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On irreducible pseudo symmetric ideals of a partially ordered ternary semigroup

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Abstract. In this paper, the concepts of irreducible and strongly irreducible pseudo symmetric ideals in a partially ordered ternary semigroup are introduced. We also studied some interesting properties of irreducible and strongly irreducible pseudo symmetric ideals of a partially ordered ternary semigroup and prove that the space of strongly irreducible pseudo symmetric ideals of a partially ordered ternary semigroup is topologized.

1. Introduction

In 1932, D. H. Lehmer [7] studied some ternary algebraic systems called triplex that appear to be commutative ternary groups. The idea of ternary semigroups was known to Banach. He showed through an example, that there exists a ternary semigroup which cannot be reduced to an ordinary semigroup. Hewitt and Zuckerman described in [3] the method of construction of ternary semigroups from binary and described various connections between such semigroups.

Ternary semigroups are a special case of n-ary (polyadic) semigroups. So many results on ternary semigroups has an analogous version for n-ary semigroups and many results on ternary semigroups is a consequence of results proved for n-ary semigroups. For example, F. M. Sioson proved in [10] some results on ideals in n-ary semigroups, next some results on ideals in ternary semigroups [9]. Also characterization of regular ternary semigroups by ideals can be deduced from general results proved in [1] for n-ary semigroups..

M. Shabir and M. Bano [8] introduced the notion of prime, semiprime and strongly prime bi-ideals in ternary semigroups and studied the space of strongly prime bi-ideals is topologized. A. Iampan [4, 5] invented the concept of ordered ternary semigroups which is the generalization of the concept of ordered semigroup as well as the concept of ternary semigroup. In [11], the ideal theory of a partially ordered ternary semigroups is introduced. The notions of complete prime ideals, prime ideals, complete semiprime ideals, semiprime ideals of po ternary semigroups is defined in [12]. The notion of semipseudo symmetric ideals and pseudo symmetric ideals of partially ordered ternary semigroups is introduced in [6].

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We introduce the concepts of irreducible and strongly irreducible pseudo symmetric ideals in a partially ordered ternary semigroup and prove that the space of such ideals is topologized.

2. Preliminaries

A non-empty set T together with a ternary operation [ ] defined on T satisfying the following associative law,

$$[x_1\ x_2\ x_3\ x_4\ x_5] = [[x_1\ x_2\ x_3]\ x_4\ x_5] = [x_1\ [x_2\ x_3\ x_4]\ x_5] = [x_1\ x_2\ [x_3\ x_4\ x_5]]$$

for all $x_i \in T, 1 \leqslant i \leqslant 5$ is called a *ternary semigroup.*

A ternary semigroup T is said to be a *partially ordered ternary semigroup* if there exist a partially ordered relation $\leqslant$ such that for any $a, b, x, y \in T, a \leqslant b \Rightarrow [xya] \leqslant [xyb],\ [xay] \leqslant [xby]$ and $[axy] \leqslant [bxy]$.

A partially ordered ternary semigroup is also referred to as a po ternary semigroup or an ordered ternary semigroup. In this article, we write T for a partially ordered ternary semigroup, unless otherwise specified.

An element e of T is said to be an *identity* of T if $[eet] = [ete] = [tee] = t$ and $t \leqslant e$ for all $t \in T$. Further, for simplicity, the value of $[xyz]$ will be denoted by xyz.

Let H be a non-empty subset of T. We denote, $\{t \in T : t \leqslant h, \text{for some } h \in H\}$ by $(H]$. If $A \subseteq T, B \subseteq T$ then $A \subseteq (A], ((A]] = (A], (A](B](C] \subseteq (ABC]$ and $A \subseteq B$ implies $(A] \subseteq (B]$.

A non-empty subset I of T is said to be a *partially ordered left* (respectively, *right, lateral*) *ideal* of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I, TIT \subseteq I$) and $(I] = I$. A non-empty subset I of T is said to be *ideal* of T if it is a left ideal, a right ideal and a lateral ideal of T. An ideal I of T is called *idempotent* if $I^3 = I$. A partially ordered ternary semigroup T is called *semisimple* if all its ideals are idempotent.

An ideal I of T is said to be a *pseudo symmetric ideal* if $x, y, z \in T,\ xyz \in I$ implies $xsytz \in I\ \forall s, t \in T$. A pseudo symmetric ideal I of T is said to be *proper pseudo symmetric ideal* of T if it differs from T. An ideal I of T is said to be *maximal pseudo symmetric ideal* if I is a proper pseudo symmetric ideal of T and is not properly contained in any proper pseudo symmetric ideal of T.

Note that, the arbitrary intersection of any family of pseudo symmetric ideals of T is an pseudo symmetric ideal of T, provided it is non-empty and the arbitrary union of pseudo symmetric ideals of T is an pseudo symmetric ideal of T.

Let A be the non-empty subset of T. The intersection of all pseudo symmetric ideals of T containing A is called as *pseudo symmetric ideal of T generated by A* and it is denoted by $(A)_{ps}$. A pseudo symmetric ideal I of T is said to be the principal pseudo symmetric ideal generated by an element a if I is a *pseudo symmetric ideal generated by* $\{a\}$ for some $a \in T$. It is denoted by $(a)_{ps}$.

A proper pseudo symmetric ideal I of T is said to be a *prime pseudo symmetric ideal* of T if $I_1 I_2 I_3 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ where I_1, I_2, I_3 are the pseudo symmetric ideals of T.

A proper pseudo symmetric ideal I of T is said to be a *strongly prime pseudo symmetric ideal* of T if $I_1 I_2 I_3 \cap I_2 I_3 I_1 \cap I_3 I_1 I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ where I_1, I_2, I_3 are pseudo symmetric ideals of T.

