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Characterization of monoids by condition (P_E)

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Abstract. In this paper first we recall condition (P_E) and then will give general properties and a characterization of monoids for which all right acts satisfy this condition. Finally, we give a characterization of monoids, by comparing this property of their acts with some others.

1. Introduction

For a monoid S, with 1 as its identity, a set A (we consider nonempty) is called a right S-act, usually denoted by A_S (or simply A), if S on A unitarian from the right, that is, there exists a mapping $A \times S \to A$, $(a,s) \mapsto as$, satisfying the conditions $a(st) = (as)t$ and $a1 = a$, for all $a \in A$ and all $s,t \in S$. Let A, B be two right S-acts. A mapping $f : A \to B$ is called a homomorphism of right S-acts or just an S-homomorphism if $f(as) = f(a)s$ for $a \in A$, $s \in S$. The set of all S-homomorphisms from A into B will be denoted by $\mathrm{Hom}(A,B)$. Also **Act-**S is the category of right S-acts.

In [4], introduced condition (P_E) and it is shown that this condition implies weak flatness, but the converse is true when S is left PP and in [3] gave a classification of monoid by this condition of (finitely generated, cyclic, monocyclic, Rees factor) right acts.

In this paper, we recall condition (P_E) and we continue the investigation of this condition. At first we give general properties of this condition. Finally, we will give a characterization of monoids S over which all right S-acts satisfy condition (P_E) and also a characterization of monoids S for which this condition of right S-acts has some other properties and vice versa.

We refer the reader to [5, 6], for basic definitions and terminologies relating to semigroups and acts over monoids and to [8], for definitions and results on flatness which are used here.

2. General properties

In this section we recall condition (P_E) and give some results of it.

Recall from [3] that a right S-act A satisfies *condition* (P), if for all $a, a' \in A$, $s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(a = a''u, a' = a''v$ and $us = vs')$.

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It satisfies condition (P_E), if for all $a, a' \in A$, $s, s' \in S$, $as = a's' \Rightarrow (\exists a'' \in A)(\exists u, v \in S)(\exists e, f \in E(S))$ $(ae = a''ue, a'f = a''vf, es = s, fs' = s'$ and $us = vs')$. It is clear that condition (P) implies condition (P_E).

We can easily seen that right S-act A satisfies condition (P_E) if and only if $as = a's'$, for $a, a' \in A$, $s, s' \in S$, implies that there exist $a'' \in A$, $u, v \in S$ and $e, f \in E(S)$ such that $ae = a''u$, $a'f = a''v$, $es = s$, $fs' = s'$ and $us = vs'$.

According to the equivalent definition for condition (P_E) expressed above, theorem 2.5 of [3], can be written as follows.

For a right congruence ρ on a monoid S, S/ρ satisfies condition (P_E) if and only if $(xs)\rho(yt)$, for $x, y, s, t \in S$, implies that there exist $u, v \in S$ and $e, f \in E(S)$ such that $(xe)\rho u$, $(yf)\rho v$, $es = s$, $ft = t$ and $us = vt$.

We recall from [6] that a monoid S is called right reversible if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$.

Theorem 2.1. *Let S be a monoid and A be a right S-act. Then:*

1. *S satisfies condition (P_E).*
2. *Θ satisfies condition (P_E) if and only if S is right reversible.*
3. *Let $I \neq \emptyset$ and $A = \coprod_{i \in I} A_i$, where A_i, $i \in I$, are right S-acts. Then A satisfies condition (P_E) if and only if each A_i, $i \in I$, satisfies condition (P_E).*
4. *Let $\{B_i | i \in I\}$ be a nonempty chain of subacts of A. If every B_i, $i \in I$, satisfies condition (P_E), then $\bigcup_{i \in I} B_i$ as a subact of A satisfies condition (P_E).*
5. *If A satisfies condition (P_E), then every retract of A satisfies condition (P_E).*

Proof. The proofs are straightforward. □

3. Characterization by condition $(\mathbf{P_E})$ of right acts

In this section we give a characterization of monoids S by condition (P_E) of right S-acts. Also, we give a characterization of monoids, by comparing condition (P_E) of their acts with some others.

