]_n$ gives $\beta_s j=[-(-\overline{k}s+\overline{k}\overline{t}mj)t]_n=[\overline{k}(st-mj)]_n$.

($\Leftarrow$): For all $i,j\in\mathbb{Z}_n$, $[i+1]_n*[j+k]_n=[mi+m+\overline{k}(st-m(j+k))]_n=[mi+m+\overline{k}(st-mj)-m]_n=[mi+\overline{k}(st-mj)]_n=i*j$. So, k-translatability follows from Lemma 2.2(ii). □

Theorem 5.8. *If a k-translatable quasigroup $(Q,*)$ is an (α,β)-isotope of the group $(Q,\oplus)$, then there is an ordering $1,2,\ldots,n$ on Q such that*

(i) $\alpha s=0=\beta n$ *for some* $s\in\overline{\{1,n\}}$,

(ii) $\alpha[i+1]_n=\alpha i\oplus\alpha[s+1]_n$ *for all* $i\in\overline{\{1,n\}}$,

(iii) $\alpha[s+i]_n=i(\alpha[s+1]_n)$ *for all* $i\in\overline{\{1,n\}}$,

(iv) $(Q,\oplus)$ *is isomorphic to* $(\mathbb{Z}_n,+)$,

(v) $\beta[jk]_n=j(-\alpha[s+1]_n)$ *for all* $j\in\overline{\{1,n\}}$.

Proof. From Lemma 2.3, there is a k-translatable sequence $1, 2, \ldots, n$ on Q such that $\beta n = 0$ and, since α is a bijection, $\alpha s = 0$ for some $s \in \overline{\{1, n\}}$. Then, using k-translatability and Lemma 2.2, $0 = s * n = [s+1]_n * k = \alpha[s+1]_n \oplus \beta k$. Hence,

$$\beta k = -\alpha[s+1]_n. \tag{2}$$

Also, $\alpha i = i * n = [i+1]_n * k = \alpha[i+1]_n \oplus \beta k = \alpha[i+1]_n - \alpha[s+1]_n$, which implies $\alpha[i+1]_n = \alpha i \oplus \alpha[s+1]_n$. This proves (ii). Then, by induction on i, we can prove (iii).

Now, $\beta j = s * j = [s+1]_n * [j+k]_n = \alpha[s+1]_n \oplus \beta[j+k]_n$. Therefore $\alpha[s+1]_n = \beta j - \beta[j+k]_n$, which together with (2) implies $\beta j - \beta[j+k]_n = -\beta k$. From this, by induction, we obtain $\beta[jk]_n = \beta k \oplus \beta k \oplus \ldots \oplus \beta k$ (with j number of summands). This, by (2), proves (v).

Since α is a bijection $Q = \{\alpha[s+i]_n : i \in \overline{\{1, n\}}\}$. So we can define a bijection $\varphi : Q \to \mathbb{Z}_n$ as $\varphi\alpha[s+i]_n = i$. Then we have $\varphi(\alpha[s+i]_n \oplus \alpha[s+j]_n) = \varphi(i\alpha[s+1]_n \oplus j\alpha[s+1]_n) = \varphi([i+j]_n\alpha[s+1]_n) = \varphi\alpha[s+[i+j]_n]_n = [i+j]_n = [\varphi\alpha[s+i]_n + \varphi\alpha[s+j]_n]_n$. Hence, φ is an isomorphism between $(Q, \oplus)$ and $(\mathbb{Z}_n, +)$. This completes the proof of Theorem 5.8. □

Proposition 5.9. *If an (α, β)-isotope of the commutative group $(Q, \oplus)$ satisfies (ii), (iii) and (v) of Theorem* 5.8 *then it is a k-translatable quasigroup.*

