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Semisymmetric quasigroups as alignments on abstract polyhedra

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Abstract. A quasigroup satisfying the identity $x(yx) = y$ is called semisymmetric; if a semisymmetric quasigroup is commutative, then it is totally symmetric. We demonstrate a bijection between totally symmetric quasigroups and directed graphs satisfying certain specifications. Further, we demonstrate a bijection between semisymmetric quasigroups and certain mappings between abstract polyhedra and directed graphs, termed alignments.

1. Introduction

As a class, the semisymmetric quasigroups arguably warrant particular interest due to both their algebraic and their combinatorial properties – commutative semisymmetric i.e. totally symmetric quasigroups have been an object of study for almost as long as quasigroups themselves [1]. There is a well-known bijection between idempotent totally symmetric quasigroups and the combinatorial block designs known as Steiner triple systems [2]; this further links totally symmetric quasigroups to finite geometry, as the Steiner triple system of order 7 is equivalent to the finite projective plane of order 2, and the Steiner triple system of order 9 is equivalent to the finite affine plane of order 3 [11]. Notably, via the semisymmetrization functor described by Smith [16], as well as the similar Mendelsohnization functor described by Krapež and Petrić [12], [17], it is possible to reduce homotopisms between arbitrary quasigroups to homomorphisms between semisymmetric quasigroups.

In this paper, we first lay groundwork by establishing a novel bijection between totally symmetric quasigroups and directed graphs meeting certain specifications. There have been several graph theoretic approaches applied to the study of quasigroups in the past [3], [9]; the main advantages of the schema implemented here are that the diagrams remain relatively simple, yet we are still able to fully recover the structure of any given (totally symmetric) quasigroup from its associated directed graph, even such that new quasigroups can be constructed starting only with a set of rules for constructing digraphs. Then, we expand this result to demonstrate a link between semisymmetric quasigroups and abstract polytopes, which are a combinatorial generalization of more traditional, geometric polytopes [5], [13].

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Specifically, we demonstrate a bijection between semisymmetric quasigroups and objects we will refer to as *alignments*, which represent mappings between abstract polyhedra and directed graphs. Likewise, up to isomorphism the full structure of a semisymmetric quasigroup will be shown to be recoverable from its associated alignment and vice versa.

2. Preliminaries

A *partial quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $(\cdot)$ such that for some $a, b \in Q$ there exist (at most) unique elements $x, y \in Q$ such that $a \cdot x = b, y \cdot a = b$; if this relation is satisfied for all $a, b \in Q$, then it is *complete* or simply a *quasigroup* [2]. For brevity, we will denote $x \cdot y$ by juxtaposition xy. An *isomorphism* between partial quasigroups is a bijection $f : Q \to Q'$ such that $f(x) \cdot f(y) = f(xy)$ for all $x, y \in Q$, in which case Q and Q' are said to be *isomorphic*.

Given a quasigroup $(Q, \cdot)$, it is possible to define 5 conjugate or *parastrophic* operations [6], [15] such that:

$$x \cdot y = z \Leftrightarrow z/y = x \tag{1}$$
$$x \cdot y = z \Leftrightarrow x\backslash z = y \tag{2}$$
$$x \cdot y = z \Leftrightarrow y \circ x = z \tag{3}$$
$$x \cdot y = z \Leftrightarrow y//z = x \tag{4}$$
$$x \cdot y = z \Leftrightarrow z\backslash\backslash x = y \tag{5}$$

If Q satisfies any of the equivalent [16] identities:

$$y \cdot xy = x \tag{6}$$
$$yx \cdot y = x \tag{7}$$
$$x/y = yx \tag{8}$$
$$x\backslash y = yx \tag{9}$$

then it is said to be *semisymmetric*. If Q is both semisymmetric and commutative, then it is *totally symmetric*, abbreviated as a *TS-quasigroup*. Equivalently, Q is totally symmetric iff all of its parastrophic operations coincide with one another.

A *partial Steiner triple system of order n* is a pair (V, B) where V is an n-element set and B is a set of 3-element subsets of V, referred to as *Steiner triples*, where any 2-element subset of V is contained in at most 1 triple. A partial Steiner triple system is *complete* if every 2-element subset of N is contained in exactly 1 triple in B, in which case it is referred to as simply a *Steiner triple system* [2].

A *cyclic order* on 3 elements is a ternary relation θ such that for distinct elements x, y, z then $\theta(x, y, z) \Leftrightarrow \theta(z, x, y) \Leftrightarrow \neg\theta(z, y, x)$ [7]. We call a pair of cyclic orders of the form $\theta_1(x, y, a), \theta_2(y, x, b)$ *partial opposites*; that is, to say, they share $\geqslant 2$ common elements which are in reversed order in regards to each other. If partial opposites share all 3 elements, then they are simply *opposites*. The scope of this paper is limited to cyclic orders on 3 elements, and so we need not consider cyclic orders on larger sets.

A *partial Mendelsohn triple system* (W, C) is a generalization of a Steiner triple system where W is a set and C is set of 3-element subsets of W with some cyclic order, referred to as *Mendelsohn triples*, such that $(\{x, y, z\}, \theta) = (x, y, z)$ contains the ordered pairs (x, y),(y, z),(z, x), and no others. Likewise, any ordered pair of distinct elements $(x, y) : x, y \in W$ can be contained in at most 1 triple in C; if every possible ordered pair of distinct elements in W is contained in exactly 1 triple in C, then the system is *complete* and simply a *Mendelsohn triple system* [3].

A *multiset* is a generalization of a set allowing for multiple instances of each element. Similarly, an *extended Steiner system of order* n is a pair (V, B) where V is an n-element set and B is a set of 3-element submultisets of V, called *extended Steiner triples* wherein each 2-element multisubset of V is contained in exactly 1 extended Steiner triple. An *extended Mendelsohn system* is a pair (W, C) where W is a set and C is a set of *extended Mendelsohn triples* such that any ordered pair of not necessarily distinct elements $(a, b) : a, b \in M$ is contained within exactly 1 triple in C. That is to say, extended Steiner and Mendelsohn triple systems are simply triple systems that allow for the repetition of elements [3]. From hereon, we will assume all Steiner and Mendelsohn systems are extended, and as such we can safely use just *triples* and *triple systems* when there is no chance of confusion. Cyclic orders also extend to multisets – note that any cyclic order of the form $\theta(x, x, y)$ or $\theta(x, x, x)$ is opposite to itself.

