

# Relative (pre-)anti-flexible algebras and associated algebraic structures

*Mafoya Landry Dassoundo*

**Abstract.** Pre-anti-flexible family algebras are introduced and used to define and describe the notions of  $\Omega_c$ -relative anti-flexible algebras, left and right pre-Lie family algebras and  $\Omega_c$ -relative Lie algebras. The notion of  $\Omega_c$ -relative pre-anti-flexible algebras are introduced and also used to characterize pre-anti-flexible family algebras, left and right pre-Lie family algebras and significant identities associated to these algebraic structures are provided. Finally, a generalization of the Rota-Baxter operators defined on an  $\Omega_c$ -relative anti-flexible algebra is introduced and it is also proved that both Rota-Baxter operators and its generalization provide  $\Omega_c$ -relative pre-anti-flexible algebras structures and related consequences are derived.

## 1. Introduction and preliminaries

Anti-flexible algebra, originally derived in the generalization of flexible algebras (algebras satisfy identity  $(xy)x = x(yx)$ ) leading to the introduction of several classes of nonassociative algebras ([23]), is a vector space  $A$  equipped with bilinear product “ $*$  :  $A \times A \rightarrow A$ ” satisfying, for any  $x, y, z \in A$ ,

$$(x * y) * z + z * (y * x) = (z * y) * x + x * (y * z), \tag{1a}$$

equivalently

$$(x, y, z) = (z, y, x), \tag{1b}$$

where,

$$(x, y, z) = (x * y) * z - x * (y * z), \tag{2}$$

is the associator of the bilinear product  $*$  :  $A \times A \rightarrow A$ . Anti-flexible algebras are also known as  $G_4$ -associative algebras ([15]) and center-symmetric algebras ([19]). Simplicity and semi-simplicity of anti-flexible algebras were investigated and characterized ([24]). Besides, simple and semisimple (totally) anti-flexible algebras over splitting fields of characteristic different to 2 and 3 were studied and classified in [6, 25, 26]. Moreover, the primitive structures and prime anti-flexible

---

2020 Mathematics Subject Classification: 17A30, 16W99, 05E16

Keywords: Rota-Baxter family operators,  $\Omega_c$ -relative Lie algebra, (pre)anti-flexible family algebra,  $\Omega_c$ -relative (pre)anti-flexible algebra

rings were investigated in [7] and it were established that a simple nearly anti-flexible algebra of characteristic prime to 30 satisfying the identity  $(x, x, x) = 0$  in which its commutator gives non-nilpotent structure possesses a unity element [10].

A Rota-Baxter operator, originally introduced in [4, 27], is a linear operator  $R_B : A \rightarrow A$  defined on an associative algebra  $(A, \cdot)$  and satisfying, for any  $x, y \in A$ ,

$$R_B(x) \cdot R_B(y) = R_B(R_B(x) \cdot y) + R_B(x \cdot R_B(y)). \quad (3)$$

It is well known from [2] that Rota-Baxter operator of weight zero on a given associative algebra induces a dendriform algebra (introduced by Loday in [20]) structures. More precisely, for a given linear map  $R : A \rightarrow A$  on an associative algebra  $(A, \cdot)$ , the two following bilinear products  $\prec, \succ : A \times A \rightarrow A$  given by, for any  $x, y \in A$ ,

$$x \succ y := R(x) \cdot y, \quad x \prec y := x \cdot R(y), \quad (4)$$

satisfy the following relations, for any  $x, y, z \in A$ ,

$$(x \succ y) \prec z - x \succ (y \prec z) = 0, \quad (5a)$$

$$(x \succ y + x \prec y) \succ z - x \succ (y \succ z) = 0, \quad (5b)$$

$$x \prec (y \succ z + y \prec z) - (x \prec y) \prec z = 0, \quad (5c)$$

if and only if  $R : A \rightarrow A$  is a Rota-Baxter operator of weight zero on  $A$ , that is,  $R$  satisfies Eq. (3). Similarly, from [8, 9], it is established that for a given a Rota-Baxter operator of weight zero defined on an anti-flexible algebra  $(A, *)$ , the bilinear products given by Eq. (4) satisfy, for any  $x, y, z \in A$ ,

$$(x \succ y) \prec z - x \succ (y \prec z) = (z \succ y) \prec x - z \succ (y \prec x), \quad (6a)$$

$$(x \succ y + x \prec y) \succ z - x \succ (y \succ z) = z \prec (y \succ x + y \prec x) - (z \prec y) \prec x, \quad (6b)$$

and the algebra  $(A, \prec, \succ)$  is known as pre-anti-flexible algebra. More generally, for a given linear map  $G_{RB} : A \rightarrow A$  defined on an anti-flexible algebra  $(A, *)$  and considering the following bilinear products given by, for any  $x, y \in A$ ,

$$x \succ' y := G_{RB}(x) * y, \quad x \prec' y := x * G_{RB}(y), \quad (7)$$

then  $(A, \prec', \succ')$  is a pre-anti-flexible algebra if and only if, for  $x, y, z \in A$ ,

$$(G_{RB}(G_{RB}(x) * y + x * G_{RB}(y)) - G_{RB}(x) * G_{RB}(y)) * z + z * (G_{RB}(y) * G_{RB}(x) - G_{RB}(G_{RB}(y) * x + y * G_{RB}(x))) = 0. \quad (8)$$

A linear map  $G_{RB} : A \rightarrow A$  given on an anti-flexible algebra  $(A, *)$  satisfying Eq. (8) is known as a generalization of the Rota-Baxter operator of weight zero.

Besides, dendriform and (di-)tri-algebras were introduced and related to Rota-Baxter operators and associated consequences were derived ([11]). Moreover, it is well known (from [16]) that Koszul duality of operad governing (di-)tri-algebras is corresponding to operad governing variety of (di-)tri-dendriform algebras which are embedded to zero's weight Rota-Baxter algebra. Furthermore, it is proved that a general operadic definition for the notion of splitting algebraic structures are equivalent with some Manin products of operads which are closely related to Rota-Baxter operators ([3]). More generally, splitting algebraic operations procedure in any algebraic operad theory were uniformed and linked to the notion of Rota-Baxter operators on operads ([22]) and other results established on Rota-Baxter algebras are surveyed in [18] and the references therein.

The notion of operated semi-group are introduced to build some algebraic structures on combinatoric elements mainly the binary rooted trees. The most relevant examples are the construction of free Rota-Baxter algebras in terms of Motzkin paths and planar rooted trees ([17]) and the use of typed decorated trees theory for describing combinatorial species ([5]). Given a (non)associative  $\mathbb{K}$  (field of characteristic zero) algebra  $A$ , a Rota-Baxter family operators of weight  $\lambda$  ( $\lambda \in \mathbb{K}$ ) is a family of linear maps  $P_\omega : A \rightarrow A$ , where  $\omega \in \Omega$ , satisfying, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega$ ,

$$P_\alpha(x)P_\beta(y) = P_{\alpha\beta}(xP_\beta(y)) + P_{\alpha\beta}(P_\alpha(x)y) + \lambda P_{\alpha\beta}(xy). \quad (9)$$

The theory of Rota-Baxter family operators takes its origins in renormalization theory of quantum field theory ([12, page 591]). Recently, free (non)commutative Rota-Baxter family is introduced and linked to (tri)dendriform family algebras ([28]). Moreover, it is proved that Rota-Baxter family algebras indexed by an associative semigroup amounts to an ordinary Rota-Baxter algebra structure on the tensor product with the semigroup algebra. Similar results are provided with (tri)dendriform family algebras ([29]), and more generally, the notion of  $\Omega$ -dendriform structures are introduced and nonassociative structures on typed binary trees are unified and generalized ([13]). Similarly, pre-Lie family algebras and free pre-Lie family algebras, are introduced and related typed decorated trees are constructed and related generalization are also derived ([21]). In addition, a general account of family algebras over a finitely presented linear operad are given and proved that this operad together with its presentation naturally define an algebraic structure on the set of parameters ([14]).