Proof. Suppose that $i, j \in Q$. By (ii), (iii) and (v) of Theorem 5.8 we see that $Q = \{i\alpha[s+1]_n : i \in \overline{\{1, n\}}\} = \{[ik]_n : i \in \overline{\{1, n\}}\}$ and so $j = [\hat{j}k]_n$ for some $\hat{j} \in \overline{\{1, n\}}$. Then, $[i+1]_n * [j+k]_n = \alpha[i+1]_n \oplus \beta[(\hat{j}+1)k]_n = \alpha i \oplus \alpha[s+1]_n \oplus [\hat{j}+1]_n(-\alpha[s+1]_n) = \alpha i \oplus \hat{j}(-\alpha[s+1]_n) = \alpha i \oplus \beta[\hat{j}k]_n = \alpha i \oplus \beta j = i * j$ and k-translatability follows from Lemma 2.2. □

The following Corollary follows directly from Theorem 5.8.

Corollary 5.10. *An (α, β)-isotope of the group $\mathbb{Z}_n$ is k-translatable if and only if there is an ordering $1', 2', \ldots, n'$ on $\mathbb{Z}_n$ such that for some $s \in \overline{\{1, n\}}$ and all $i \in \overline{\{1, n\}}$*

(i) $\alpha s' = n = \beta n'$,

(ii) $(\alpha([s+1]'_n), n) = 1$,

(iii) $\alpha([i+1]'_n) = \alpha i' + \alpha([s+1]'_n)$,

(iv) $(\alpha([s+i]'_n) = i\alpha([s+1]'_n)$,

(v) $\beta([ik]'_n) = -i\alpha([s+1]'_n)$.

Theorem 5.11. *If an (α, β)-isotope $(\mathbb{Z}_n, *)$ of the group $\mathbb{Z}_n$ is right linear over $\mathbb{Z}_n$, then it is k-translatable if and only if there exist $m, s.t \in \overline{\{1, n\}}$ such that $(t, n) = 1 = (m, n)$ and $\alpha i = [-st - mki]_n$ for all $i \in \overline{\{1, n\}}$, where $\beta j = [mj]_n$.*

Proof. ($\Rightarrow$): Since β is an automorphism of the group $\mathbb{Z}_n$, $\beta j = [mj]_n$ for some $(m,n) = 1$. Using Corollary 5.10(ii) with $t = \alpha([s+1]'_n)$ and $\alpha s' = n$, for all $i \in \overline{\{1,n\}}$ we have $[m([ik]'_n)]_n = [-it]_n$ and so $[i']_n = -[\overline{m}\overline{k}it]_n$, where $[\overline{m}m]_n = 1$. By Corollary 5.10(iv), $[jt]_n = \alpha([\overline{m}\overline{k}t(s+j)]_n$, which for $j = [-s - mk\overline{t}i]_n$ gives $\alpha i = [-st - mki]_n$.

($\Leftarrow$): For all $i, j \in \overline{\{1,n\}}$ we have $[i+1]_n * [j+k]_n = [-st - mki + mj]_n = [\alpha i + \beta j]_n = i * j$. Therefore, by Lemma 2.2(ii), $(\mathbb{Z}_n, *)$ is k-translatable. $\square$

Corollary 5.12. *For any ordering $1', 2', \ldots, n'$ on $\mathbb{Z}_n$ and any $k, t \in \overline{\{1,n\}}$ such that $(k,n) = (t,n) = 1$ there is a bijection β_t on $\mathbb{Z}_n$ and bijections α_s, $s \in \overline{\{1,n\}}$, such that the quasigroups $(\mathbb{Z}_n, *_s)$ defined by $i *_s j = [\alpha_s i + \beta_t j]_n$ are k-translatable with respect to this ordering.*