We recall [6] that a right ideal K of a monoid S satisfies condition (LU) if for every $k \in K$, there exists $l \in K$ such that $lk = k$.

Theorem 3.1. *Let K be any proper right ideal of a monoid S. If the right S-act $S \coprod^K S$ satisfies condition (P_E), then K satisfies condition (LU).*

Proof. All right S-acts satisfying condition (P_E) are weakly flat, by [4, Theorem 2.3]. Thus K satisfies condition (LU), by [6, III, Proposition 12.19]. □

Recall from [4, 6] that a monoid S is called a left PP monoid if every principal left ideal of S is projective. Therefore, a monoid S is left PP if and only if for every $s \in S$ there exists an idempotent e of S such that $es = s$ and $ker\rho_s \leqslant ker\rho_e$. A right S-act A is called weakly flat (WF), if the functor $A \otimes_S -$ preserves all embeddings of left ideals into S.

Theorem 3.2. *For a proper right ideal K of a monoid S, the following statements are equivalent:*

(1) *All right S-acts of the form $S \coprod^K S$ satisfy condition (P_E).*

(2) *S is regular.*

Proof. (1) $\Rightarrow$ (2). Suppose $s \in S$. If $sS = S$ then it is obvious that s is regular. Otherwise sS is a proper right ideal of monoid S and $S \coprod^{sS} S$ satisfies condition (P_E), by assumption. So by Theorem 3.1, sS satisfies condition (LU). Then there exists $l \in sS$ such that $ls = s$. Hence s is regular, and so, S is regular.

(2) $\Rightarrow$ (1). Suppose K is any proper right ideal of the monoid S and $k \in K$. By assumption there exists $k' \in S$ such that $k = kk'k$, that is, the ideal K satisfies condition (LU). So by [6, III, Proposition 12.19], $S \coprod^K S$ is weakly flat. Since S is regular, it is left PP. Then by [4, Theorem 2.5], $S \coprod^K S$ satisfies condition (P_E). □

Recall from [6], [8], [10] and [2] that a right S-act A satisfies condition (E), if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \text{ and } us = us').$$

A satisfies condition (E'), if for all $a \in A$, $s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u \in S)(a = a'u \text{ and } us = us').$$

A satisfies condition (EP), if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's').$$

A satisfies condition $(E'P)$, if for all $a \in A$, $s, s', z \in S$,

$$(as = as', sz = s'z) \Rightarrow (\exists a' \in A)(\exists u, u' \in S)(a = a'u = a'u' \text{ and } us = u's').$$

Theorem 3.3. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfying condition $(E'P)$ satisfy condition (P_E).*

(2) *All right S-acts satisfying condition (E') satisfy condition (P_E).*

(3) *All right S-acts satisfying condition (EP) satisfy condition (P_E).*

(4) *All right S-acts satisfying condition (E) satisfy condition (P_E).*

(5) *S is regular.*

Proof. Since $(E) \Rightarrow (EP) \Rightarrow (E'P)$ and $(E) \Rightarrow (E') \Rightarrow (E'P)$, implications $(1) \Rightarrow (3) \Rightarrow (4)$ and $(1) \Rightarrow (2) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (5)$. If $s \in S$, and $sS = S$ then s is regular. Now let $sS \neq S$ Then

$$A = S \coprod^{sS} S = \{(l, x) | l \in S \setminus sS\} \dot\cup sS \dot\cup \{(t, y) | t \in S \setminus sS\}$$

is a right S-act and

$$B = \{(l, x) | l \in S \setminus sS\} \dot\cup sS \cong S \cong \{(t, y) | t \in S \setminus sS\} \dot\cup sS = C.$$

Since $A = B \cup C$ is generated by exactly two elements $(1, x)$ and $(1, y)$ and S satisfies condition (E), subacts B and C of A satisfy condition (E), so A satisfies condition (E). Then by assumption A satisfies condition (P_E). Thus the equality $(1, x)s = (1, y)s$, implies that there exist $a \in A$, $u, v \in S$ and $e, f \in E(S)$, such that $(1, x)e = au$, $(1, y)f = av$, $us = vs$ and $es = s = fs$. Now $(1, x)e = au$ and $(1, y)f = av$ imply that at least one of elements e or f is belong to sS. If $e \in sS$ then there exists $s' \in S$ such that $e = ss'$, and so, $s = es = ss's$, that is, s is regular. Similarly, we can show that s is regular, if $f \in sS$. Therefore S is regular.