Suppose some graded partially ordered set $(P, \leqslant)$ with strictly monotone rank function $\rho : P \to \{-1, 0, 1, 2, ..., n\}$ sending elements $f_i \in P$, called *faces*, to integer values such that there is some unique least face f_{-1} and some unique greatest face f_n such that $\rho(f_{-1}) = -1$ and $\rho(f_n) = n$. Faces of rank n are *n-faces* – we call 0-faces *vertices* and 1-faces *edges*. Faces f_1, f_2 are *incident* if $f_1 \leqslant f_2$ or $f_2 \leqslant f_1$. Any maximal totally ordered subset $F_i \subset P$ is a *flag*; each flag contains exactly $n + 2$ faces. 2 flags are *adjacent* if they differ by exactly 1 face. P is *strongly flag-connected* if for any 2 flags F_x, F_y in P, there is some sequence of flags $(F_0, F_1, ..., F_n)$ such that any 2 successive F_i, F_{i+1} are adjacent to each other, where $F_x = F_0, F_y = F_n$ and $F_x \cap F_y \subseteq F_i$ for all i. If for any pair of faces $f_x \leqslant f_z$ in P where $\rho(f_x) = i-1, \rho(f_z) = i+1$, there are exactly 2 faces f_{y1}, f_{y2} such that $f_x \leqslant f_{y1,2} \leqslant f_z$ and $\rho(f_{y1,2}) = i$, then P is said to satisfy the *diamond condition*; that is to say, any pair of incident faces that differ in rank by 2 have exactly 2 incident faces strictly between them.

A graded poset $(P, \leqslant)$ is an *abstract n-polytope* [5], [13], [14] if it has a unique least face of rank -1 and a unique greatest face of rank n, is strongly flag-connected, all flags contain exactly $n + 2$ faces, and it satisfies the diamond condition. An abstract 3-polytope is an *abstract polyhedron*. We will call a polyhedron *cubic* if its graph is 3-regular – that is to say, each vertex is incident to exactly 3 edges.

An *automorphism* on an abstract polytope P is an order-preserving bijection

$\varphi : P \to P$. From hereon, all polytopes will be assumed to be abstract and all quasigroups will be assumed to be finite.

3. Totally symmetric quasigroups and digraphs

3.1 Constructing didgraphs from quasigroups

There is a natural bijection between Steiner triple systems and totally symmetric quasigroups given by $S : Q \to S(Q)$ where Q is some partial TS-quasigroup and $S(Q) = (V, B)$ is the partial Steiner system over the same underlying set such that for $x, y, z \in Q$ then $\{x, y, z\} \in B$ if and only if $xy = z, yx = z, xz = y$. We will refer to partial Steiner systems as *isomorphic* to each other iff their corresponding partial quasigroups are isomorphic to each other, and likewise for individual Steiner triples.

Lemma 3.1. *There are exactly* 3 *isomorphism classes of Steiner triples: triples of the form* $\{x, x, x\}$ (*type* 1)*, of the form* $\{x, x, y\}$ (*type* 2)*, and of the form* $\{x, y, z\}$ (*type* 3)*, where* $x \neq y \neq z$.

Proof. Any 2 triples $\{x, x, x\}, \{a, a, a\}$ are isomorphic by $\varphi(x) = a, \varphi(a) = x$. Any 2 triples $\{x, x, y\}, \{a, a, b\}$ are isomorphic by $\varphi(x) = a, \varphi(a) = x, \varphi(y) = b, \varphi(b) = y$. Any 2 triples $\{x, y, z\}, \{a, b, c\}$ are isomorphic by $\varphi(x) = a, \varphi(a) = x, \varphi(y) = b, \varphi(b) = y, \varphi(z) = c, \varphi(c) = z$. No isomorphism between triples of different types is possible because any mapping would necessarily either map unique values x, y to the same value a or map the a single value x to different values a, b. □

A partial triple system can be constructed through the union of any 2 triples with less than 2 elements in common. Necessarily then, said triples must either have exactly 1 element in common, in which case we will refer to them as *intersecting*, or they have no elements in common, making them *disjoint*. If 2 triples t_1, t_2 are intersecting such that t_1 has more instances of the intersecting element than t_2, we will say that t_2 *binds* to t_1 e.g. $\{1, 2, 3\}$ binds to $\{1, 1, 4\}$.

Proposition 3.2. *A partial Steiner triple system is uniquely determined up to isomorphism by the types of its constituent triples and the intersection between them.*

Proof. Given partial triple systems (V_1, B_1) where $B_1 = \{\{x_1, y_1, z_1\}, \{a_1, b_1, c_1\}\}$ and (V_2, B_2) where $B_2 = \{\{x_2, y_2, z_2\}, \{a_2, b_2, c_2\}\}$ there exists an isomorphism $\varphi(x_1) = x_2, \varphi(x_2) = x_1, \varphi(a_1) = a_2, \varphi(a_2) = a_1$ et cetera iff $\forall d_1, e_1 \in \cup B_1 \exists d_2, e_2 \in \cup B_2((d_1 = e_1) \Rightarrow (d_2 = e_2))$. This process can be continued inductively for the union of triple systems of arbitrarily greater (finite) order. □

Corollary 3.3. *Any given totally symmetric quasigroup is uniquely determined up to isomorphism by the types of its corresponding triples and the intersection between them.*

In light of this, we can devise a schema to represent totally symmetric quasigroups as directed graphs: for given partial totally symmetric quasigroup Q, let $D : Q \to D(Q)$ take it to the directed graph $D(Q)$ such that for every Steiner triple $t_i \in S(Q)$ there is exactly 1 vertex $v_i \in D(Q)$ and where for any $t_1, t_2 \mapsto v_1, v_2$ then v_1 directly succeeds v_2 if and only if t_2 binds to t_1. For example, given an example quasigroup Q_4 of order 4 with the Cayley table:

	1	**2**	**3**	**4**
1	1	2	4	3
2	2	1	3	4
3	4	3	2	1
4	3	4	1	2

we can derive the corresponding triples: $\{1,1,1\}, \{2,2,1\}, \{3,3,2\}, \{4,4,2\}, \{1,3,4\}$, producing the directed graph:

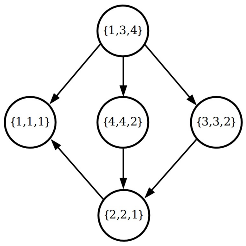

Figure 1: Labeled digraph of Q_4

The labels in figure 1 are purely for illustrative purposes; the final, unlabeled digraph is:

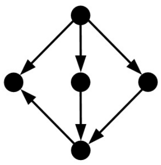

Figure 2: Unlabeled digraph $D(Q_4)$

We will refer to vertices in $D(Q)$ as being of the same *type* as the triples in $S(Q)$ they correspond to e.g. a type 1 vertex represents some triple of the form $\{x, x, x\}$. In general, if there is little chance for confusion we will use the same terminology between vertices in $D(Q)$ and the triples in $S(Q)$ which they represent.