Proof. Suppose that $1', 2', \ldots, n'$ is an order on $\mathbb{Z}_n$ and that $k, t \in \overline{\{1,n\}}$, with $(k,n) = 1 = (t,n)$. Then, we define $\alpha_s([s+i]'_n) = [it]_n$. It follows that for all $i \in \overline{\{1,n\}}$, $\alpha([i+1]'_n) = [\alpha i' + t]_n$. Then, we define $\beta_t([ik]'_n) = [-it]_n$. It follows that for all $i \in \overline{\{1,n\}}$, $\beta_t[j+k]'_n = [\beta_t j' - t]_n$. Then, $[i+1]'_n *_s [j+k]'_n = [\alpha_s([i+1]'_n) + \beta_t([j+k]'_n)]_n = [\alpha_s i' + t + \beta_t j - t]_n = i' *_s j'$. The required result then follows from Lemma 2.2(ii). $\square$

Theorem 5.13. *A k-translatable quasigroup left or right linear over $\mathbb{Z}_n$ is medial and linear over $\mathbb{Z}_n$. If $[k^2]_n = 1$ then it is also paramedial.*

Proof. By Theorem 5.7 a k-translatable quasigroup left linear over $\mathbb{Z}_n$ has the operation $i * j = [\alpha i + \overline{k}st + \delta j]_n$, where $\alpha i = [mi]_n$ and $\delta j = [-\overline{k}mj]_n$. A k-translatable quasigroup right linear over $\mathbb{Z}_n$ has, by Theorem 5.11, the operation $i * j = [\gamma i - st + \beta j]_n$, where $\gamma i = [-mki]_n$ and $\beta j = [mj]_n$. Since $(k,n) = (m,n) = 1$, $\alpha, \beta, \delta, \gamma$ are automorphisms of the group $\mathbb{Z}_n$. If $[k^2]_n = 1$ then $\alpha^2 = \delta^2$ and $\gamma^2 = \beta^2$. This means (cf. [6, Theorem 9]) that this quasigroup is paramedial. $\square$

We have seen in Theorem 5.13 that k-translatable left linear and k-translatable right linear quasigroups over $\mathbb{Z}_n$ are linear. This leads to the question of whether there are k-translatable isotopes over $\mathbb{Z}_n$ of the form $x * y = [\alpha x + \beta y]_n$ where both α and β are not automorphisms of $\mathbb{Z}_n$ and $(\mathbb{Z}_n, *)$ cannot be written as $x * y = [\hat{\alpha} x + c + \hat{\beta} y]_n$, where either $\hat{\alpha}$ or $\hat{\beta}$ are automorphisms of $\mathbb{Z}_n$. (That is, the k-translatable quasigroup $(\mathbb{Z}_n, *)$ has no representation as a linear, k-translatable quasigroup over $\mathbb{Z}_n$.) In fact, there are many such k-translatable quasigroups over $\mathbb{Z}_4$, as we show in the example below.

The proofs of Theorem 5.14 and Corollary 5.15 are similar to the proofs of Theorems 5.1 and 5.8 and Corollaries 5.3 and 5.10 and are therefore omitted. Corollary 5.15 will be applied to give the examples just referred to in the preceding paragraph.

Theorem 5.14. *If an (α, β)-isotope $(Q, *)$ of a group $(Q, \oplus)$ of order n is k-translatable then there is an ordering $1, 2, \ldots, n$ on Q such that for some $r, s \in \overline{\{1,n\}}$ and all $i, j \in \overline{\{1,n\}}$*

(*i*) $\alpha r = 0 = \beta s$,

(*ii*) $\alpha[i+1]_n = \alpha i \oplus \alpha[r+1]_n$,

(*iii*) $\alpha[r+i]_n = \underbrace{\alpha[r+1]_n \oplus \alpha[r+1]_n \oplus \ldots \oplus \alpha[r+1]_n}_{i\ times} = i(\alpha[r+1]_n)$,

(*iv*) $(Q, \oplus)$ *is isomorphic to the group* $\mathbb{Z}_n$,

(*v*) $\beta[j+k]_n = \beta j \oplus \beta[s+k]_n$ *and* $\beta[s+jk]_n = j(-\alpha[r+1]_n)$.