Throughout this article,  $\Omega$  is an associative semi-group and  $\Omega_c$  is a commutative associative semi-group, algebras are defined over a field of characteristic zero. We end this introductory section by describing the content flowchart of this paper as follows. In section , we introduce the notion of pre-anti-flexible family algebras, establish their relations with dendriform family algebras and use them to construct  $\Omega_c$ -relative anti-flexible algebras as well as  $\Omega_c$ -relative Lie algebras,

left and right pre-Lie family algebras and related consequences are derived. In section , the notion of  $\Omega_c$ -relative pre-anti-flexible algebras is introduced and viewed as a generalization of the  $\Omega_c$ -relative dendriform algebras and used to build  $\Omega_c$ -relative pre-anti-flexible algebras,  $\Omega_c$ -relative pre-Lie and right pre-Lie algebras,  $\Omega_c$ -relative Lie algebras and other associated structures are derived. In section , we prove that a Rota-Baxter family operators of weight zero defined on an  $\Omega_c$ -relative anti-flexible algebra and its generalization induce an  $\Omega_c$ -relative pre-anti-flexible algebra structure. Under some assumptions on  $\Omega_c$ -relative anti-flexible algebra, we prove that a Rota-Baxter family operators defined on a related  $\Omega_c$ -relative Lie algebra of an  $\Omega_c$ -relative anti-flexible algebra also induces an  $\Omega_c$ -relative pre-anti-flexible algebra structure.

## 2. Pre-anti-flexible family algebras

In this section, pre-anti-flexible family algebras are introduced and related consequences are established. Associated family algebras are derived as well as its  $\Omega_c$ -relative algebraic structures.

**Definition 2.1.** A pre-anti-flexible family algebra is a quadruple  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  such that  $A$  is a vector space equipped with two families of bilinear products  $\prec_\alpha, \succ_\alpha: A \times A \rightarrow A$  with  $\alpha \in \Omega_c$  and satisfying for any  $x, y, z \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,

$$(x \succ_\alpha y) \prec_\beta z - x \succ_\alpha (y \prec_\beta z) = (z \succ_\beta y) \prec_\alpha x - z \succ_\beta (y \prec_\alpha x), \quad (10a)$$

$$(x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z) = (z \prec_\beta y) \prec_\alpha x - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x). \quad (10b)$$

**Remark 2.2.** If the LHS and the RHS of each Eq. (10a) and Eq. (10b) become zero, then pre-anti-flexible family algebra is dendriform family algebra ([28, 21, 29]). Consequently, pre-anti-flexible family algebras can be considered as a generalization of dendriform family algebras.

**Definition 2.3.** An  $\Omega_c$ -relative anti-flexible algebra is a triple  $(A, \cdot_{\alpha,\beta}, \Omega_c)$  in which  $A$  is a vector space equipped with the family of bilinear products “ $\cdot_{\alpha,\beta}: A \times A \rightarrow A$ ” with  $(\alpha, \beta) \in \Omega_c^2$  and satisfying, for any  $x, y, z \in A$  and any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z + z \cdot_{\gamma,\beta\alpha} (y \cdot_{\beta,\alpha} x) - (z \cdot_{\gamma,\beta} y) \cdot_{\gamma\beta,\alpha} x - x \cdot_{\alpha,\beta\gamma} (y \cdot_{\beta,\gamma} z) = 0, \quad (11)$$

equivalently

$$(x, y, z)_{\alpha,\beta,\gamma} = (z, y, x)_{\gamma,\beta,\alpha}, \quad (12)$$

where, for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x, y, z)_{\alpha,\beta,\gamma} := (x \cdot_{\alpha,\beta} y) \cdot_{\alpha\beta,\gamma} z - x \cdot_{\alpha,\beta\gamma} (y \cdot_{\beta,\gamma} z). \quad (13)$$

**Theorem 2.4.** *Let  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  be a pre-anti-flexible family algebra, the following family of bilinear products given by, for any  $\alpha, \beta \in \Omega_c$  and for any  $x, y \in A$ ,*

$$x *_{\alpha, \beta} y = x \succ_\alpha y + x \prec_\beta y \quad (14)$$

*is such that  $(A, *_{\alpha, \beta}, \Omega_c)$  is an  $\Omega_c$ -relative anti-flexible algebra.*

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . We have

$$\begin{aligned} (x, y, z)_{\alpha, \beta, \gamma} &:= (x *_{\alpha, \beta} y) *_{\alpha, \beta, \gamma} z - x *_{\alpha, \beta, \gamma} (y *_{\beta, \gamma} z) \\ &= (x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z + (x \succ_\alpha y + x \prec_\beta y) \prec_\gamma z \\ &\quad - x \succ_\alpha (y \succ_\beta z + y \prec_\gamma z) - x \prec_{\beta\gamma} (y \succ_\beta z + y \prec_\gamma z) \\ &= \{(x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z)\} \\ &\quad + \{(x \prec_\beta y) \prec_\gamma z - x \prec_{\beta\gamma} (y \succ_\beta z + y \prec_\gamma z)\} \\ &\quad + \{(x \succ_\alpha y) \prec_\gamma z - x \succ_\alpha (y \prec_\gamma z)\} \\ &= \{(z \prec_\beta y) \prec_\alpha x - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x)\} \\ &\quad + \{(z \succ_\gamma y + z \prec_\beta y) \succ_{\gamma\beta} x - z \succ_\gamma (y \succ_\beta x)\} \\ &\quad + \{(z \succ_\gamma y) \prec_\alpha x - z \succ_\gamma (y \prec_\alpha x)\} \\ &= (z \succ_\gamma y + z \prec_\beta y) \succ_{\gamma\beta} x + (z \succ_\gamma y + z \prec_\beta y) \prec_\alpha x \\ &\quad - z \succ_\gamma (y \succ_\beta x + y \prec_\alpha x) - z \prec_{\beta\alpha} (y \succ_\beta x + y \prec_\alpha x) \\ (x, y, z)_{\alpha, \beta, \gamma} &= (z, y, x)_{\gamma, \beta, \alpha}. \end{aligned}$$

Therefore, the family of bilinear products given by Eq. (14) satisfies Eq. (12). Thus  $(A, *_{\alpha, \beta}, \Omega_c)$  is an  $\Omega_c$ -relative anti-flexible algebra where, “ $*_{\alpha, \beta} : A \times A \rightarrow A$ ” is derived by Eq. (14).  $\square$

**Theorem 2.5.** *Let  $A$  be a  $\mathbf{k}$  vector space and consider on  $A \otimes \mathbf{k}\Omega_c$ , two bilinear products given by  $\prec, \succ : A \otimes \mathbf{k}\Omega_c \times A \otimes \mathbf{k}\Omega_c \rightarrow A \otimes \mathbf{k}\Omega_c$ . The triple  $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$  is a pre-anti-flexible algebra if and only if  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  is a pre-anti-flexible family algebra and for any  $x, y, \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$(x \otimes \alpha) \prec (y \otimes \beta) := (x \prec_\beta y) \otimes \alpha\beta, \quad (15a)$$

$$(x \otimes \alpha) \succ (y \otimes \beta) := (x \succ_\alpha y) \otimes \alpha\beta. \quad (15b)$$

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . We have

$$\begin{aligned} ((x \otimes \alpha) \succ (y \otimes \beta)) \prec (z \otimes \gamma) - (x \otimes \alpha) \succ ((y \otimes \beta) \prec (z \otimes \gamma)) = \\ ((x \succ_\alpha y) \prec_\gamma z - x \succ_\alpha (y \prec_\gamma z)) \otimes \alpha\beta\gamma, \end{aligned} \quad (16a)$$

$$\begin{aligned} ((x \otimes \alpha) \succ (y \otimes \beta) + (x \otimes \alpha) \prec (y \otimes \beta)) \succ (z \otimes \gamma) \\ - (x \otimes \alpha) \succ ((y \otimes \beta) \succ (z \otimes \gamma)) = ((x \succ_\alpha y + x \prec_\beta y) \succ_{\alpha\beta} z \\ - x \succ_\alpha (y \succ_\beta z)) \otimes \alpha\beta\gamma. \end{aligned} \quad (16b)$$

Similarly, using the commutativity of  $\Omega_c$ , we have

$$((z \otimes \gamma) \succ (y \otimes \beta)) \prec (x \otimes \alpha) - (z \otimes \gamma) \succ ((y \otimes \beta) \prec (x \otimes \alpha)) =$$

$$((z \succ_{\gamma} y) \prec_{\alpha} x - z \succ_{\gamma} (y \prec_{\alpha} x)) \otimes \alpha\beta\gamma, \quad (17a)$$

$$\begin{aligned} & ((z \otimes \gamma) \prec (y \otimes \beta)) \prec (x \otimes \alpha) - (z \otimes \gamma) \prec ((y \otimes \beta) \succ (x \otimes \alpha)) \quad (17b) \\ & + (y \otimes \beta) \prec (x \otimes \alpha) = ((z \prec_{\beta} y) \prec_{\alpha} x - z \prec_{\beta\alpha} (y \succ_{\beta} x + y \prec_{\alpha} x)) \otimes \alpha\beta\gamma. \end{aligned}$$

Hence, if  $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$  is a pre-anti-flexible algebra i.e., the RHS of Eq. (16a) and Eq. (17a) are respectively equal to that of Eq. (16b) and (17b), then  $(A, \prec_{\omega}, \succ_{\omega}, \Omega_c)$  is a pre-anti-flexible family algebra.