Proposition 3.4. *Up to isomorphism, the full structure of any TS-quasigroup Q can be recovered from its directed graph D(Q).*

Proof. It is clear from the definition of an extended Steiner triple system that in any complete system (V, B) each element of its underlying set $x \in V$ must occur in exactly 1 triple either of the form $\{x, x, x\}$ or of the form $\{x, x, y\}$. It follows then that for given triples t_1, t_2 the only possible case in which t_2 can contain less instances of some shared element $x \in t_1, t_2$ is if t_2 contains exactly 1 instance of x and t_1 contains either 2 or 3 instances of x. That is to say, a given triple binds exactly once for each element it contains exactly 1 instance of. Therefore, the type of triple each vertex represents can be inferred from its outdegree: vertices with outdegree 0 map to type 1 triples, outdegree 1 to type 2 triples, and outdegree 3 to type 3 triples.

Given the digraph $D(Q)$, once the type of each vertex is identified, we may arbitrarily assign some bijective mapping between the type 1 and 2 vertices of the digraph and the elements of Q; that is to say, we label each type 1 and 2 vertex with a unique element of Q. Now, each vertex can be mapped to some triple as follows: type 1 vertices with label x are sent to $\{x, x, x\}$, type 2 vertices with label x binding to some vertex with label y are sent to $\{x, x, y\}$, type 3 vertices binding to some vertices with labels x, y, z (respectively) are sent to $\{x, y, z\}$. The union of these triples forms a triple system and thus a totally symmetric quasigroup. For example:

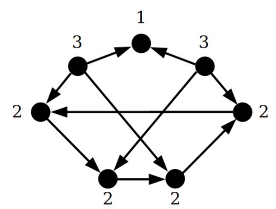

Figure 3: The type of each vertex in example diagram $D(Q_5)$

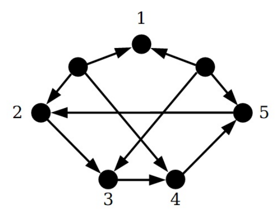

Figure 4: Arbitrary labeling of type 1 and type 2 vertices of $D(Q_5)$

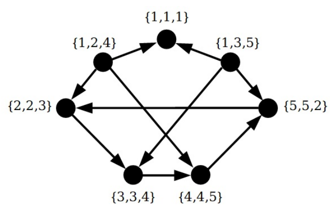

Figure 5: Deriving the corresponding triples for each vertex of $D(Q_5)$

Our choice in assigning type 1 and 2 vertices to elements of Q does not matter, because the type of each triple and the intersection between them are preserved and so by Corollary 3.3 any quasigroup produced by this method will be isomorphic to Q. In fact, every quasigroup isomorphic to Q on the same underlying set can be produced via permutations on the labels of the type 1 and 2 vertices of its digraph $D(Q)$. □

Corollary 3.5. *Every automorphism of a given TS-quasigroup Q corresponds to some graph isomorphism between permutations of labelings on the type 1 and 2 vertices of its directed graph $D(Q)$.*

3.2 Constructing quasigroups from digraphs

A complete extended Steiner triple system of order n contains:

$$\binom{n+2-1}{2} = \frac{1}{2}n(n+1) \tag{10}$$

(unordered) pairs of elements. As shown by Johnson and Mendelsohn in Section 3 of [8], given a triple system of order n, fixing the number of triples of any type also fixes the number of triples of each of the other 2 types. More specifically, where i is the number of type 1 triples, the number of type 3 triples must be equal to:

$$\frac{\frac{1}{2}n(n+1) - (i + 2(n-i))}{3} = n^2/6 - n/2 + i/3 \tag{11}$$

and therefore the number of type 1 triples i in a given triple system of order n must be such that:

$$3 \mid \frac{1}{2}n^2 - \frac{3}{2}n + i \tag{12}$$

A given element of a quasigroup $x \in Q$ such that $xx = x$ is called an *idempotent element* or simply an *idempotent* [3]; a quasigroup wherein all elements are idempotent is an *idempotent quasigroup.* It is readily apparent that each type 1 triple in a Steiner system specifies an element of its corresponding TS-quasigroup to be idempotent, and that each type 2 triple specifies an element not to be idempotent. By definition:

$$xx = y \Leftrightarrow xy = x \Leftrightarrow yx = x \tag{13}$$

and so these triples define not only the squares for each element $x^2 = y$ but also the local identities for each element $xy = x$. Let us define the subset: $U = \{y \in Q \mid x^2 = y, x \in Q\}$ as the *unique squares* of Q. On $D(Q)$, the unique squares correspond to the type 1 vertices together with the type 2 vertices which have at least 1 other type 2 vertex bound to them – this is equivalent to saying the unique squares are the elements that are either their own squares or the square of some other element.

Lemma 3.6. *For a TS-quasigroup of odd order n, $|U| = n$; all elements are unique squares.*

Proof. For elements of a TS-quasigroup $x, y, z \in Q$, by definition $xy = z \Leftrightarrow xz = y$. Then for any fixed x, we can define an involution $\varphi : Q \to Q$ sending $y \mapsto xy$. If n is odd, because φ is an involution there then must be some element z for which $\varphi(z) = z$ i.e. $xz = z$. Because Q is a quasigroup, there can be no y such that $xz = z, yz = z, x \neq y$; that is to say, if x acts as a local identity element for z, then it must be the only identity element for z. There being exactly n elements in Q, if some x were to act as an identity for more than 1 element, then there must be some y that cannot be an identity for any element – but as we established, every element of Q must be an identity for some other element. Therefore, each z maps uniquely to some local identity x, or alternatively, every element x is the unique square of some z. □

In informal terms, every row and column of the Cayley table for Q is some involution on the underlying set of Q, which means each row can be represented as the product of disjoint transpositions, but because n is odd for any row x there always must be some cell left over that cannot be swapped with any other cell. This defines the local identity for x and thus it also defines x^2; this must be unique because if another row had the same local identity for x there would be multiple instances of the same element in a single column.