$(5) \Rightarrow (1)$. Since S is regular, by [2, Theorem 2.8] all right S-acts satisfying condition $(E'P)$ are weakly flat. Also every regular monoid is left PP, and so, by [4, Theorem 2.5] condition (P_E) and weakly flat are equivalent. Hence all right S-acts satisfying condition $(E'P)$ satisfy condition (P_E). □

By the proof of Theorem 3.3, we conclude that the above theorem is true for finitely generated right S-acts and also right S-acts generated by at most (exactly) two elements. Moreover, by [4, Theorem 2.5], if in Theorem 3.3, we replace condition (P_E) by weakly flat, then theorem is still true. In addition this theorem is true for finitely generated right S-acts, and also, right S-acts generated by at most (exactly) two elements.

Recall from [6] that a right S-act A is called principally weakly flat (PWF) if the functor $A \otimes_S -$, preserves all embeddings of principal left ideals into S. An act A_S is called torsion free (TF) if for any $a, b \in A$ and for any right cancellable element $u \in S$, the equality $au = bu$ implies $a = b$.

Also we recall from [10] that a right S-act A is called $\mathfrak{R}$-torsion free ($\mathfrak{R}$-TF) if for any $a, b \in A$ and $u \in S$, u right cancellable, $au = bu$ and $a \,\mathfrak{R}\, b$ ($\mathfrak{R}$ is Green's equivalence) imply that $a = b$.

Theorem 3.4. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts satisfy condition (P_E).*

(2) *All $\mathfrak{R}$-torsion free right S-acts satisfy condition (P_E).*

(3) *S is regular and satisfies condition*

(R) : *for any elements $s, t \in S$ there exists $w \in Ss \cap St$ such that $w\rho(s, t)s$.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). By [10, Proposition 1.2] all right S-acts satisfying condition (E) are $\mathfrak{R}$-torsion free. Therefore all right S-acts satisfying condition (E) satisfy condition (P_E), by assumption. Hence, S is regular, by Theorem 3.3. Then by [6, IV, Theorem 6.6], all right S-acts are PWF. Since $PWF \Rightarrow TF \Rightarrow \mathfrak{R}$-$TF$, then all right S-acts are $\mathfrak{R}$-torsion free, and so, by assumption, all right S-acts satisfy condition (P_E). Therefore by [3, Theorem 2.1], S is regular and satisfies condition (R).

(3) $\Rightarrow$ (1). It is obvious, by [3, Theorem 2.1]. □

It is clear that the theorem above is also true for finitely generated right S-acts and right S-acts generated by at most (exactly) two elements.

Recall from [6] that an element $a \in A_S$ is called divisible by $s \in S$ if there exists $b \in A$, such that $bs = a$. An act A_S is called divisible if every element of A is divisible by any left cancellable element of S.

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S

Theorem 3.5. *For any monoid S the following statements are equivalent:*

(1) *All right S-acts are divisible.*

(2) *All monocyclic right S-acts satisfying condition (P_E), are divisible.*

(3) *$Sc = S$ for every $c \in C_l$.*

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). By part (2.1) of Theorem 2.1, S_S satisfies condition (P_E) therefore by assumption, for every $x \in S$, $S_S \cong S/\Delta_S = S/\rho_{(x,x)}$ is divisible, that is, $Sc = S$ for all $c \in C_l$.