Conversely, if  $(A, \prec_{\omega}, \succ_{\omega}, \Omega_c)$  is a pre-anti-flexible family algebra in which “ $\prec_{\omega}, \succ_{\omega}: A \times A \rightarrow A$ ” are given by Eqs. (15a) and (15b), according to Eqs. (16a), (16b), (17a) and (17b), we deduce that  $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$  is a pre-anti-flexible algebra.  $\square$

**Definition 2.6.** (cf. [21]) A left pre-Lie family algebra is a triple  $(A, \triangleright_{\omega}, \Omega_c)$  in which  $A$  is a vector space equipped with the family of bilinear products  $\triangleright_{\omega}: A \times A \rightarrow A$  with  $\omega \in \Omega_c$ , satisfying for any  $x, y, z \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,

$$(x \triangleright_{\alpha} y) \triangleright_{\alpha\beta} z - x \triangleright_{\alpha} (y \triangleright_{\beta} z) = (y \triangleright_{\beta} x) \triangleright_{\beta\alpha} z - y \triangleright_{\beta} (x \triangleright_{\alpha} z). \quad (18)$$

**Definition 2.7.** A right pre-Lie family algebra is a triple  $(A, \triangleleft_{\omega}, \Omega_c)$  in which  $A$  is a vector space equipped with the family of bilinear products  $\triangleleft_{\omega}: A \times A \rightarrow A$  with  $\omega \in \Omega_c$ , satisfying for any  $x, y, z \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,

$$x \triangleleft_{\alpha\beta} (y \triangleleft_{\beta} z) - (x \triangleleft_{\alpha} y) \triangleleft_{\beta} z = x \triangleleft_{\beta\alpha} (z \triangleleft_{\alpha} y) - (x \triangleleft_{\beta} z) \triangleleft_{\alpha} y. \quad (19)$$

**Remark 2.8.** For a given left pre-Lie family algebra  $(A, \cdot_{\alpha}, \Omega_c)$ , setting for any  $x, y \in A$  and for any  $\alpha \in \Omega_c$ ,  $x \cdot_{\alpha}^{opp} y = y \cdot_{\alpha} x$ , we have for any  $x, y, z \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,

$$\begin{aligned} x \cdot_{\alpha\beta}^{opp} (y \cdot_{\beta}^{opp} z) - (x \cdot_{\alpha}^{opp} y) \cdot_{\beta}^{opp} z &= (z \cdot_{\beta} y) \cdot_{\alpha\beta} x - z \cdot_{\beta} (y \cdot_{\alpha} x) \\ &= (y \cdot_{\alpha} z) \cdot_{\beta\alpha} x - y \cdot_{\alpha} (z \cdot_{\beta} x) \\ &= x \cdot_{\beta\alpha}^{opp} (z \cdot_{\alpha}^{opp} y) - (x \cdot_{\beta}^{opp} z) \cdot_{\alpha}^{opp} y. \end{aligned}$$

Therefore,  $(A, \cdot_{\alpha}^{opp}, \Omega_c)$  is a right pre-Lie family algebra.

**Theorem 2.9.** Let  $(A, \prec_{\omega}, \succ_{\omega}, \Omega_c)$  be a pre-anti-flexible family algebra, the two following families of bilinear products

$$x \triangleright_{\omega} y := x \succ_{\omega} y - y \prec_{\omega} x, \quad \forall \omega \in \Omega_c, \quad (20a)$$

$$x \triangleleft_{\omega} y := x \prec_{\omega} y - y \succ_{\omega} x, \quad \forall \omega \in \Omega_c, \quad (20b)$$

for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ , are such that  $(A, \triangleright_{\omega}, \Omega)$  is a left pre-Lie family algebra and  $(A, \triangleleft_{\omega}, \Omega)$  is a right pre-Lie family algebra.

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta \in \Omega_c$ . We have

$$\begin{aligned}
& (x \triangleright_\alpha y) \triangleright_{\alpha\beta} z - x \triangleright_\alpha (y \triangleright_\beta z) = (x \succ_\alpha y - y \prec_\alpha x) \triangleright_{\alpha\beta} z \\
& - x \triangleright_\alpha (y \succ_\beta z - z \prec_\beta y) = (x \succ_\alpha y - y \prec_\alpha x) \succ_{\alpha\beta} z \\
& - z \prec_{\alpha\beta} (x \succ_\alpha y - y \prec_\alpha x) - x \succ_\alpha (y \succ_\beta z - z \prec_\beta y) \\
& + (y \triangleright_\beta z - z \prec_\beta y) \prec_\alpha x = \{(x \succ_\alpha y) \succ_{\alpha\beta} z - x \succ_\alpha (y \succ_\beta z) \\
& - (z \prec_\beta y) \prec_\alpha x - z \prec_{\alpha\beta} (y \prec_\alpha x)\} - (y \prec_\alpha x) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (x \succ_\alpha y) \\
& + x \succ_\alpha (z \prec_\beta y) + (y \succ_\beta z) \prec_\alpha x \\
& = -(x \prec_\beta y) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (y \succ_\beta x) - (y \prec_\alpha x) \succ_{\alpha\beta} z \\
& - z \prec_{\alpha\beta} (x \succ_\alpha y) + x \succ_\alpha (z \prec_\beta y) + (y \succ_\beta z) \prec_\alpha x \\
& = -(x \prec_\beta y + y \prec_\alpha x) \succ_{\alpha\beta} z - z \prec_{\alpha\beta} (x \succ_\alpha y + y \succ_\beta x) \\
& + (y \succ_\beta z) \prec_\alpha x + x \succ_\alpha (z \prec_\beta y) = (y \triangleright_\beta x) \triangleright_{\beta\alpha} z - y \triangleright_\beta (x \triangleright_\alpha z).
\end{aligned}$$

Note that the third equal sign above upwards is due to Eq. (10b) while the last equal sign one is due to Eq. (10a). Therefore,  $(A, \triangleright_\omega, \Omega_c)$  is a left pre-Lie family algebra.

Similarly to the above calculations, we prove that  $(A, \triangleleft_\omega, \Omega_c)$  is a right pre-Lie family algebra.  $\square$

**Definition 2.10.** An  $\Omega_c$ -relative Lie algebra is a triple  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$  in which  $A$  is a vector space equipped with a family of bilinear products  $[\cdot, \cdot]_{\alpha, \beta} : A \otimes A \rightarrow A$  with  $(\alpha, \beta) \in \Omega_c$ , and satisfying, for any  $x, y, z \in A$ , and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$[x, y]_{\alpha, \beta} + [y, x]_{\beta, \alpha} = 0, \quad (21a)$$

$$[[x, y]_{\alpha, \beta}, z]_{\alpha\beta, \gamma} + [[y, z]_{\beta, \gamma}, x]_{\beta\gamma, \alpha} + [[z, x]_{\gamma, \alpha}, y]_{\gamma\alpha, \beta} = 0. \quad (21b)$$

**Theorem 2.11.** Let  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  be a pre-anti-flexible family algebra. The following family of bilinear products given by, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega$ ,

$$[x, y]_{\alpha, \beta} = x *_{\alpha, \beta} y - y *_{\beta, \alpha} x = (x \succ_\alpha y + x \prec_\beta y) - (y \succ_\beta x + y \prec_\alpha x), \quad (22)$$

is such that  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$  is an  $\Omega_c$ -relative Lie algebra.