Corollary 3.7. *For any TS-quasigroup Q of odd order, all type 2 vertices in $D(Q)$ are partitioned into cycles of length $\geqslant 3$.*

Proof. If all elements are unique squares, then each type 2 vertex must have at least 1 other type 2 vertex bound to it. Given that type 2 vertices have outdegree 1, they all must bind to other type 2 vertices, else there necessarily would be some type 2 vertex left over with no type 2 vertex bound to it. Assuming the number of vertices is finite, they will therefore be partitioned into cycles. There can be no 2-cycles as that would imply $\{\{x, x, y\}, \{y, y, x\}\}$, thus the pair $\{x, y\}$ would occur in more than 1 triple. □

Lemma 3.8. *For a TS-quasigroup of even order n, $1 \leqslant |U| \leqslant n/2$.*

Proof. As above, on TS-quasigroup Q we define an involution $\varphi : y \mapsto xy$ for some fixed x where $x, y \in Q$. If n is even, because φ is an involution for every y such that $\varphi(y) = y$ there must also be another distinct element $z \in Q$ where $\varphi(z) = z$; that is, any x must act as a local identity for an even number of elements in Q (0 being even). Conversely, every x must be the square of an even number of elements. It follows then that the maximum possible number of unique squares is $n/2$; trivially, there must be at least 1 unique square. □

Informally, because n is even there cannot be an odd number of unswapped cells in a given row of the Cayley table for Q.

Corollary 3.9. *In the digraph $D(Q)$ for a TS-quasigroup Q of even order, every type 1 vertex must have an odd number of type 2 vertices bound to it and every* *type* 2 *vertex must have an even number of type* 2 *vertices bound to it* (0 *being even*).

To summarize, for a TS-quasigroup Q of order n: the number of type 1 vertices i must be such that $3 \mid \frac{1}{2}n^2 - \frac{3}{2}n + i$. The number of type 2 vertices must be $n - i$. If n is odd, the type 2 vertices are partitioned into cycles of length $\geqslant 3$. If n is even, every type 1 vertex must have an odd number of type 2 vertices bound to it and every type 2 vertex must have an even number of type 2 vertices bound to it. We will refer to a given configuration of type 1 and 2 vertices meeting the aforementioned specifications as a *diagonal subgraph*.

Proposition 3.10. *For any TS-quasigroup Q, $D(Q)$ contains a diagonal subgraph as an induced subgraph. Further, up to isomorphism every diagonal subgraph can be mapped to some unique partial TS-quasigroup.*

Proof. By Corollaries 3.7 and 3.9, the induced subgraph containing only the type 1 and type 2 vertices of the digraph of a TS-quasigroup will always be a diagonal subgraph. Using the method specified in Proposition 3.4, we can always produce a partial Steiner system and therefore a partial TS-quasigroup with any arbitrary labeling of the vertices bijective with some set. Because this method preserves the types of triples and the intersections between them, by Corollary 3.3 this partial quasigroup is unique up to isomorphism for each unique diagonal subgraph. Triples in a diagonal subraph are all either of the form $\{x, x, x\}$ or $\{x, x, y\}$, and thus the only way for a given pair to show up more than once would be to label more than 1 vertex with the same element, which goes against the definition. □

However, not every diagonal subgraph can be made into a complete TS-quasigroup. There must be $n^2/6 - n/2 + i/3$ type 3 triples in a complete Steiner system, and each corresponding type 3 vertex must bind to exactly 3 type 1 or type 2 vertices. Further, no 2 type 3 vertices may bind to more than 1 shared vertex, as this would imply 2 triples that shared more than 1 common element. Finally, no type 3 vertex may bind to 2 vertices a, b where a is bound to b; this would imply some $\{\{x, y, z\}, \{x, x, y\}, \{y, y, w\}\}$ and thus the pair $\{x, y\}$ is contained in more than 1 triple. A directed graph composed (solely) of a diagonal subgraph and a set of type 3 vertices meeting the aforementioned specifications is *complete*.

Theorem 3.11. *Up to isomorphism, there exists a bijection between complete digraphs and totally symmetric quasigroups such that the full structure of a unique totally symmetric quasigroup can be recovered from any complete digraph and vice versa.*

Proof. Given any diagonal subgraph and some bijective labeling from some set to the vertices, it is readily apparent that any completion via the addition of bound type 3 vertices is equivalent to the specification of a set of triples, each containing exactly 3 distinct elements of the set. If any 2 of these type 3 triples shared more than 1 common element between them, they would necessarily bind to more than

1 shared vertex and thus violate the definition of a complete digraph. If any of these type 3 triples shared more than 1 common element with some type 2 triple, it would also necessarily bind to the triple said type 2 binds to and thus violate the definition of a complete digraph. Clearly, a type 3 triple cannot share more than 1 common element with a type 1 triple. There being $n^2/6 - n/2 + i/3$ type 3 triples ensures by the pigeonhole principle that every possible pair of elements of the set is accounted for in some triple. By Corollary 3.3, any 2 digraphs corresponding to isomorphic quasigroups are necessarily isomorphic to each other. By Proposition 3.4, every totally symmetric quasigroup corresponds to a directed graph, and thus the bijection is complete. □

Corollary 3.12. *Every subquasigroup of any TS-quasigroup Q appears as an induced subgraph of $D(Q)$.*

The methodology described here for constructing digraphs from TS-quasigroups is compatible with that of Khatirinejad et al. in [10] for constructing digraphs from Mendelsohn triple systems, which are equivalent to idempotent, semisymmetric quasigroups [17]. Specifically, given any idempotent TS-quasigroup Q, we can construct a Khatirinejad et al. digraph from $D(Q)$ by replacing each type 3 vertex with a set of 6 vertices arranged into 2 cyclically ordered triangles (as each Steiner triple is equivalent to 2 Mendelsohn triples).

Remark 3.13. There is known to exist a bijection between idempotent TS-quasigroups of order n and TS-quasigroups of order $n+1$ with a (global) identity element [4]. This can be represented graphically as follows: given the digraph of some idempotent, TS-quasigroup, add 1 additional type 1 vertex, then bind every other type 1 vertex to the added vertex, converting them to type 2 vertices.

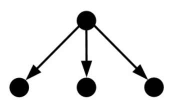

Figure 6: Example idempotent quasigroup Q_3

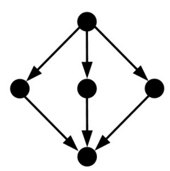

Figure 7: Derived quasigroup with identity V_4 (the Klein 4-group)

Note that 1 of the arrows in Figure 2 is in the opposite orientation to that of its counterpart in Figure 7, distinguishing Q_4 and V_4 as nonisomorphic quasigroups.