Corollary 5.15. *An* (α, β)*-isotope of the group* $\mathbb{Z}_n$ *is* k*-translatable for some* k *if and only if there is an ordering* $1', 2', \ldots, n'$ *of* $\mathbb{Z}_n$ *such that for some* $r, s \in \overline{\{1, n\}}$ *and all* $i \in \overline{\{1, n\}}$

(*i*) $\alpha r' = n = \beta s'$,

(*ii*) $\alpha([i+1]'_n) = [\alpha i' + \alpha([r+1]'_n)]_n$,

(*iii*) $\alpha([r+i]'_n) = [i\alpha([r+1])'_n]_n$,

(*iv*) $\beta([i+k]'_n) = [\beta i' + \beta([s+k]'_n)]_n$ *and* $\beta([s+ik]'_n) = [i(-\alpha([r+1]'_n)]_n$,

(*v*) $(\alpha([r+1]'_n), n) = 1$.

Theorem 5.16. *If an* (α, β)*-isotope of the group* $\mathbb{Z}_n$ *is* $(n-1)$*-translatable for some ordering* $1', 2', \ldots, n'$ *with* $\beta s' = n$*, then* $\beta i' = [\alpha i' - \alpha s']_n$ *for all* $i \in \overline{\{1, n\}}$.

Proof. An $(n-1)$-translatable quasigroup of order n is commutative. Hence in an $(n-1)$-translatable (α, β)-isotope of the group $\mathbb{Z}_n$ we have $[\alpha i + \beta j]_n = [\alpha j + \beta j]_n$. In particular, $\alpha i' = [\alpha i' + \beta s']_n = [\alpha s' + \beta i']_n$. So, $\beta i' = [\alpha i' - \alpha s']_n$. □

6. 3-translatable isotopes of $\mathbb{Z}_4$

We proceed to calculate the 3-translatable (α, β)-isotopes of the group $\mathbb{Z}_4$. By Theorem 5.16, for all $i \in \overline{\{1, 4\}}$, $\beta i' = [\alpha i' - \alpha s']_4$. Using Corollary 5.15, there is a 3-translatable ordering $1', 2', 3', 4'$ on $\mathbb{Z}_4$ and $r, s \in \overline{\{1, 4\}}$ such that $\alpha r' = 4 = \beta s'$, $(\alpha([r+1]'_4), 4) = 1$ and $\alpha([r+i]'_4) = i\alpha([r+1]'_4)$ for all $i \in \overline{\{1, 4\}}$. So, $\alpha([r+1]_4)' \in \{1, 3\}$. If we choose $\alpha([r+1]'_4) = 1$ then $\alpha([r+i]'_4) = i\alpha([r+1]'_4) = i$ for all $i \in \overline{\{1, 4\}}$. Therefore, $\beta([r+i]'_4) = [\alpha([r+i]'_4) - \alpha s']_4 = [i - \alpha s']_4$. Since $\alpha s' = \alpha([r-(r-s)]'_4) = [s-r]_4$ we have $\beta([r+i]'_4) = [i - \alpha s']_4 = [i + r - s]_4$.

Note that since $1', 2', 3', 4'$ is a 3-translatable ordering, by Lemma 2.3 so is the ordering $[r+1]'_4, [r+2]'_4, [r+3]'_4, r'$. If we define $x_i = [r+i]'_4$ then we obtain the following 3-translatable Cayley table for $\mathbb{Z}_4$, where $d = [r - s]_4$ and each entry is calculated modulo 4.

$*$	x_1	x_2	x_3	x_4
x_1	$2+d$	$3+d$	d	$1+d$
x_2	$3+d$	d	$1+d$	$2+d$
x_3	d	$1+d$	$2+d$	$3+d$
x_4	$1+d$	$2+d$	$3+d$	d

Note that in the Cayley table above, changing the ordering to $x_3x_4x_1x_2$ in the leftmost column and also in the top row gives exactly the same quasigroup. That is, not only is the main body of the Cayley table the same, all the products are the same. For a fixed value of d, any other ordering gives a different quasigroup.