*Proof.* For any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ , we have

- Skew symmetric,

$$[x, y]_{\alpha, \beta} + [y, x]_{\beta, \alpha} = x *_{\alpha, \beta} y - y *_{\beta, \alpha} x + y *_{\beta, \alpha} x - x *_{\alpha, \beta} y = 0.$$

Thus, Eq. (21a) is satisfied.

- Family of Jacobi identity,

$$\begin{aligned}
& [[x, y]_{\alpha, \beta}, z]_{\alpha\beta, \gamma} + [[y, z]_{\beta, \gamma}, x]_{\beta\gamma, \alpha} + [[z, x]_{\gamma, \alpha}, y]_{\gamma\alpha, \beta} \\
& = (x *_{\alpha, \beta} y) *_{\alpha\beta, \gamma} z - z *_{\gamma, \alpha\beta} (x *_{\alpha, \beta} y)
\end{aligned}$$

$$\begin{aligned}
& - (y *_{\beta, \alpha} x) *_{\alpha, \beta, \gamma} z + z *_{\gamma, \alpha, \beta} (y *_{\beta, \alpha} x) \\
& + (y *_{\beta, \gamma} z) *_{\beta, \gamma, \alpha} x - x *_{\alpha, \beta, \gamma} (y *_{\beta, \gamma} z) \\
& - (z *_{\gamma, \beta} y) *_{\beta, \gamma, \alpha} x + x *_{\alpha, \beta, \gamma} (z *_{\gamma, \beta} y) \\
& + (z *_{\gamma, \alpha} x) *_{\gamma, \alpha, \beta} y - y *_{\beta, \gamma, \alpha} (z *_{\gamma, \alpha} x) \\
& - (x *_{\alpha, \gamma} z) *_{\gamma, \alpha, \beta} y + y *_{\beta, \gamma, \alpha} (x *_{\alpha, \gamma} z) \\
& = (x, y, z)_{\alpha, \beta, \gamma} + (y, z, x)_{\beta, \gamma, \alpha} + (z, x, y)_{\gamma, \alpha, \beta} \\
& - (z, y, x)_{\gamma, \beta, \alpha} - (x, z, y)_{\alpha, \gamma, \beta} - (y, x, z)_{\beta, \alpha, \gamma} = 0
\end{aligned}$$

due to Theorem 2.4 and Eq. (12). Hence, Eq. (21b) is satisfied.

Therefore,  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$  is an  $\Omega_c$ -relative Lie algebra.  $\square$

**Proposition 2.12.** *Let  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  be a pre-anti-flexible family algebra. The following family of bilinear products given on  $A$  by, for any  $\alpha, \beta \in \Omega_c$  and for any  $x, y \in A$ ,*

$$[x, y]_{\alpha, \beta} := x \triangleright_\alpha y - y \triangleright_\beta x, \quad (23)$$

where, “ $\triangleright$ ” is defined by Eq. (20a), turns  $A$  into an  $\Omega_c$ -relative Lie algebra which is the same as that given in Theorem 2.11.

*Proof.* Let  $x, y \in A$  and  $\alpha, \beta \in \Omega_c$ . We have

$$\begin{aligned}
[x, y]_{\alpha, \beta} & := x \triangleright_\alpha y - y \triangleright_\beta x = x \succ_\alpha y - y \prec_\alpha x - y \succ_\beta x + x \prec_\beta y \\
& = (x \succ_\alpha y + x \prec_\beta y) - (y \succ_\beta x + y \prec_\alpha x) = x *_{\alpha, \beta} y - y *_{\beta, \alpha} x
\end{aligned}$$

which is the commutator given by Eq. (22).  $\square$

**Theorem 2.13.** *Let  $(A, \prec_\omega, \succ_\omega, \Omega_c)$  be a pre-anti-flexible family algebra. The following family of bilinear products given by, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$x \circ_{\alpha, \beta} y = x *_{\alpha, \beta} y + y *_{\beta, \alpha} x, \quad (24)$$

in which “ $*_{\alpha, \beta} : A \times A \rightarrow A$ ” is given by Eq. (14) is such that, for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x, y, z)_{\circ_{\alpha, \beta, \gamma}} = [y, [x, z]_{\alpha, \gamma}]_{\beta, \alpha, \gamma}, \quad (25)$$

where,

$$(x, y, z)_{\circ_{\alpha, \beta, \gamma}} = (x \circ_{\alpha, \beta} y) \circ_{\alpha, \beta, \gamma} z - x \circ_{\alpha, \beta, \gamma} (y \circ_{\beta, \gamma} z), \quad (26)$$

and “ $[\cdot, \cdot]_{\alpha, \beta}$ ” is given by Eq. (22).

*Proof.* Let  $x, y, z \in A$ , and for all  $\alpha, \beta, \gamma \in \Omega_c$ , we have

$$\begin{aligned}
(x, y, z)_{\circ_{\alpha, \beta, \gamma}} & = (x \circ_{\alpha, \beta} y) \circ_{\alpha, \beta, \gamma} z - x \circ_{\alpha, \beta, \gamma} (y \circ_{\beta, \gamma} z) \\
& = (x *_{\alpha, \beta} y + y *_{\beta, \alpha} x) \circ_{\alpha, \beta, \gamma} z + z *_{\gamma, \alpha, \beta} (x *_{\alpha, \beta} y + y *_{\beta, \alpha} x)
\end{aligned}$$



$$\begin{aligned}
& -x *_{\alpha, \beta \gamma} (y *_{\beta, \gamma} z + z *_{\gamma, \beta} y) - (y *_{\beta, \gamma} z + z *_{\gamma, \beta} y) *_{\beta \gamma, \alpha} x \\
& = \{(x *_{\alpha, \beta} y) *_{\alpha \beta, \gamma} z - x *_{\alpha, \beta \gamma} (y *_{\beta, \gamma} z)\} \\
& - \{(z *_{\gamma, \beta} y) *_{\beta \gamma, \alpha} x - z *_{\gamma, \alpha \beta} (y *_{\beta, \alpha} x)\} \\
& + (y *_{\beta, \alpha} x) *_{\alpha \beta, \gamma} z + z *_{\gamma, \alpha \beta} (x *_{\alpha, \beta} y) \\
& - x *_{\alpha, \beta \gamma} (z *_{\gamma, \beta} y) - (y *_{\beta, \gamma} z) *_{\beta \gamma, \alpha} x \\
& = (y *_{\beta, \alpha} x) *_{\alpha \beta, \gamma} z + z *_{\gamma, \alpha \beta} (x *_{\alpha, \beta} y) \\
& - x *_{\alpha, \beta \gamma} (z *_{\gamma, \beta} y) - (y *_{\beta, \gamma} z) *_{\beta \gamma, \alpha} x \\
& = y *_{\beta, \alpha \gamma} (x *_{\alpha, \gamma} z) + (z *_{\gamma, \alpha} x) *_{\gamma \alpha, \beta} y \\
& - y *_{\beta, \gamma \alpha} (z *_{\gamma, \alpha} x) - (x *_{\alpha, \gamma} z) *_{\alpha \gamma, \beta} y = [y, [x, z]_{\alpha, \gamma}]_{\beta, \alpha \gamma}.
\end{aligned}$$

Note that the three last equals sign upwards are due to Eq. (11).  $\square$

**Proposition 2.14.** *Let  $(A, \cdot_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative anti-flexible algebra. We have for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,*

$$(x, y, z)_{\circ_{\alpha, \beta, \gamma}} + (z, x, y)_{\circ_{\gamma, \alpha, \beta}} + (y, z, x)_{\circ_{\beta, \gamma, \alpha}} = 0, \quad (27)$$

where, for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,  $(x, y, z)_{\circ_{\alpha, \beta, \gamma}}$  is given by Eq. (26) and  $x \circ_{\alpha, \beta} y = x \cdot_{\alpha, \beta} y + y \cdot_{\beta, \alpha} x$ .

*Proof.* According to Eq. (26), Theorem 2.11 and Theorem 2.13, Eq. (27) holds.  $\square$

### 3. Associated $\Omega_c$ -relative algebras

In this section,  $\Omega_c$ -relative pre-anti-flexible algebras structures are introduced and associated  $\Omega_c$ -relative algebras structures are derived. Moreover,  $\Omega_c$ -relative pre-anti-flexible algebras are viewed as a generalization of pre-anti-flexible family algebras and associated consequences are deduced.