(3) $\Rightarrow$ (1). It is obvious, by [6, III, Proposition 2.2]. □

We recall from [6] that a right S-act A is (strongly) faithful if for $s, t \in S$ the equality $as = at$, for (some) all $a \in A$, implies that $s = t$. It is obvious that every strongly faithful right S-act is faithful, but the converse is not true in generally.

In [1, Lemma 2.10], there exists at least one strongly faithful cyclic right (left) S-act if and only if S_S (${}_SS$) is a strongly faithful right (left) S-act, which it is equivalent to S is left (right) cancellative monoid.

Theorem 3.6. *For any monoid S the following statements are equivalent:*

(1) *All strongly faithful right S-acts satisfy condition (P_E).*

(2) *S is not a left cancellative monoid or S is regular.*

(3) *S is not a left cancellative monoid or S is group.*

Proof. (1) $\Rightarrow$ (2). If S is not left cancellative, the result follows. Otherwise, if $sS = S$, for $s \in S$, then s is regular. Now let $sS \neq S$. Then

$$A = S\coprod^{sS} S = \{(l,x) | l \in S \setminus sS\} \,\dot{\cup}\, sS \,\dot{\cup}\, \{(t,y) | t \in S \setminus sS\}$$

is a right S-act and

$$B = \{(l,x) | l \in S \setminus sS\} \,\dot{\cup}\, sS \cong S \cong \{(t,y) | t \in S \setminus sS\} \,\dot{\cup}\, sS = C.$$

Since S is left cancellative, it is strongly faithful. Therefore B and C are strongly faithful as subacts of A. Thus A is strongly faithful, and so, it satisfies condition (P_E), by assumption. Now by the proof (4) $\Rightarrow$ (5) of Theorem 3.3, S is regular.

(2) $\Rightarrow$ (3). If S is left cancellative, then it is regular. Thus for every $s \in S$, there exists $x \in S$ such that $sxs = s$, which implies $xs = 1$. Hence for every $s \in S$, $Ss = S$, and so, S is group.

(3) $\Rightarrow$ (1). If S is not left cancellative, then we are done, because, by [1, Lemma 2.10], there exists no strongly faithful right S-act. Otherwise, since S is group, all right S-acts satisfy condition (P), by [6, IV, Theorem 9.10], and so, all right S-acts satisfy condition (P_E). □

It is clear that above theorem is true for finitely generated right S-acts and also right S-acts generated by two elements.

We recall from [9] that a right S-act A is called almost weakly flat if A is principally weakly flat and satisfies condition

(W') If $as = a't$ and $Ss \cap St \neq \emptyset$ for $a, a' \in A$, $s, t \in S$, then there exist $a'' \in A$, $u \in Ss \cap St$ such that $as = a't = a''u$.

Theorem 3.7. *For any monoid S the following statements are equivalent:*

(1) *All generator right S-acts satisfy condition (P_E).*

(2) *All finitely generated generator right S-acts satisfy condition (P_E).*

(3) *All generator right S-acts generated by at most three elements satisfy condition (P_E).*

(4) *$S \times A$ satisfies condition (P_E), for every generator right S-act A.*

(5) *$S \times A$ satisfies condition (P_E), for every finitely generated generator right S-act A.*

(6) *$S \times A$ satisfies condition (P_E), for every generator right S-act generated by at most three elements A.*

(7) *$S \times A$ satisfies condition (P_E), for every right S-act A.*

(8) *$S \times A$ satisfies condition (P_E), for every finitely generated right S-act A.*

(9) *$S \times A$ satisfies condition (P_E), for every right S-act A generated by at most two elements.*

(10) *A right S-act A satisfies condition (P_E), if $Hom(A,S) \neq \emptyset$.*

(11) *A finitely generated right S-act A satisfies condition (P_E), if $Hom(A,S) \neq \emptyset$.*

(12) *A right S-act A generated by at most two elements satisfies condition (P_E), if $Hom(A,S) \neq \emptyset$.*

(13) *All right S-acts are almost weakly flat.*

(14) *S is regular and S satisfies condition bellow:*

$$(\forall s,t\in S)\big(Ss\cap St\neq\emptyset\Rightarrow\big(\exists w\in Ss\cap St; 1(ker\lambda_s\vee ker\lambda_t)w\big)\big).$$

Proof. The implications (1) ⇒ (2) ⇒ (3), (4) ⇒ (5) ⇒ (6), (7) ⇒ (8) ⇒ (9), (10) ⇒ (11) ⇒ (12) and (7) ⇒ (4) are obvious.