4. Quasigroups and abstract polyhedra

4.1 Constructing polyhedra from quasigroups

Similarly to Steiner systems and totally symmetric quasigroups, there exists a natural bijection between Mendelsohn triple systems and semisymmetric quasigroups given by $M : Q \to M(Q)$ where Q is a given partial semisymmetric quasigroup and $M(Q) = (W, C)$ is the partial Mendelsohn system over the same underlying set such that for elements $x, y, z \in Q$ then $(x, y, z) \in C$ if and only if $xy = z, yz = x, zx = y$; note that because semisymmetric quasigroups are not necessarily commutative, this does not necessarily imply $yx = z, zy = x, xz = y$.

Lemma 4.1. *There exist exactly* 3 *isomorphism classes of extended Mendelsohn triples.*

Proof. The same reasoning applied to Steiner systems in Lemma 3.1 equally applies to Mendelsohn systems. □

Indeed, type 1 and type 2 Mendelsohn triples behave similarly to their Steiner counterparts in that they specify squares and local identities and are also commutative: type 1 triples (x, x, x) trivially imply $xx = x$, type 2 triples (x, x, y) imply $xx = y, yx = x, xy = x$. Type 3 Mendolsohn triples, however, have a more complex structure in that $(x, y, z) \neq (z, y, x)$. As such, we will need to devise a new schema to represent type 3 Mendelsohn triples.

For given partial semisymmetric quasigroup Q, let $G : Q \to G(Q)$ take it to the (undirected) multigraph $G(Q)$ such that for every type 3 Mendelsohn triple $t_i \in M(Q)$ there is exactly 1 vertex $v_i \in G(Q)$ and where for any $t_1, t_2 \mapsto v_1, v_2$ then there is exactly 1 edge linking v_1 to v_2 for every pair of elements t_1 and t_2 have in common. Thus, 2 vertices are adjacent if and only if the triples they represent share at least 2 elements in common e.g. $(1, 2, 3)$ is adjacent to $(2, 1, 4)$ but not to $(1, 5, 6)$. As above, we will use the same terminology between vertices in $G(Q)$ and the triples they represent in $M(Q)$ when expedient.

To illustrate, from an example semisymmetric quasigroup Q_{4s} with Cayley table:

	1	**2**	**3**	**4**
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

we can derive 4 type 1 triples $\{(1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4)\}$ and 4 type 3 triples $\{(1, 2, 3), (1, 3, 4), (1, 4, 2), (2, 4, 3)\}$. This would produce the graph:
or unlabeled:

For given semisymmetric quasigroup Q, let us define a relation $\rightleftharpoons$ on the type 3 triples of $M(Q)$ such that $a \rightleftharpoons b$ for $a, b \in M(Q)$ if and only if their corresponding

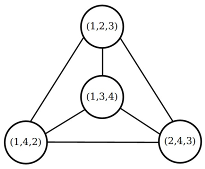

Figure 8: Labelled graph of Q_{4s}

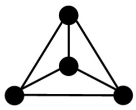

Figure 9: Unlabeled graph $G(Q_{4s})$

vertices in $G(Q)$ are connected. Because connectivity is reflexive, symmetric, and transitive, $\rightleftharpoons$ is then an equivalence relation; we will refer to the partial quasigroups corresponding to the equivalence classes of type 3 triples in $M(Q)$ under $\rightleftharpoons$ as the *components* of Q. A partial quasigroup q such that any $t_1, t_2 \in M(q)$ are type 3 triples corresponding to vertices of degree 3 in $G(q)$ and $t_1 \rightleftharpoons t_2$ we will call a *free component*. That is to say, a free component q is a partial quasigroup composed of type 3 triples where $G(q)$ is connected and where adding any further type 3 triple to $M(q)$ would make $G(q)$ disconnected.

Lemma 4.2. *Given a complete semisymmetric quasigroup Q, $G(Q)$ will be 3-regular; further, $G(q)$ will be 3-regular for every component q of Q.*

Proof. Each type 3 Mendelsohn triple contains exactly 3 ordered pairs of elements, and because Q is complete then for each ordered pair (x, y) in a type 3 triple there also must be some triple containing (y, x). If there were some type 2 triple containing (y, x), then necessarily (y, x, x) or (y, y, x), which would make (x, y) appear in more than 1 triple, and trivially no type 1 triple can contain (y, x), so (y, x) must be contained in some other type 3 triple, which will be adjacent by definition. Therefore, every vertex must be incident to exactly 3 edges, each edge corresponding to an unordered pair $\{x, y\}$. By definition any vertices in $G(Q)$ connected to any vertex in $G(q)$ of any component q are also within $G(q)$, thus $G(q)$ for every component of Q must also be 3-regular. $\square$

Corollary 4.3. *Every component of a complete semisymmetric quasigroup is isomorphic to some free component.*

In some cases, $M(Q)$ may contain triples of the form $\{(x, y, z), (z, y, x)\}$, that is to say, pairs of triples containing the same elements but in opposite order; we will call these *commutative pairs*. In $G(Q)$, these pairs correspond to the multigraph:

Figure 10: Multigraph of a commutative pair

Remark 4.4. A semisymmetric quasigroup is totally symmetric if and only if all of its components are commutative pairs.

Lemma 4.5. *For any free component q, if q is not a commutative pair, then $G(q)$ is a simple graph.*

Proof. By definition, any vertex $v \in G(q)$ must have 3 incident edges. If all edges connect to 1 other vertex, then their corresponding triples in $M(q)$ have all 3 pairs of elements in common and thus q is a commutative pair. If v were linked to some other vertex by exactly 2 edges, this would imply there are 2 triples that have 2 pairs of elements in common, but not the 3rd, which is clearly combinatorially impossible. Then if q is not a commutative pair, any $v \in G(q)$ will have 3 edges linking to 3 separate vertices, thus $G(q)$ is a simple graph. □

For a given free component q, let a cycle $c_x \in G(q)$ be an *element-cycle* for x iff for every vertex in c_x, its corresponding triple in $M(q)$ contains x. Define a *cycle structure* on q to be a surjection $C : G(q) \to q$ sending each element-cycle in $G(q)$ to an element of q such that if $c_x \mapsto x$ then c_x is an element-cycle for x.