Note also that, given a fixed r, s and $t = \alpha([r+1]'_n)$, any chosen ordering $x_1x_2x_3x_4$ determines precisely one bijection α which in turn by Corollary 5.15 and Theorem 5.16 determines the bijection β, as indicated in the table below, the entries of which are calculated modulo 4.

There are all 24 possible orderings listed in the table below, twelve pairs of which give 12 distinct 3-translatable quasigroups induced by $\mathbb{Z}_4$. The first 4 pairs of those are linear over $\mathbb{Z}_4$, namely, the quasigroups determined by the orderings 1234, 3412, 2341, 4123, 4321, 2143, 1432 and 3214, as will be shown below. None of the quasigroups determined by the eight other pairs of orderings is linear over $\mathbb{Z}_4$.

$x_1x_2x_3x_4$	α	$\beta 1$	$\beta 2$	$\beta 3$	$\beta 4$
1234	ε	$1+d$	$2+d$	$3+d$	d
3412	(13)(24)	$3+d$	d	$1+d$	$2+d$
2341	(1432)	d	$1+d$	$2+d$	$3+d$
4123	(1234)	$2+d$	$3+d$	d	$1+d$
4321	(14)(23)	d	$3+d$	$2+d$	$1+d$
2143	(12)(34)	$2+d$	$1+d$	d	$3+d$
1432	(24)	$1+d$	d	$3+d$	$2+d$
3214	(13)	$3+d$	$2+d$	$1+d$	d
1243	(34)	$1+d$	$2+d$	d	$3+d$
4312	(1324)	$3+d$	d	$2+d$	$1+d$
1321	(23)	$1+d$	$3+d$	$2+d$	d
2413	(1342)	$3+d$	$1+d$	d	$2+d$

1342	(243)	$1+d$	d	$2+d$	$3+d$
4213	(134)	$3+d$	$2+d$	d	$1+d$
1423	(234)	$1+d$	$3+d$	d	$2+d$
2314	(132)	$3+d$	$1+d$	$2+d$	d
2134	(12)	$2+d$	$1+d$	$3+d$	d
3421	(1423)	d	$3+d$	$1+d$	$2+d$
2431	(142)	d	$1+d$	$3+d$	$2+d$
3124	(123)	$2+d$	$3+d$	$1+d$	d
3142	(1243)	$2+d$	d	$1+d$	$3+d$
4231	(14)	d	$2+d$	$3+d$	$1+d$
3241	(143)	d	$2+d$	$1+d$	$3+d$
4132	(124)	$2+d$	d	$3+d$	$1+d$

Given that the only automorphisms of $\mathbb{Z}_4$ are of the form $\varphi i = i$ and $\varphi i = 3i$, using Lemma 2.1 it is easy to calculate that the only 3-translatable quasigroups linear over $\mathbb{Z}_4$ are of the form $i*j = [\varphi i+\varphi j+c]_4$, where $c \in \mathbb{Z}_4$ is fixed. Examining the Cayley table of the quasigroups determined by the first eight pairs in the table, in their natural ordering, shows that they each are of one of these linear forms.

In particular, the orderings 1234 and 3412 give $i*j = [i+j-d]_4$, 2341 and 4123 give $i*j = [i+j+2-d]_4$, 4321 and 2143 give $i*j = [3i+3j+2-d]_4$ and 1432 and 3214 give $i*j = [3i+3j-d]_4$.

Any of the other quasigroups determined by the remaining 8 pairs of orderings is not of a linear form because, in their natural ordering, there is always an increase in the value of a particular two consecutive, increasing entries by a value of 2. This is not possible for a 3-translatable quasigroup linear over $\mathbb{Z}_4$, where the values of two consecutive, increasing entries always increases by a value of 1 or 3.