(13) ⇔ (14). By [9, Theorem 3.4] all generators are weakly flat, and so, it is obvious, by [9, Theorem 3.8].

(1) ⇒ (7). Let A be a right S-act. It is obvious that the mapping $\pi:S\times A\to S$, where $\pi(s,a)=s$, for $a\in A$ and $s\in S$, is an epimorphism in Act-S, and so by [6, II, Theorem 3.16], $S\times A$ is a generator. Thus $S\times A$ satisfies condition (P_E), by assumption.

(12) ⇒ (1). Let A be a generator right S-act and $as=a't$ for $a,a'\in A$ and $s,t\in S$. If $B=aS\cup a'S$. It is obvious that B is a subact of A and generated by at most two elements. Since A is generator, there exists an epimorphism $\pi:A\to S$ in Act-S. So $\pi_{|_B}:B\to S$ is a homomorphism in Act-S, and so, $Hom(B,S)\neq\emptyset$. Thus, by assumption, B satisfies condition (P_E). Therefore equality $as=a't$ in B, implies that there exist $a''\in B\subseteq A$, $u,v\in S$ and $e,f\in E(S)$ such that $ae=a''u$, $a'f=a''v$, $us=vt$, $es=s$ and $ft=t$. Hence, A satisfies condition (P_E).

(6) ⇒ (1). Let A be a generator right S-act and $as=a't$ for $a,a'\in A$ and $s,t\in S$. Since A is generator, there exists an epimorphism $\pi:A\to S$ in Act-S. Let $\pi(z)=1$ and $B=aS\cup a'S\cup zS$. It is obvious that B is a subact of A and generated by at most three elements. Since $\pi_{|_B}:B\to S$ is an epimorphism in Act-S, by [6, II, Theorem 3.16], B is a generator. Thus, by assumption, $S\times B$ satisfies condition (P_E). If $\pi(a)=l$, $\pi(a')=l'$ then equality $as=a't$ in B, implies equality $(l,a)s=(l',a')t$ in $S\times B$. Hence, by definition, there exist $(w,a'')\in S\times B$, $u,v\in S$ and $e,f\in E(S)$ such that $(l,a)e=(w,a'')u$, $(l',a')f=(w,a'')v$, $us=vt$, $es=s$ and $ft=t$. Thus, $ae=a''u$, $a'f=a''v$, $us=vt$, $es=s$ and $ft=t$. Hence, A satisfies condition (P_E).

(3) ⇒ (1). Let A be a generator right S-act and $as=a't$ for $a,a'\in A$ and $s,t\in S$. Since A is generator, there exists an epimorphism $\pi:A\to S$ in Act-S. Let $\pi(z)=1$ and $B=aS\cup a'S\cup zS$. It is obvious that B is a subact of A and generated by at most three elements. Since $\pi_{|_B}:B\to S$ is an epimorphism in Act-S, by [6, II, Theorem 3.16], B is a generator. Thus, by assumption, B satisfies condition (P_E). Therefore equality $as=a't$ in B, implies that there exist $a''\in B\subseteq A$, $u,v\in S$ and $e,f\in E(S)$ such that $ae=a''u$, $a'f=a''v$, $us=vt$, $es=s$ and $ft=t$. Hence, A satisfies condition (P_E).

(1) ⇒ (13). By [4, Theorem 2.3], condition (P_E) implies weakly flat. So by assumption all generator right S-acts are weakly flat. Then by [9, Theorem 3.4] all right S-acts are almost weakly flat.