Lemma 4.6. *For any commutative pair q, up to isomorphism there exists exactly* 1 *cycle structure on q.*

Proof. All vertices in $G(q)$ represent triples in $M(q)$ containing all elements of q, so all cycles qualify as element-cycles. There are 3 elements of q and there are 3 cycles in $G(q)$, so any surjection must assign 1 cycle to each element. $G(q)$ is vertex transitive and edge transitive, therefore any such assignment will be equivalent up to isomorphism. □

Proposition 4.7. *For any free component q, if q is not a commutative pair, then there exists exactly* 1 *cycle structure on q.*

Proof. For a given triple $t_1 = (x, y, z) \in M(q)$, consider an element x; by definition, $G(q)$ is 3-regular, therefore there exist edges linking t_1 to vertices containing (y, x) and (x, z). $G(q)$ is simple, therefore these edges link to distinct vertices $t_2 = (y, x, a)$ and $t_3 = (x, z, b)$ where $a \neq b$. The 3rd edge must link to some vertex containing (z, y), and this vertex cannot contain x, else the pairs (y, x) or (x, z) would appear in more than 1 triple. Now, t_2 must be adjacent to t_1, some vertex $t_4 = (x, d, a)$, and some 3rd vertex which also cannot contain x else (x, a) or (a, x) would appear in more than 1 triple. Likewise, t_3 is adjacent to t_1, some vertex $t_5 = (x, b, e)$, and a 3rd vertex not containing x. So then t_4 must be adjacent to some vertex containing (d, x), and t_5 must be adjacent to a vertex containing

(x, e), and so on. Assuming the number of triples and therefore vertices is finite, there must eventually be some vertex (x, e, d) linking these 2 trails into a closed cycle c_x.

All vertices in c_x contain x, so then c_x is an element-cycle for x; thus for any triple in $M(q)$ and any element contained in that triple, there exists an element-cycle in $G(q)$ for that element. Further, as demonstrated, any vertex adjacent to a vertex in c_x which is not contained in c_x cannot contain x, so c_x is the only possible element cycle for x for any vertex in c_x. If there were some element $f \in q$ such that c_x was also an element cycle for f, then there would be multiple triples containing (x, f) or (f, x). Therefore, any cycle structure C has only 1 possible mapping from cycles to elements. By definition, any element in q must be represented in some vertex of $G(q)$, so then C is a surjection. □

Given that for any free component q there always exists a cycle structure on $G(q)$ unique up to isomorphism for commutative pairs and fully unique for simple $G(q)$, from hereon we can safely assume the cycle structure on any free component. It is therefore meaningful to speak of *the* element-cycles of a given q.

Corollary 4.8. *Each vertex of $G(q)$ is contained within exactly 3 element-cycles.*

Lemma 4.9. *For some free component q, any 2 element-cycles in $G(q)$ either share exactly 2 common vertices that are adjacent to each other, or they share no common vertices.*

Proof. Given graph $G(q)$ containing element-cycles c_x, c_y for elements $x, y \in q$, if they share a common vertex it must be representative of some triple containing the pair (x, y) or the pair (y, x). The existence of a triple containing (x, y) necessarily implies the existence of some triple containing (y, x) and vice versa, and because they share 2 common elements by definition they are adjacent. There cannot be any more triples containing (x, y) or (y, x) and thus there are no more common vertices shared by c_x and c_y. □

Lemma 4.10. *For some free component q, each edge in $G(q)$ is contained within exactly 2 element-cycles.*

Proof. By definition, every edge in $G(q)$ links 2 vertices representing triples containing 2 shared elements, and by Proposition 4.7 there can be no adjacent vertices sharing a common element not contained within a shared element-cycle. An edge cannot be in more than 2 element-cycles for any graph with > 2 vertices because that would imply 2 triples sharing more than 2 common elements, and it cannot be in more than 2 element cycles for any graph with 2 vertices because that would necessitate a cycle with length > 2. □

Proposition 4.11. *The graph of any free component is isomorphic to the graph of some cubic abstract polyhedron.*

Proof. We use the work of Murty in [13]: Lemmas 4.9 and 4.10 satisfy Murty's Lemmas 2.2 (i) and (ii), therefore by Murty's Theorem 2.11, the graph of any free component satisfies the necessary and sufficient conditions to be that of a cubic abstract 3-polytope i.e. an abstract polyhedron, where each element-cycle is equivalent to some 2-face. □

Further, by Murty's Theorem 2.8, any 2 abstract polytopes with the same 2 dimensional skeleton are isomorphic, thus we can specify the polyhedron associated with any given free component via its element-cycles. Define $P : q \to P(q)$ taking some free component q to the cubic polyhedron $P(q)$ such that each element-cycle $c_i \in G(q)$ is sent to its equivalent 2-face in $P(q)$. For any cubic polyhedron p, we define a *labeling* on p to be a function $L : p \to X$ sending each 2-face of p to an element of some set X such that for every edge in p incident to 2-faces f_1, f_2, the (unordered) pair $\{L(f_1), L(f_2)\}$ is unique.

4.2 Constructing quasigroups from polyhedra

For quasigroup $(Q, \cdot)$ we will refer to the parastrophic quasigroup $(Q, \circ)$ such that $x \cdot y = z \Leftrightarrow y \circ x = z$ as the *transpose* of $(Q, \cdot)$; or alternatively, Q^T is the transpose of Q. A totally symmetric quasigroup and its transpose are exactly identical (indeed, this is true for any commutative quasigroup). By definition, a strictly semisymmetric quasigroup and its transpose are not identical, but sometimes they are isomorphic. This is somewhat problematic, as heretofore our procedure cannot distinguish between a semisymmetric quasigroup and its transpose – both will produce the same graph, even if they are not isomorphic to each other. We must devise a way to differentiate between parastrophes, but also a way to identify when they are essentially the same.

Conveniently, because we can now map components of semisymmetric quasigroups to polyhedra, we can also assign them an orientation. Define an *oriented vertex* to be the pair $\hat{v} = (v, \theta)$, where v is a vertex of some polyhedron p and θ is some cyclic order on the 2-faces incident to v, called an *orientation* on v. Let an *oriented polyhedron* be the pair $\hat{p} = (p, \Theta)$ where p is some cubic polyhedron and $\Theta : V \to \Theta(V)$ is a function on the vertices $V \subset P$ sending each vertex $v_i \mapsto \hat{v_i}$ to an oriented vertex such that the orientation for any $\hat{v_1}$ is a partial opposite that of any adjacent vertex $\hat{v_2}$. We will refer to Θ as an *orientation* on p.