If we had chosen $\alpha([r+1]'_4) = 3$ then by Corollary 5.15, for all $i \in \overline{\{1,4\}}$, $(\alpha[r+1]'_4) = [3i]_4$ and $\beta([s+3i]'_4) = [-3i]_4 = i = \beta([s-i]'_4)$. Therefore, $\beta([r+i]'_4) = [s-r-i]_4$. As previously, if we define $x_i = [r+i]'_4$ then any ordering $x_1x_2x_3x_4$ gives the following 3-translatable Cayley table.

$*$	x_1	x_2	x_3	x_4
x_1	$2-d$	$1-d$	$-d$	$3-d$
x_2	$1-d$	$-d$	$3-d$	$2-d$
x_3	$-d$	$3-d$	$2-d$	$1-d$
x_4	$3-d$	$2-d$	$1-d$	$-d$

The first eight orderings of the table below give different values of the mapping α, but for each ordering the value of βi, $i \in \overline{\{1,4\}}$ is the additive inverse of the corresponding entries in the table on the previous page.

$x_1x_2x_3x_4$	α	$\beta 1$	$\beta 2$	$\beta 3$	$\beta 4$
1234	(13)	$-d-1$	$-d-2$	$-d-3$	$-d$
3412	(24)	$-d-3$	$-d$	$-d-1$	$-d-2$
2341	(14)(23)	$-d$	$-d-1$	$-d-2$	$-d-3$
4123	(12)(34)	$-d-2$	$-d-3$	$-d$	$-d-1$
4321	(1432)	$-d$	$-d-3$	$-d-2$	$-d-1$
2143	(1234)	$-d-2$	$-d-1$	$-d$	$-d-3$
1432	(13)(24)	$-d-1$	$-d$	$-d-3$	$-d-2$
3214	ε	$-d-3$	$-d-2$	$-d-1$	$-d$

In particular, the orderings 1234 and 3412 give $i * j = [3i + 3j - d]_4$, 2341 and 4123 give $i * j = [3i + 3j + 2 - d]_4$, 4321 and 2143 give $i * j = [i + j + 2 - d]_4$ and 1432 and 3214 give $i * j = [i + j - d]_4$.

Note that, whether $\alpha([r+1]'_4) = 1$ or $\alpha([r+1]'_4) = 3$, since $[r-s]_4 \in \overline{\{1,4\}}$ every possible 3-translatable linear isotope appears for any of the first eight orderings in Tables 2 or 4. The remainder of the non-linear, 3-translatable isotopes are of one of the following 8 forms in their natural ordering.

$*_1$	1	2	3	4
1	2	1	3	4
2	1	4	2	3
3	3	2	4	1
4	4	3	1	2

$*_2$	1	2	3	4
1	2	4	1	3
2	4	2	3	1
3	1	3	4	2
4	3	1	2	4

$*_3$	1	2	3	4
1	2	3	1	4
2	3	4	2	1
3	1	2	4	3
4	4	1	3	2

$*_4$	1	2	3	4
1	2	4	3	1
2	4	2	1	3
3	3	1	4	2
4	1	3	2	4

$*_5$	1	2	3	4
1	4	1	3	2
2	1	2	4	3
3	3	4	2	1
4	2	3	1	4

$*_6$	1	2	3	4
1	4	3	1	2
2	3	2	4	1
3	1	4	2	3
4	2	1	3	4

$*_7$	1	2	3	4
1	4	2	1	3
2	2	4	3	1
3	1	3	2	4
4	3	1	4	2

$*_8$	1	2	3	4
1	4	2	3	1
2	2	4	1	3
3	3	1	2	4
4	1	3	4	2

The quasigroups $(\mathbb{Z}_4, *_1)$, $(\mathbb{Z}_4, *_3)$, $(\mathbb{Z}_4, *_7)$ and $(\mathbb{Z}_4, *_8)$ are isomorphic to each other, as are the quasigroups $(\mathbb{Z}_4, *_2)$, $(\mathbb{Z}_4, *_4)$, $(\mathbb{Z}_4, *_5)$ and $(\mathbb{Z}_4, *_6)$.

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