**Definition 3.1.** An  $\Omega_c$ -relative pre-anti-flexible algebra is a quadruple  $(A, \prec_{\alpha, \beta}, \succ_{\alpha, \beta}, \Omega_c)$  in which  $A$  is a vector space equipped with two families of bilinear products  $\prec_{\alpha, \beta}; \succ_{\alpha, \beta}: A \times A \rightarrow A$  for  $(\alpha, \beta) \in \Omega_c^2$  and satisfying for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x \succ_{\alpha, \beta} y) \prec_{\alpha \beta, \gamma} z - x \succ_{\alpha, \beta \gamma} (y \prec_{\beta, \gamma} z) = (z \succ_{\gamma, \beta} y) \prec_{\gamma \beta, \alpha} x - z \succ_{\gamma, \beta \alpha} (y \prec_{\beta, \alpha} x), \quad (28a)$$

$$(x \prec_{\alpha, \beta} y + x \succ_{\alpha, \beta} y) \succ_{\alpha \beta, \gamma} z - x \succ_{\alpha, \beta \gamma} (y \succ_{\beta, \gamma} z) = (z \prec_{\gamma, \beta} y) \prec_{\gamma \beta, \alpha} x - z \prec_{\gamma, \beta \alpha} (y \prec_{\beta, \alpha} x + y \succ_{\beta, \alpha} x). \quad (28b)$$

**Remark 3.2.** If the LSH and RHS of Eq. (28a) and Eq. (28b) vanish, then the  $\Omega_c$ -relative pre-anti-flexible algebra  $(A, \prec_{\alpha, \beta}, \succ_{\alpha, \beta}, \Omega_c)$  is called  $\Omega_c$ -relative dendriform algebra ([1]). Hence,  $\Omega_c$ -relative pre-anti-flexible algebras are a generalization of  $\Omega_c$ -relative dendriform algebras.

Similarly to Theorem 2.5, we have

**Theorem 3.3.** *Let  $A$  be a  $\mathbf{k}$  vector space and consider the bilinear products given on  $A \otimes \mathbf{k}\Omega_c$  by  $\prec, \succ: A \otimes \mathbf{k}\Omega_c \times A \otimes \mathbf{k}\Omega_c \rightarrow A \otimes \mathbf{k}\Omega_c$ . The triple  $(A \otimes \mathbf{k}\Omega_c, \prec, \succ)$  is a pre-anti-flexible algebra if and only if the quadruple  $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$  is an  $\Omega_c$ -relative pre-anti-flexible algebra where, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$(x \otimes \alpha) \prec (y \otimes \beta) := (x \prec_{\alpha,\beta} y) \otimes \alpha\beta, \quad (29a)$$

$$(x \otimes \alpha) \succ (y \otimes \beta) := (x \succ_{\alpha,\beta} y) \otimes \alpha\beta. \quad (29b)$$

**Proposition 3.4.** *Let  $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra. The following family of bilinear products given by, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$x \circledast_{\alpha,\beta} y = x \prec_{\alpha,\beta} y + x \succ_{\alpha,\beta} y, \quad (30)$$

turns  $A$  into an  $\Omega_c$ -relative anti-flexible algebra.

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . We have

$$\begin{aligned} (x, y, z)_{\circledast_{\alpha,\beta,\gamma}} &= (x \succ_{\alpha,\beta} y + x \prec_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z + (x \succ_{\alpha,\beta} y + x \prec_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z \\ &\quad - x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z + y \prec_{\beta,\gamma} z) - x \prec_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z + y \prec_{\beta,\gamma} z) \\ &= \{(x \succ_{\alpha,\beta} y + x \prec_{\alpha,\beta} y) \succ_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z)\} \\ &\quad - \{x \prec_{\alpha,\beta\gamma} (y \succ_{\beta,\gamma} z + y \prec_{\beta,\gamma} z) - (x \prec_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z\} \\ &\quad + \{(x \succ_{\alpha,\beta} y) \prec_{\alpha\beta,\gamma} z - x \succ_{\alpha,\beta\gamma} (y \prec_{\beta,\gamma} z)\} \\ &= \{(z \prec_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x - z \prec_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x + y \succ_{\beta,\alpha} x)\} \\ &\quad + \{(z \prec_{\gamma,\beta} y + z \succ_{\gamma,\beta} y) \succ_{\gamma\beta,\alpha} x - z \succ_{\gamma,\beta\gamma} (y \succ_{\beta,\alpha} x)\} \\ &\quad + \{(z \succ_{\gamma,\beta} y) \prec_{\gamma\beta,\alpha} x - z \succ_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x)\} \\ &= (z \circledast_{\gamma,\beta} y) \circledast_{\gamma\beta,\alpha} x - z \circledast_{\gamma,\beta\alpha} (y \circledast_{\beta,\alpha} x) = (z, y, x)_{\circledast_{\gamma,\beta,\alpha}}, \end{aligned}$$

the third equal sign upwards above is due to Eq. (28a) and Eq. (28b).

Therefore,  $(A, \circledast_{\omega_1, \omega_2}, \Omega_c)$  is an  $\Omega_c$ -relative anti-flexible algebra.  $\square$

**Definition 3.5.** An  $\Omega_c$ -relative pre-Lie (left-symmetric) algebra is a vector space  $A$  equipped with a family of bilinear products  $*_{\alpha,\beta}: A \otimes A \rightarrow A$  with  $(\alpha, \beta) \in \Omega_c^2$ , such that for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x, y, z)_{\alpha,\beta,\gamma} = (y, x, z)_{\beta,\alpha,\gamma}, \quad (31)$$

or equivalently

$$(x *_{\alpha,\beta} y) *_{\alpha\beta,\gamma} z - x *_{\alpha,\beta\gamma} (y *_{\beta,\gamma} z) - (y *_{\beta,\alpha} x) *_{\beta\alpha,\gamma} z + y *_{\beta,\alpha\gamma} (x *_{\alpha,\gamma} z) = 0. \quad (32)$$

**Definition 3.6.** An  $\Omega_c$ -relative right-symmetric algebra is a vector space  $A$  equipped with a family of bilinear products  $*_{\alpha,\beta}: A \otimes A \rightarrow A$  for  $(\alpha, \beta) \in \Omega_c^2$  such that for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x, y, z)_{\alpha,\beta,\gamma} = (x, z, y)_{\alpha,\gamma,\beta}, \quad (33)$$

or equivalently

$$(x *_{\alpha,\beta} y) *_{\alpha,\beta,\gamma} z - x *_{\alpha,\beta,\gamma} (y *_{\beta,\gamma} z) - (x *_{\alpha,\gamma} z) *_{\alpha,\beta,\gamma} y + x *_{\alpha,\beta,\gamma} (z *_{\gamma,\beta} y) = 0. \quad (34)$$

**Theorem 3.7.** *Let  $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra, defining for all  $\alpha, \beta \in \Omega$  and for any  $x, y \in A$ ,*

$$x \blacktriangleright_{\alpha,\beta} y = x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x, \quad (35a)$$

$$x \blacktriangleleft_{\alpha,\beta} y = x \prec_{\alpha,\beta} y - y \succ_{\beta,\alpha} x, \quad (35b)$$

then  $(A, \blacktriangleright_{\alpha,\beta}, \Omega_c)$  is an  $\Omega_c$ -relative pre-Lie algebra and  $(A, \blacktriangleleft_{\alpha,\beta}, \Omega_c)$  is an  $\Omega_c$ -relative right symmetric algebra.