Lemma 4.12. *There are at most* 2 *possible orientations on any given polyhedron* p*.*

Proof. Suppose we fix the orientation for some vertex $\hat{v_1}$ such that $\theta_1(f_1, f_2, f_3)$. Then any adjacent vertex $\hat{v_2}$ sharing incident 2-faces f_1, f_2 must be partial opposite such that $\theta_2(f_2, f_1, -)$, and likewise for all other adjacent vertices. So fixing a single vertex therefore fixes all connected vertices, and since all vertices in p are connected and there are only 2 possible cyclic orders on a set of 3 elements, there are at most 2 possible orientations on p. □

Proposition 4.13. *Given some oriented polyhedron $\hat{p}$, any labeling on $\hat{p}$ specifies a unique free component q_p.*

Proof. A labeling on $\hat{p}$ identifies each 2-face with some set element such that every edge is incident to a unique pair of elements, and each oriented vertex $\hat{v_i} \in \hat{p}$ specifies a cyclic order on its incident 2-faces, thus each $\hat{v_i}$ specifies a cyclic order on 3 distinct set elements and is therefore equivalent to a type 3 Mendelsohn triple. By the definition of a cubic polyhedron, there are exactly 2 vertices $\hat{v_1}, \hat{v_2}$ incident to any pair of 2-faces $\{f_1, f_2\}$, and by the definition of an orientation on p then $\hat{v_1}$ and $\hat{v_2}$ must have opposite orientations relative to f_1 and f_2; therefore no ordered pair (f_1, f_2) occurs in $\hat{p}$ more than once. The graph of $\hat{p}$ is connected and 3-regular so necessarily the partial quasigroup q_p it defines is a free component. Any other polyhedron that defines the same q_p would necessarily have the same faces, labels, and orientation as $\hat{p}$ and thus be identical to $\hat{p}$; therefore q_p is unique. □

For given free component q, define $\hat{P} : q \to \hat{P}(q)$ as the function taking q to the oriented polyhedron $\hat{P}(q)$ such that for each triple $t_i \in M(q)$ is sent to a corresponding oriented vertex $t_i \mapsto \hat{v_i}$.

For given cubic polyhedron p, consider the action of its automorphism group $\mathrm{Aut}\,(p)$ on its 2-faces; let us denote the orbit of a 2-face f_i under this action as $\mathrm{Aut}\,(p) \cdot f_i$. Given any 2 vertices $v_1, v_2 \in p$ with incident 2-faces $\{f_1, f_2, f_3\}$ and $\{f_4, f_5, f_6\}$, respectively, then by definition if there is some $\varphi \in \mathrm{Aut}\,(p)$ sending $v_1 \mapsto v_2$ then necessarily

$$\{\mathrm{Aut}\,(p) \cdot f_1, \mathrm{Aut}\,(p) \cdot f_2, \mathrm{Aut}\,(p) \cdot f_3\} = \{\mathrm{Aut}\,(p) \cdot f_4, \mathrm{Aut}\,(p) \cdot f_5, \mathrm{Aut}\,(p) \cdot f_6\} \quad (14)$$

that is to say, for any vertices in the same orbit, the set of orbits of their incident 2-faces must also be the same. However, given some orientation on p, the order of incident 2-faces relative to $\hat{v_1}$ and $\hat{v_2}$ may be different. If some $\varphi \in \mathrm{Aut}\,(p) : v_1 \mapsto v_2$ and the orbits of the faces incident to corresponding oriented vertices $\hat{v_1}$ and $\hat{v_2}$ are in opposite order, we will call them *opposite vertices*. Any vertex which is opposite to itself is a *self-opposite vertex*.

Proposition 4.14. *A free component q is isomorphic to its transpose q^T if and only if there exists some automorphism $\varphi : P(q) \to P(q)$ taking every vertex in $\hat{P}(Q)$ to some opposite vertex.*

Proof. By definition, q and q^T are identical in all respects except for the order of the elements in their constituent triples in $M(q), M(q^T)$, so as the (unordered) sets of elements and their intersections are preserved, $P(q) = P(q^T)$ without some orientation to distinguish between them. Therefore $\hat{P}(q^T)$ is simply $\hat{P}(q)$ with its orientation reversed. If there exists some $\varphi \in \mathrm{Aut}\,(P(q))$ taking every vertex to some opposite, it follows that $\hat{P}(q)$ is isomorphic to itself with reversed orientation i.e. $\hat{P}(q^T)$; then by Proposition 4.13 q is isomorphic to q^T. □

Let $D : Q \to D(Q)$ take semisymmetric quasigroup Q to directed graph $D(Q)$ such that for every type 1 or 2 triple in $t_i \in M(Q)$ there is exactly 1 vertex $v_i \in D(Q)$ and where for any $t_1, t_2 \mapsto v_1, v_2$ then v_1 directly succeeds v_2 if and only if t_2 binds to t_1. That is to say, D applies to the type 1 and 2 triples of semisymmetric quasigroups in the same way it does for totally symmetric quasigroups; as above, the number of type 1 and 2 vertices is equal to $|Q|$. Let any digraph such that each vertex has outdegree $\leqslant 1$ be a *semisymmetric diagonal subgraph.*

For a given oriented polyhedron $\hat{p}$ and a given semisymmetric diagonal subgraph d, let $\psi : \hat{p} \to d$ be any function taking each 2-face of $\hat{p}$ to some vertex of d such that for every edge in $\hat{p}$ incident to 2-faces f_1, f_2, the (unordered) pair $\{\psi(f_1), \psi(f_2)\}$ is unique and $\psi(f_1)$ does not bind to $\psi(f_2)$ or vice versa.

Lemma 4.15. *Given some oriented polyhedron $\hat{p}$ and some semisymmetric diagonal subgraph d, any ψ_i from $\hat{p}$ to d specifies a unique partial semisymmetric quasigroup q up to isomorphism.*

Proof. Suppose some bijective mapping between the vertices of d and the elements of some set X – it is clear that this is equivalent to a labeling on $\hat{p}$ given by $L : \hat{p} \to X$ maps each face of $\hat{p}$ to an element of X iff ψ_i sends that face to the vertex in d mapped to X. Therefore by Proposition 4.13 we now have a unique free component, and we derive all type 1 and 2 triples from d in the same way as we did for TS-quasigroups to produce a unique partial semisymmetric quasigroup q. The derived type 3 triples in $M(q)$ are self-consistent by Proposition 4.13 and the type 1 and 2 triples are self-consistent by Proposition 3.10. Supposing, then, there were some pair (x, y) contained in a type 3 triple (x, y, a) and a type 2 triple (x, x, y) – necessarily there would then be some other type 3 triple (y, x, b) forming an edge in $\hat{p}$ incident to faces f_x, f_y such that $\psi_i(f_x) = (x, x, y), \psi_i(f_y) = (y, y, -)$, meaning $\psi_i(f_x)$ binds to $\psi_i(f_y)$, which would violate the definition of the ψ function. It follows then that for any ψ_j that specifies a quasigroup isomorphic to q then the image of $\hat{p}$ under ψ_j must be isomorphic to the image of $\hat{p}$ under ψ_i; therefore, the mapping ψ_i is unique up to isomorphism. □