(13) ⇒ (1). By [9, Theorem 3.4] all generator right S-acts are weakly flat. Thus, by [9, Theorem 3.8], S is regular, and so, S is left PP. Then by [4, Theorem 2.5], condition (P_E) and weakly flat are equivalent, the result follows.

(9) ⇒ (10). Let for right S-act A, $Hom(A,S)\neq\emptyset$ and $as=a't$ for $a,a'\in A$ and $s,t\in S$. If $B=aS\cup a'S$ then B is a subact of A and generated by at most two elements. Since $Hom(A,S)\neq\emptyset$ let $f:A\to S$ be a homomorphism in Act-S.

Now equality $as = a't$ in A implies equality $\big(f(a), a\big)s = \big(f(a'), a'\big)t$ in $S \times B$. Since by assumption $S \times B$, satisfies condition (P_E), there exist $(w, a'') \in S \times B$, $u, v \in S$ and $e, f \in E(S)$ such that $\big(f(a), a\big)e = (w, a'')u$, $\big(f(a'), a'\big)f = (w, a'')v$, $us = vt$, $es = s$ and $ft = t$. Then $ae = a''u$, $a'f = a''v$, $us = vt$, $es = s$ and $ft = t$. Hence, A satisfies condition (P_E). □

Recall [6] that a right S-act Q is injective (Inj), if for any monomorphism $\iota : A \to B$ and any homomorphism $f : A \to Q$ there exists a homomorphism $\overline{f} : B \to Q$ such that $f = \overline{f}\iota$. It is called (fg-) weakly injective ((fg-)WI), if it is injective relative to all embeddings of (finitely generated) right ideals into S.

Recall [7] that for elements $u, v \in S$, the relation $P_{u,v}$ is defined on S as

$$P_{u,v} = \big\{(x, y) \in S \times S | ux = vy\big\}.$$

For $s, t \in S$, let $\mu_{s,t} = ker\lambda_s \vee ker\lambda_t$.

For any right ideal I of S let ρ_I denote the right Rees congruence, i.e., for x, y in S,

$$(x, y) \in \rho_I \Longleftrightarrow x = y \text{ or } x, y \in I.$$

Theorem 3.8. *For any monoid S the following statements are equivalent:*

(1) *All fg-weakly injective right S-acts satisfy condition (P_E).*

(2) *All weakly injective right S-acts satisfy condition (P_E).*

(3) *All injective right S-acts satisfy condition (P_E).*

(4) *All cofree right S-acts satisfy condition (P_E).*

(5) $(\forall s, t \in S)$ $(\exists u, v \in S)\big(\exists e_1, e_2 \in E(S)\big)$; $(e_1 s = s, e_2 t = t \wedge us = vt)$ *and the following conditions hold*

  (i) $P_{ue_1, ve_2} \subseteq P_{e_1,s} \circ \mu_{s,t} \circ P_{t,e_2}$,

  (ii) $ker\lambda_u \cap (e_1 S \times e_1 S) \subseteq \rho_{sS}$,

  (iii) $ker\lambda_v \cap (e_2 S \times e_2 S) \subseteq \rho_{tS}$.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious, because $cofree \Rightarrow Inj \Rightarrow WI \Rightarrow fg - WI$.

(4) ⇒ (5). Let $s, t \in S$ and S_1, S_2 be two distinct sets, where $|S_1| = |S_2| = |S|$, and $\alpha : S \to S_1$, $\beta : S \to S_2$ are bijections. Put $X = (S/\mu_{s,t}) \ \dot\cup \ S_1 \ \dot\cup \ S_2$, and define the mappings $f, g : S \to X$ as follows:

$$f(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = sy \\ \alpha(x) & if\ x \in S \setminus sS \end{cases}$$

$$g(x) = \begin{cases} [y]_{\mu_{s,t}} & \text{if there exists } y \in S;\ x = ty \\ \beta(x) & if\ x \in S \setminus tS. \end{cases}$$

We show that f is well-defined. For this, suppose that $sy_1 = sy_2$, for $y_1, y_2 \in S$, hence $(y_1, y_2) \in ker\lambda_s \subseteq ker\lambda_s \vee ker\lambda_t = \mu_{s,t}$ and so $[y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}}$, that is, $f(sy_1) = [y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}} = f(sy_2)$ and so f is well-defined. Similarly, g is well-defined.