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . We have

$$\begin{aligned} (x, y, z)_{\blacktriangleright_{\alpha,\beta,\gamma}} &= (x \blacktriangleright_{\alpha,\beta} y) \blacktriangleright_{\alpha,\beta,\gamma} z - x \blacktriangleright_{\alpha,\beta,\gamma} (y \blacktriangleright_{\beta,\gamma} z) \\ &= (x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x) \succ_{\alpha,\beta,\gamma} z \\ &\quad - z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y - y \prec_{\beta,\alpha} x) \\ &\quad - x \succ_{\alpha,\beta,\gamma} (y \succ_{\beta,\gamma} z - z \prec_{\gamma,\beta} y) \\ &\quad + (y \succ_{\beta,\gamma} z - z \prec_{\gamma,\beta} y) \prec_{\beta\gamma,\alpha} x \\ &= \{(x \succ_{\alpha,\beta} y) \succ_{\alpha,\beta,\gamma} z - x \succ_{\alpha,\beta,\gamma} (y \succ_{\beta,\gamma} z)\} \\ &\quad - \{(z \prec_{\gamma,\beta} y) \prec_{\beta\gamma,\alpha} x - z \prec_{\gamma,\beta\alpha} (y \prec_{\beta,\alpha} x)\} \\ &\quad + \{(y \succ_{\beta,\gamma} z) \prec_{\beta\gamma,\alpha} x + x \succ_{\alpha,\beta,\gamma} (z \prec_{\gamma,\beta} y)\} \\ &\quad - \{(y \prec_{\beta,\alpha} x) \succ_{\alpha,\beta,\gamma} z + z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y)\} \\ &= \{(x \succ_{\alpha,\gamma} z) \prec_{\alpha\gamma,\beta} y + y \succ_{\beta,\alpha\gamma} (z \prec_{\gamma,\alpha} x)\} \\ &\quad - \{(x \succ_{\alpha,\beta} y + y \prec_{\beta,\alpha} x) \succ_{\alpha,\beta,\gamma} z\} \\ &\quad - \{z \prec_{\gamma,\alpha\beta} (x \succ_{\alpha,\beta} y + y \succ_{\beta,\alpha} x)\} = (y, x, z)_{\blacktriangleright_{\beta,\alpha,\gamma}}, \end{aligned}$$

the second equal sign upwards in the above successive relations is due to Eq. (28a) and Eq. (28b) while the last one is due to Eq. (28a).

Therefore,  $(A, \blacktriangleright_{\alpha,\beta}, \Omega_c)$  is an  $\Omega_c$ -relative pre-Lie algebra.

Similarly, we prove that  $(A, \blacktriangleleft_{\alpha,\beta}, \Omega_c)$  is an  $\Omega_c$ -relative right symmetric algebra.  $\square$

Moreover, we have

**Theorem 3.8.** *Let  $(A, \prec_{\alpha,\beta}, \succ_{\alpha,\beta}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra. There is an  $\Omega_c$ -relative Lie algebra structure on  $A$  given by for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$[x, y]_{\alpha,\beta} := (x \succ_{\alpha,\beta} y + x \prec_{\alpha,\beta} y) - (y \succ_{\beta,\alpha} x + y \prec_{\beta,\alpha} x). \quad (36)$$

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . In view of Eq. (36) and Eq. (30) we have

$$[x, y]_{\alpha,\beta} + [y, x]_{\beta,\alpha} = x \otimes_{\alpha,\beta} y - y \otimes_{\beta,\alpha} x + y \otimes_{\beta,\alpha} x - x \otimes_{\alpha,\beta} y = 0.$$

In addition, we have

$$\begin{aligned} & [[x, y]_{\alpha, \beta}, z]_{\alpha, \beta, \gamma} + [[y, z]_{\beta, \gamma}, x]_{\beta, \gamma, \alpha} + [[z, x]_{\gamma, \alpha}, y]_{\gamma, \alpha, \beta} \\ &= (x, y, z)_{\alpha, \beta, \gamma} \otimes_{\alpha, \beta, \gamma} + (y, z, x)_{\beta, \gamma, \alpha} \otimes_{\beta, \gamma, \alpha} + (z, x, y)_{\gamma, \alpha, \beta} \otimes_{\gamma, \alpha, \beta} \\ & - (z, y, x)_{\alpha, \beta, \gamma} \otimes_{\alpha, \beta, \gamma} - (x, z, y)_{\alpha, \beta, \gamma} \otimes_{\alpha, \beta, \gamma} - (y, x, z)_{\alpha, \beta, \gamma} \otimes_{\alpha, \beta, \gamma}. \end{aligned}$$

According to Proposition 3.4, we deduce the  $\Omega_c$ -relative Jacobi identity. Therefore,  $A$  contains an  $\Omega_c$ -relative Lie algebra structure.  $\square$

**Theorem 3.9.** *Let  $(A, \prec_{\alpha, \beta}, \succ_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra such that its related  $\Omega_c$ -relative anti-flexible algebra derived in Proposition 3.4 is  $(A, \otimes_{\omega_1, \omega_2}, \Omega_c)$ . The family of bilinear products given by, for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,*

$$x \odot_{\alpha, \beta} y = x \otimes_{\alpha, \beta} y + y \otimes_{\beta, \alpha} x, \quad (37)$$

satisfies the following relation, for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$(x, y, z)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} = [y, [x, z]_{\alpha, \gamma}]_{\beta, \alpha, \gamma}, \quad (38)$$

where  $(x, y, z)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} = (x \odot_{\alpha, \beta} y) \odot_{\alpha, \beta, \gamma} z - x \odot_{\alpha, \beta, \gamma} (y \odot_{\beta, \gamma} z)$  and  $[x, y]_{\alpha, \beta} = x \otimes_{\alpha, \beta} y - y \otimes_{\beta, \alpha} x$ .

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . We have

$$\begin{aligned} (x, y, z)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} &= (x \odot_{\alpha, \beta} y) \odot_{\alpha, \beta, \gamma} z - x \odot_{\alpha, \beta, \gamma} (y \odot_{\beta, \gamma} z) \\ &= (x \otimes_{\alpha, \beta} y) \otimes_{\alpha, \beta, \gamma} z + (y \otimes_{\beta, \alpha} x) \otimes_{\beta, \alpha, \gamma} z \\ &+ z \otimes_{\gamma, \alpha, \beta} (x \otimes_{\alpha, \beta} y) + z \otimes_{\gamma, \alpha, \beta} (y \otimes_{\beta, \alpha} x) \\ &- x \otimes_{\alpha, \beta, \gamma} (y \otimes_{\beta, \gamma} z) - x \otimes_{\alpha, \beta, \gamma} (z \otimes_{\gamma, \beta} y) \\ &- (y \otimes_{\beta, \gamma} z) \otimes_{\beta, \gamma, \alpha} x - (z \otimes_{\gamma, \beta} y) \otimes_{\beta, \gamma, \alpha} x \\ &= (y \otimes_{\beta, \alpha} x) \otimes_{\beta, \alpha, \gamma} z + z \otimes_{\gamma, \alpha, \beta} (x \otimes_{\alpha, \beta} y) \\ &- x \otimes_{\alpha, \beta, \gamma} (z \otimes_{\gamma, \beta} y) - (y \otimes_{\beta, \gamma} z) \otimes_{\beta, \gamma, \alpha} x \\ &= y \otimes_{\beta, \alpha, \gamma} (x \otimes_{\alpha, \beta} z) + (z \otimes_{\gamma, \alpha} x) \otimes_{\gamma, \alpha, \beta} y \\ &- (x \otimes_{\alpha, \beta} z) \otimes_{\alpha, \beta, \gamma} y - y \otimes_{\beta, \gamma, \alpha} (z \otimes_{\gamma, \alpha} x) = [y, [x, z]_{\alpha, \gamma}]_{\beta, \alpha, \gamma}. \end{aligned}$$

The second and third equal sign upward are due to Proposition 3.4.  $\square$

**Proposition 3.10.** *Let  $(A, \prec_{\alpha, \beta}, \succ_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra. Consider the algebra  $(A, \odot_{\alpha, \beta}, \Omega_c)$  derived above. We have for any  $x, y, z \in A$  and for any  $\alpha, \beta, \gamma \in \Omega_c$ ,*

$$(x, y, z)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} + (z, x, y)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} + (y, z, x)_{\alpha, \beta, \gamma} \odot_{\alpha, \beta, \gamma} = 0. \quad (39)$$

*Proof.* According to Theorem 3.8 and Proposition 3.4 and Theorem 3.9, the above equation is satisfied.  $\square$

**Proposition 3.11.** *Let  $(A, \prec_{\omega_1, \omega_2}, \succ_{\omega_1, \omega_2}, \Omega_c)$  be an  $\Omega_c$ -relative pre-anti-flexible algebra. Suppose for any  $\omega_1, \omega_2 \in \Omega_c$  the family of bilinear operations  $\prec_{\omega_1, \omega_2}: A \otimes A \rightarrow A$  is independent of  $\omega_1$  and  $\succ_{\omega_1, \omega_2}: A \otimes A \rightarrow A$  is independent of  $\omega_2$ . Then  $A$  possesses:*