Suppose some diagonal subgraph d; each vertex of d represents an element of some semisymmetric quasigroup Q, and for every element $x \in Q$ there must be $|Q|$ unordered pairs $\{x, y\}$ represented within $M(Q)$. Each type 1 triple contains 1 pair and each type 2 triple contains 2 pairs, so we shall say that a type 1 vertex starts with a *bound weight* of 1 and a type 2 vertex starts with a bound weight of 2. Every type 2 triple bound to a given vertex corresponds to another pair of elements, so we add +1 bound weight to a vertex for every other type 2 vertex bound to it. Finally, for each face of a polyhedron 1 pair is represented for every edge, so we add the number of edges mapped to a vertex in d to its bound weight.

Define an *alignment* to be the ordered triple (d, O, Ψ) where d is some semisymmetric diagonal subgraph, $O = \{\hat{p_1}, \hat{p_2}, ..., \hat{p_n}\}$ some set of oriented polyhedra, and $\Psi = \{\psi_1, \psi_2, ..., \psi_n\}$ some set of functions $\psi_i : \hat{p_i} \to d$ taking each 2-face of its respective $\hat{p_i} \in O$ to some vertex in d such that for every edge in $\hat{p_i}$ incident to 2-faces f_1, f_2, the unordered pair $\{\Psi^{-1}(\psi_i(f_1)), \Psi^{-1}(\psi_i(f_2))\}$ is unique, where $\Psi^{-1}(v_i) = \{f_x | \psi_x(f_x) = v_i\}$, that is to say Ψ^{-1} is the preimage of $v_i \in d$ across all $\psi_x \in \Psi$. Further, there is no v_1 binding to v_2 such that some face $f_1 \in \Psi^{-1}(v_1)$ shares an incident edge with some $f_2 \in \Psi^{-1}(v_2)$, and the total bound weight for each $v_i \in d$ across all of Ψ is equal to $|d|$, the number of vertices in d. We will call 2 alignments A_1, A_2 *isomorphic* iff their sets of polyhedra O_1, O_2 are isomorphic to each other and the image of Ψ_1 in d_1 is isomorphic to the image of Ψ_2 in d_2.

Theorem 4.16. *Up to isomorphism, there exists a bijection between alignments and semisymmetric quasigroups such that the full structure of a unique semisymmetric quasigroup can be recovered from any alignment and vice versa.*

Proof. Suppose some alignment $A = (d, O, \Psi)$: by Lemma 4.15 each $\psi_i \in \Psi$ yields a unique partial semisymmetric quasigroup, so then the union of these partial quasigroups also produces a semisymmetric quasigroup Q. Because the bound weight of each $v_i \in d$ is equal to $|d|$, every possible pair of elements in Q must be represented and therefore Q is complete. If there were 2 type 3 triples $t_1, t_2 \in M(Q)$ both containing some ordered pair of elements (x, y), then this would imply there are faces $f_{1-4} \in \cup O$ such that f_1, f_2 share an incident edge and f_3, f_4 share an incident edge and there are some $\psi_i, \psi_j \in \Psi$ where $\psi_i(f_1) = \psi_j(f_3), \psi_i(f_2) = \psi_j(f_4)$, but this would violate the definition of an alignment because for any edge in $\cup O$ the image of its pair of incident faces must be unique across all Ψ. If there were a type 3 triple t_1 and a type 2 triple t_2 in $M(Q)$ both containing some ordered pair of elements (x, y), then this would imply some faces $f_1, f_2 \in \cup O$ such that $\psi_i(f_1)$ binds to $\psi_j(f_2)$, which also violates the definition of an alignment. Any alignment that yields a quasigroup isomorphic to Q would necessarily have a set of oriented polyhedra isomorphic to O mapping to an image isomorphic to $\Psi(O)$ and therefore be equivalent to A, thus A corresponds to a unique Q up to isomorphism.

Conversely, suppose some semisymmetric quasigroup Q': the diagonal subgraph is given by $D(Q')$. For each component $q'_i \in Q'$, we can derive an oriented polyhedron $\hat{P}(q'_i)$; let the set of all such $\hat{P}(q'_i)$ be $\hat{P}(Q')$. Finally, ψ_i for each $\hat{P}(q'_i)$ is given by simply mapping each 2-face corresponding to an element $x \in Q'$ to the vertex in $D(Q')$ corresponding to x; let the set of all such ψ_i be $\Psi_{Q'}$. Now we can define function $\alpha : Q' \to A' = (D(Q'), \hat{P}(Q'), \Psi_{Q'})$ taking any given semisymmetric quasigroup Q' to a unique alignment A' up to isomorphism, thus, the bijection is complete. □

For example, given an alignment A_5 on a triangular prism:

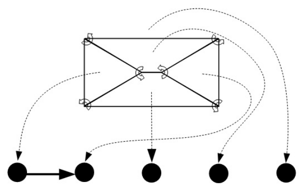

Figure 11: Diagram of alignment A_5

We can assign an arbitrary labeling to the type 1 and 2 vertices:

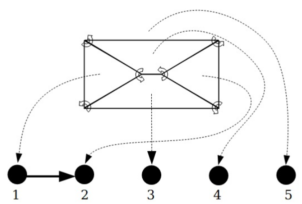

Figure 12: Arbitrary labeling on A_5

And derive the Mendelsohn triples corresponding to each vertex:

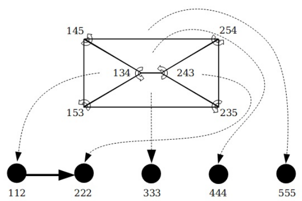

Figure 13: A_5 with derived triples

Yielding a semisymmetric quasigroup with the Cayley table:

	1	**2**	**3**	**4**	**5**
1	2	1	4	5	3
2	1	2	5	3	4
3	5	4	3	1	2
4	3	5	2	4	1
5	4	3	1	2	5

Remark 4.17. Any labeling on a triangular prism produces a free component isomorphic to its transpose, so in the previous example the orientations on the vertices could have been omitted, but we retain them for illustrative purposes.

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