Since $fs = gt$, and $X^S = \{h : S \to X\}$ satisfies condition (P_E), there exist mapping $h : S \to X$, $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $fe_1 = hu$, $ge_2 = hv$, $e_1 s = s$, $e_2 t = t$ and $us = vt$. So $fe_1 = hue_1$, $ge_2 = hve_2$, $e_1 s = s$, $e_2 t = t$ and $us = vt$. Let $(l_1, l_2) \in P_{ue_1, ve_2}$, for $l_1, l_2 \in S$, then:

$$(fe_1)l_1 = (hue_1)l_1 = h(ue_1 l_1) = h(ve_2 l_2) = (hve_2)l_2 = (ge_2)l_2 = g(e_2 l_2)$$

So by the definitions f and g there exist $y_1, y_2 \in S$ such that $e_1 l_1 = sy_1$ and $e_2 l_2 = ty_2$ so $f(e_1 l_1) = f(sy_1) = [y_1]_{\mu_{s,t}}$ and $g(e_2 l_2) = g(ty_2) = [y_2]_{\mu_{s,t}}$. Then $[y_1]_{\mu_{s,t}} = [y_2]_{\mu_{s,t}}$ implies $(y_1, y_2) \in \mu_{s,t}$. Therefore

$$\left.\begin{array}{l} e_1 l_1 = sy_1 \Rightarrow (l_1, y_1) \in P_{e_1,s} \\ (y_1, y_2) \in \mu_{s,t} \\ e_2 l_2 = ty_2 \Rightarrow (y_2, l_2) \in P_{t,e_2} \end{array}\right. \Rightarrow (l_1, l_2) \in P_{e_1,s} \circ \mu_{s,t} \circ P_{t,e_2}$$

that is, $P_{ue_1,ve_2} \subseteq P_{e_1,s} \circ \mu_{s,t} \circ P_{t,e_2}$, and so (i) is proved.
Now let $(t_1, t_2) \in ker\lambda_u \cap (e_1 S \times e_1 S)$, for $t_1, t_2 \in S$. So $ut_1 = ut_2$ and there exist $w_1, w_2 \in S$ such that $t_1 = e_1 w_1$ and $t_2 = e_1 w_2$. So $ue_1 w_1 = ut_1 = ut_2 = ue_1 w_2$. Then $f(e_1 w_1) = (fe_1)w_1 = (hue_1)w_1 = h(ue_1 w_1) = h(ue_1 w_2) = (hue_1)w_2 = (fe_1)w_2 = f(e_1 w_2)$.

According to the definition of f of the last equality, two cases can be considered.

Case 1. $e_1 w_1, e_1 w_2 \in S \backslash sS$ then: $f(e_1 w_1) = f(e_1 w_2) \Rightarrow \alpha(e_1 w_1) = \alpha(e_1 w_2) \Rightarrow e_1 w_1 = e_1 w_2 \Rightarrow t_1 = t_2 \Rightarrow (t_1, t_2) \in \rho_{sS}$

Case 2. $e_1 w_1, e_1 w_2 \in sS$ then there exist $y_1, y_2 \in S$ such that $e_1 w_1 = sy_1$ and $e_1 w_2 = sy_2$ so $(t_1, t_2) = (e_1 w_1, e_1 w_2) = (sy_1, sy_2) \in (sS \times sS) \cup \Delta_S = \rho_{sS}$. Thus $ker\lambda_u \cap (e_1 S \times e_1 S) \subseteq \rho_{sS}$. Similarly (iii).

(5) $\Rightarrow$ (1). Suppose that A is an fg-weakly injective right S-act and let $as = a't$, for $a, a' \in A$ and $s, t \in S$. By assumption, there exist $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $e_1 s = s$, $e_2 t = t$ and $us = vt$ and conditions (i), (ii) and (iii) hold.