- (1) *a pre-anti-flexible family algebra structure which is  $(A, \prec_{\omega_2}, \succ_{\omega_1}, \Omega_c)$ . Conversely, if the quadruple  $(A, \prec_{\omega}, \succ_{\omega}, \Omega_c)$  is a pre-anti-flexible family algebra, then it can be regarded as an  $\Omega_c$ -relative pre-anti-flexible algebra  $(A, \prec_{\omega_1, \omega_2}, \succ_{\omega_1, \omega_2}, \Omega_c)$  in which for any  $\omega_1, \omega_2 \in \Omega_c$  “ $\prec_{\omega_1, \omega_2}$ ” is independent of  $\omega_1$  and “ $\succ_{\omega_1, \omega_2}$ ” is independent of  $\omega_2$ .*
- (2) *a left pre-Lie family algebra structure and conversely, if  $A$  possesses a left pre-Lie family algebra structure, then it can be regarded as an  $\Omega_c$ -relative pre-Lie algebra structure (given in Theorem 3.7) in which for any  $\omega_1, \omega_2 \in \Omega_c$ , “ $\prec_{\omega_1, \omega_2}$ ” is independent of  $\omega_1$  and “ $\succ_{\omega_1, \omega_2}$ ” is independent of  $\omega_2$ .*
- (3) *similarly to (3.11), a right pre-Lie family algebra structure and conversely, if  $A$  possesses a right pre-Lie family algebra structure (given in Theorem 3.7), then it can be viewed as an  $\Omega_c$ -relative right-symmetric algebra in which, for any  $\omega_1, \omega_2 \in \Omega_c$ , “ $\prec_{\omega_1, \omega_2}$ ” is independent of  $\omega_1$  and “ $\succ_{\omega_1, \omega_2}$ ” is independent of  $\omega_2$ .*

*Proof.* Under divers assumptions, Eq. (10a) is expressed by Eq. (28a) and Eq. (10b) is translated by Eq. (28b).  $\square$

## 4. Rota-Baxter operators

This section deals with the use of the Rota-Baxter operators defined on  $\Omega_c$ -relative anti-flexible and Lie algebras to build  $\Omega_c$ -relative pre-anti-flexible algebras. It is proved that a Rota-Baxter operator define the  $\Omega_c$ -relative Lie algebra derived from a given  $\Omega_c$ -relative anti-flexible algebra induces an  $\Omega_c$ -relative pre-anti-flexible algebra.

**Definition 4.1.** Let  $(A, \cdot_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative anti-flexible algebra. A Rota-Baxter operator on  $A$  is a family of linear operators  $R_{B_\alpha}: A \rightarrow A$ , with  $\alpha \in \Omega_c$ , satisfying for any  $x, y \in A$  and any  $\alpha, \beta \in \Omega_c$ ,

$$R_{B_\alpha}(x) \cdot_{\alpha, \beta} R_{B_\beta}(y) = R_{B_{\alpha\beta}}(R_{B_\alpha}(x) \cdot_{\alpha, \beta} y + x \cdot_{\alpha, \beta} R_{B_\beta}(y)). \quad (40)$$

**Definition 4.2.** Let  $(A, \cdot_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative anti-flexible algebra. A generalized Rota-Baxter operator on  $A$  is a family of linear operators  $G_{RB_\alpha}: A \rightarrow A$ , with  $\alpha \in \Omega_c$ , satisfying for any  $\alpha, \beta, \gamma \in \Omega_c$  and any  $x, y, z \in A$ ,

$$0 = (G_{RB_{\alpha\beta}}(G_{RB_\alpha}(x) \cdot_{\alpha, \beta} y + x \cdot_{\alpha, \beta} G_{RB_\beta}(y)) - G_{RB_\alpha}(x) \cdot_{\alpha, \beta} G_{RB_\beta}(y)) \cdot_{\alpha\beta, \gamma} z + z \cdot_{\gamma, \beta\alpha} (G_{RB_\beta}(y) \cdot_{\beta, \alpha} G_{RB_\alpha}(x) - G_{RB_{\beta\alpha}}(G_{RB_\beta}(y) \cdot_{\beta, \alpha} x + y \cdot_{\beta, \alpha} G_{RB_\alpha}(x))). \quad (41)$$

**Proposition 4.3.** *Let  $(A, \cdot_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative anti-flexible algebra and  $G_{RB_\alpha} : A \rightarrow A$  a generalized Rota-Baxter operator. Defining for any  $x, y \in A$  and any  $\alpha, \beta \in \Omega_c$ ,*

$$x \prec_{\alpha, \beta} y := x \cdot_{\alpha, \beta} G_{RB_\beta}(y), \quad x \succ_{\alpha, \beta} y := G_{RB_\alpha}(x) \cdot_{\alpha, \beta} y, \quad (42)$$

*then,  $(A, \prec_{\alpha, \beta}, \succ_{\alpha, \beta}, \Omega_c)$  is turns to an  $\Omega_c$ -relative pre-anti-flexible algebra. The converse is true.*

*Proof.* Let  $x, y, z \in A$  and  $\alpha, \beta, \gamma \in \Omega_c$ . In view of Eq. (42) we have

$$(x \succ_{\alpha, \beta} y) \prec_{\alpha, \beta, \gamma} z - x \succ_{\alpha, \beta, \gamma} (y \prec_{\alpha, \beta} z) = (G_{RB_\alpha}(x), y, G_{RB_\gamma}(z))_{\alpha, \beta, \gamma},$$

$$(z \succ_{\gamma, \beta} y) \prec_{\gamma, \beta, \alpha} x - z \succ_{\gamma, \beta, \alpha} (y \prec_{\beta, \alpha} x) = (G_{RB_\gamma}(z), y, G_{RB_\alpha}(x))_{\gamma, \beta, \alpha},$$

$$(x \prec_{\alpha, \beta} y + x \succ_{\alpha, \beta} y) \succ_{\alpha, \beta, \gamma} z - x \succ_{\alpha, \beta, \gamma} (y \succ_{\beta, \gamma} z) = (z, G_{RB_\beta}(y), G_{RB_\alpha}(x))_{\gamma, \beta, \alpha} + (G_{RB_{\alpha\beta}}(G_{RB_\alpha}(x) \cdot_{\alpha, \beta} y + x \cdot_{\alpha, \beta} G_{RB_\beta}(y)) - (G_{RB_\alpha}(x) \cdot_{\alpha, \beta} G_{RB_\beta}(y))) \cdot_{\alpha, \beta, \gamma} z,$$

$$(z \prec_{\gamma, \beta} y) \prec_{\gamma, \beta, \alpha} x - z \prec_{\gamma, \beta, \alpha} (y \succ_{\beta, \alpha} x + y \prec_{\beta, \alpha} x) = (G_{RB_\alpha}(x), G_{RB_\beta}(y), z)_{\alpha, \beta, \gamma} + z \cdot_{\gamma, \beta, \alpha} (G_{RB_\beta}(y) \cdot_{\beta, \alpha} G_{RB_\alpha}(x) - G_{RB_{\beta\alpha}}(G_{RB_\beta}(y) \cdot_{\beta, \alpha} x + y \cdot_{\beta, \alpha} G_{RB_\alpha}(x))).$$

Therefore, Eq.(42) turns  $A$  into an  $\Omega_c$ -relative pre-anti-flexible algebra if and only if  $G_{RB_\alpha}$  satisfy Eq. (41).  $\square$

**Corollary 4.4.** *Any Rota-Baxter operator on an  $\Omega_c$ -relative anti-flexible algebra induces an  $\Omega_c$ -relative pre-anti-flexible algebra.*

In the sequel of this section, we consider the  $\Omega_c$ -relative anti-flexible algebra  $(A, \cdot_{\omega_1, \omega_2}, \Omega_c)$  in which for any  $\alpha \in \Omega_c$ , the linear map  $\varphi_\alpha : A \rightarrow A$  is such that the elements

$$\varphi_\alpha(x) \cdot_{\alpha, \beta} \varphi_\beta(y) - \varphi_{\alpha\beta}(x \cdot_{\alpha, \beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha, \beta} y), \quad \forall x, y \in A, \forall \alpha, \beta \in \Omega_c, \quad (43)$$

satisfy the following, for any  $\alpha, \beta, \gamma \in \Omega_c$  and for any  $x, y, z \in A$ ,

$$z \cdot_{\gamma, \beta, \alpha} (\varphi_\alpha(x) \cdot_{\alpha, \beta} \varphi_\beta(y) - \varphi_{\alpha\beta}(x \cdot_{\alpha, \beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha, \beta} y)) = (\varphi_{\alpha\beta}(x \cdot_{\alpha, \beta} \varphi_\beta(y) + \varphi_\alpha(x) \cdot_{\alpha, \beta} y) - \varphi_\alpha(x) \cdot_{\alpha, \beta} \varphi_\beta(y)) \cdot_{\alpha, \beta, \gamma} z. \quad (44)$$

**Definition 4.5.** By a Rota-Baxter operator on an  $\Omega_c$ -relative Lie algebra  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$  we mean a family of linear operators  $R_{B_\alpha} : A \rightarrow A$  with  $\alpha \in \Omega_c$ , satisfying for any  $x, y \in A$ , and for any  $\alpha, \beta \in \Omega_c$ ,

$$[R_{B_\alpha}(x), R_{B_\beta}(y)]_{\alpha, \beta} = R_{B_{\alpha\beta}}([x, R_{B_\beta}(y)]_{\alpha, \beta} + [R_{B_\alpha}(x), y]_{\alpha, \beta}). \quad (45)$$

**Proposition 4.6.** *Let  $(A, \cdot_{\alpha, \beta}, \Omega_c)$  be an  $\Omega_c$ -relative anti-flexible algebra equipped with a family of linear maps  $\varphi_\alpha : A \rightarrow A$ , with  $\alpha \in \Omega_c$ , in which the elements as the form given in Eq. (43) satisfy Eq. (44). The family linear products given by, for any  $x, y \in A$  and any  $\alpha, \beta \in \Omega_c$ ,*

$$x \prec_{\alpha, \beta} y = x \cdot_{\alpha, \beta} \varphi_\beta(y); \quad x \succ_{\alpha, \beta} y = \varphi_\alpha(x) \cdot_{\alpha, \beta} y \quad (46)$$

defines an  $\Omega_c$ -relative pre-anti-flexible structures on  $A$  if and only if the family of linear maps “ $\varphi_\alpha$ ” is a Rota-Baxter operator on the  $\Omega_c$ -relative Lie algebra  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$ , where for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,  $[x, y]_{\alpha, \beta} = x \cdot_{\beta, \alpha} y - y \cdot_{\beta, \alpha} x$ .

*Proof.* According to Proposition 4.3, the family of linear maps  $\varphi_\alpha$  satisfy Eq. (41) and Eq. (44) if and only if any  $x, y, z \in A$  and any  $\alpha, \beta, \gamma \in \Omega_c$ ,

$$z \cdot_{\gamma, \alpha \beta} ([\varphi_\alpha(x), \varphi_\beta(y)]_{\alpha, \beta} - \varphi_{\alpha\beta}([x, \varphi_\beta(y)]_{\alpha, \beta} + [\varphi_\alpha(x), y]_{\alpha, \beta})) = 0. \quad (47)$$

Therefore,  $\varphi_\alpha$  is a Rota-Baxter operator on the  $\Omega_c$ -relative Lie algebra  $(A, [\cdot, \cdot]_{\alpha, \beta}, \Omega_c)$ , where for any  $x, y \in A$  and for any  $\alpha, \beta \in \Omega_c$ ,  $[x, y]_{\alpha, \beta} = x \cdot_{\beta, \alpha} y - y \cdot_{\beta, \alpha} x$ .  $\square$

**Acknowledgments.** This work is supported the LPMC of Nankai University and Nankai ZhiDe Foundation.

## References

- [1] M. Aguiar, *Dendriform algebras relative to a semigroup*, SIGMA **16** (2020), 1 – 15.
- [2] M. Aguiar, *Pre-Poisson algebras*, Lett. Math. Phys., **54** (2000), 263 – 277.
- [3] C. Bai, O. Bellier, L. Guo, X. Ni, *Splitting of operations, Manin products, and Rota-Baxter operators*, Int. Math. Res. Not., **2013** (2013), 485 – 524.
- [4] G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math., **10** (1960), 731 – 742.
- [5] F. Bergeron, G. Labelle, P. Leroux, *Combinatorial species and tree-like structures*, *Encyclopedia of Mathematics and its Applications*, 67 Cambridge University Press, 1998.
- [6] M.C. Bhandari, *On the classification of simple anti-flexible algebras*, Trans. Amer. Math. Soc., **173** (1972), 159 – 181.
- [7] H.A. Çelik, *On primitive and prime anti-flexible ring*, J. Algebra, **20** (1972) 428 – 440.
- [8] M.L. Dassoundo, *Pre-anti-flexible bialgebras*, Commun. Algebra, **49** (2021), 4050 – 4078.
- [9] M.L. Dassoundo, C. Bai, M.N. Hounkonnou, *Anti-flexible bialgebras*, J. Algebra and Its Appl., (2021), <https://doi.org/10.1142/S0219498822502127>.

- 
- [10] **L.W. Davis, D.J. Rodabaugh**, *Simple nearly anti-flexible algebras have unity elements*, London Math. Soc. Stud. Texts, **2** (1969) 69 – 72.
- [11] **K. Ebrahimi-Fard**, *Loday-type algebras and the Rota-Baxter relation*, Lett. Math. Phys., **61** (2002), 139 – 147.
- [12] **K. Ebrahimi-Fard, J.M. Gracia-Bondía, F. Patras**, *A Lie theoretic approach to renormalization*, Commun. Math. Phys., **276** (2007), 519 – 549.
- [13] **L. Foissy**, *Typed binary trees and generalized dendriform algebras*, J. Algebra, **586** (2021), 1 – 61.
- [14] **L. Foissy, D. Manchon and Y. Zhang**, *Families of algebraic structures*, (2020), ArXiv:2005.05116v1.
- [15] **M. Goze, E. Remm**, *Lie-admissible algebras and operads*, J. Algebra, **273** (2004), 129 – 152.
- [16] **V.Yu. Gubarev, P.S. Kolesnikov**, *Embedding of dendriform algebras into Rota-Baxter algebras*, Cent. Eur. J. Math., **11** (2013), 226 – 245.
- [17] **L. Guo**, *Operated semigroups, Motzkin paths and rooted trees*, J. Algebr. Comb., **29** (2009), 35 – 62.
- [18] **L. Guo**, *An introduction to Rota-Baxter Algebras*, Surveys of Modern Mathematics 4. Somerville: International Press; Beijing: Higher Education Press, 2012.
- [19] **M.N. Hounkonnou, M.L. Dassoundo**, *Center-symmetric algebras and bialgebras: Relevant properties and consequences*. Geometric Methods in Physics. Trends in Math., Birkhäuser, Cham. (2016), 281 – 293.
- [20] **J.-L. Loday**, *Dialgebras and related operads*, Lecture Notes Math. **1763**, Springer, Berlin 2001, 7 – 66.
- [21] **D. Manchon and Y. Zhang**, *Free pre-Lie family algebras*, (2020), arXiv:2003.00917.
- [22] **J. Pei, C. Bai and L. Guo**, *Splitting of operads and Rota-Baxter operators on operads*, Appl. Cate. Stru., **25** (2017), 505 – 538.
- [23] **D.J. Rodabaugh**, *A generalization of the flexible law*, Trans. Amer. Math. Soc., **114** (1965), 468 – 487.
- [24] **D.J. Rodabaugh**, *On semisimple antiflexible algebras*, Portugal. Math., **6** (1967), 261 – 271.
- [25] **D.J. Rodabaugh**, *On anti-flexible algebras*, Trans. Amer. Math. Soc., **169** (1972), 219 – 235.
- [26] **D.J. Rodabaugh**, *A theorem of semisimple anti-flexible algebras*, Comm. Algebra, **6** (1978), 1081 – 1090.
- [27] **G.-C. Rota**, *Baxter algebras and combinatorial identities I, II*, Bull. Amer. Math. Soc., **75** (1969), 325 – 334.
- [28] **Y. Zhang, X. Gao**, *Free Rota-Baxter family algebras and (tri)dendriform family algebras*, Pacific J. Math., **301** (2019), 742 – 766.
- [29] **Y. Zhang, X. Gao and D. Manchon**, *Free (tri)dendriform family algebras*, J. Algebra, **547** (2020), 456 – 493.

Received July 14, 2021

Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, China  
E-mail: dassoundo@yahoo.com