Define

$$\varphi : ue_1 S \cup ve_2 S \to A \qquad x \mapsto \begin{cases} ae_1 p & \exists p \in S : \ x = ue_1 p, \\ a'e_2 q & \exists q \in S : \ x = ve_2 q. \end{cases}$$

First we show that φ is well-defined. If there exist $p, q \in S$ such that $ue_1 p = ve_2 q$, then $(p, q) \in P_{ue_1,ve_2}$. So by condition (i), there exist $y_1, y_2 \in S$ such that $(p, y_1) \in P_{e_1,s}$, $(y_1, y_2) \in \mu_{s,t}$ and $(y_2, q) \in P_{t,e_2}$. Then $e_1 p = sy_1$, $e_2 q = ty_2$ and $(y_1, y_2) \in ker\lambda_s \vee ker\lambda_t = \mu_{s,t}$. By this last relation, there exist $z_1, \dots, z_n \in S$ such that:

$$sy_1 = sz_1, \quad sz_2 = sz_3, \quad \dots \quad sz_{n-1} = sz_n, \quad tz_1 = tz_2, \quad \dots \quad \dots \quad \dots \quad tz_n = ty_2$$

and so

$$ae_1 p = asy_1 = asz_1 = a'tz_1 = a'tz_2 = \dots = a'tz_n = a'ty_2 = a'e_2 q.$$

Now let $p_1, p_2 \in S$ such that $ue_1 p_1 = ue_1 p_2$, then $(e_1 p_1, e_1 p_2) \in ker\lambda_u \cap (e_1 S \times e_1 S)$ and so by (ii), $e_1 p_1 = e_1 p_2$ or there exist $y_1, y_2 \in S$ such that $e_1 p_1 = sy_1$ and

$e_1p_2 = sy_2$. If $e_1p_1 = e_1p_2$ then $ae_1p_1 = ae_1p_2$. If $e_1p_1 = sy_1$ and $e_1p_2 = sy_2$ then $usy_1 = ue_1p_1 = ue_1p_2 = usy_2 = vty_2$ and so, $ue_1sy_1 = ve_2ty_2$ then $(sy_1, ty_2) \in P_{ue_1,ve_2}$. So by condition (i), there exist $l_1, l_2 \in S$ such that $(sy_1, l_1) \in P_{e_1,s}$, $(l_1, l_2) \in \mu_{s,t}$ and $(l_2, ty_2) \in P_{t,e_2}$. Then $sy_1 = e_1sy_1 = sl_1$, $ty_2 = e_2ty_2 = tl_2$ and $(l_1, l_2) \in ker\lambda_s \vee ker\lambda_t = \mu_{s,t}$. Thus, there exist $z'_1, ..., z'_m \in S$ such that:

$$sl_1 = sz'_1, \quad sz'_2 = sz'_3, \quad ... \quad sz'_{m-1} = sz'_m, \qquad tz'_1 = tz'_2, \quad ... \quad ... \quad ... \quad tz'_m = tl_2$$

and so

$ae_1p_1 = asy_1 = asl_1 = asz'_1 = a'tz'_1 = a'tz'_2 = ... = a'tz'_m = a'tl_2 = a'ty_2 = asy_2 = ae_1p_2$

If there exist $q_1, q_2 \in S$ such that $ve_2q_1 = ve_2q_2$, then by conditions (i) and (iii), with a similar argument $a'e_2q_1 = a'e_2q_2$. Thus, φ is well-defined, and obviously it is a homomorphism. Since by assumption, A is an fg-weakly injective right S-act, there exists a homomorphism $\psi : S \to A$ such that $\psi|_{ue_1S \cup ve_2S} = \varphi$.

Let $a'' = \psi(1)$, then $ae_1 = \varphi(ue_1) = \psi(ue_1) = \psi(1)ue_1 = a''ue_1$ and $a'e_2 = \varphi(ve_2) = \psi(ve_2) = \psi(1)ve_2 = a''ve_2$. that is, A satisfies condition (P_E). □

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