A proper pseudo symmetric ideal I of T is said to be a *semiprime pseudo symmetric ideal* of T if P is a pseudo symmetric ideal of T and $P^n \subseteq I \Rightarrow P \subseteq I$ for some odd natural number n.

We will use the following simple facts.

Theorem 2.1. *Every strongly prime pseudo symmetric ideal of T is a prime pseudo symmetric ideal of T.*

Theorem 2.2. *The non-empty intersection of an arbitrary collection of prime pseudo symmetric ideals of T is a semiprime pseudo symmetric ideal of T.*

Theorem 2.3. *Every prime pseudo symmetric ideal of T is a semiprime pseudo symmetric ideal of T.*

Theorem 2.4. *Every strongly prime pseudo symmetric ideal of T is a semiprime pseudo symmetric ideal of T.*

3. Irreducible pseudo symmetric ideals

In this section, we define irreducible and strongly irreducible pseudo symmetric ideals of T and obtain some properties.

Definition 3.1. A proper pseudo symmetric ideal I of T is said to be *irreducible pseudo symmetric ideal* of T if $I_1 \cap I_2 \cap I_3 = I$ implies $I_1 = I$ or $I_2 = I$ or $I_3 = I$ for any pseudo symmetric ideals I_1, I_2, I_3 of T.

Definition 3.2. A proper pseudo symmetric ideal I of T is said to be *strongly irreducible pseudo symmetric ideal* of T if $I_1 \cap I_2 \cap I_3 \subseteq I$ implies $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$ for any pseudo symmetric ideals I_1, I_2, I_3 of T.

Remark 3.3. *Every strongly irreducible pseudo symmetric ideal of T is an irreducible pseudo symmetric ideal of T but converse is not true in general.*

Definition 3.4. A non-empty subset A of T is called an *i-system* if $a, b \in A$ implies that $(a)_{ps} \cap (b)_{ps} \cap A \neq \emptyset$ where $(a)_{ps}$ and $(b)_{ps}$ are principal pseudo symmetric ideals of T generated by a and b respectively.

Theorem 3.5. *The following statements for a proper pseudo symmetric ideal I of T are equivalent,*

(i) *I is a strongly irreducible pseudo symmetric ideal of T.*

(*ii*) *If* $a, b \in T$ *such that* $(a)_{ps} \cap (b)_{ps} \subseteq I$ *then* $a \in I$ *or* $b \in I$.

(*iii*) *The complement of* I *that is* $I^c = T - I$ *is an i-system.*

Proof. $(i) \Rightarrow (ii)$: Suppose I is a strongly irreducible pseudo symmetric ideal of T. Let $a, b \in T$ such that$(a)_{ps} \cap (b)_{ps} \subseteq I$. Since I is a strongly irreducible pseudo symmetric ideal, $(a)_{ps} \subseteq I$ or $(b)_{ps} \subseteq I \Rightarrow a \in I$ or $b \in I$.

$(ii) \Rightarrow (iii)$: If possible, let $a, b \in I^c = T - I$ such that $(a)_{ps} \cap (b)_{ps} \cap I^c = \emptyset$. Then, $(a)_{ps} \cap (b)_{ps} \subseteq I$. Hence by $(ii), a \in I$ or $b \in I$; which is contradiction. Consequently $(a)_{ps} \cap (b)_{ps} \cap I^c \neq \emptyset$ and hence $I^c = T - I$ is an i-system.

$(iii) \Rightarrow (i)$: Let X and Y be two pseudo symmetric ideals of T such that $X \nsubseteq I$ and $Y \nsubseteq I$. Then there exist elements $a \in X - I$ and $b \in Y - I$. Therefore $a, b \in I^c$. Now from (iii), it follows that $(a)_{ps} \cap (b)_{ps} \cap I^c \neq \emptyset$. That is there exists an element $c \in \{(a)_{ps} \cap (b)_{ps}\} - I$. In particular $c \in X \cap Y$ and $c \notin I$. Hence $X \cap Y \nsubseteq I$. This shows that I is a strongly irreducible pseudo symmetric ideal of T. □

Theorem 3.6. *If* H *is a proper pseudo symmetric ideal of* T *and* $t(\neq 0) \in T$ *such that* $t \notin H$ *then there exists an irreducible pseudo symmetric ideal* I *of* T *such that* $H \subseteq I$ *and* $t \notin I$.

Proof. Let $\mathcal{L} = \{H_\alpha : H_\alpha$ is a pseudo symmetric ideal of $T, H \subseteq H_\alpha, t \notin H_\alpha\}$, where $\alpha \in \Delta$ is any indexing set. As H is a pseudo symmetric ideal of T and $t \notin H$, we have $H \in \mathcal{L}$, so $\mathcal{L} \neq \emptyset$. The set $\mathcal{L}$ is partially ordered set under the inclusion of sets. If $\{H_i : i \in \Delta\}$ is any totally ordered subset (chain) of $\mathcal{L}$ then $\bigcup_{i \in \Delta} H_i$ is an pseudo symmetric ideal of T containing H and $t \notin \bigcup_{i \in \Delta} H_i$. Therefore $\bigcup_{i \in \Delta} H_i$ is an upper bound of $\{H_i : i \in \Delta\}$. Thus every chain in $\mathcal{L}$ has an upper bound in $\mathcal{L}$. Hence by Zorn's lemma [2], there exists a maximal element say I in the collection $\mathcal{L}$. This shows that I is a pseudo symmetric ideal of T such that $H \subseteq I$ and $a \notin I$. Now we show that I is an irreducible pseudo symmetric ideal of T. Let X, Y and Z be any three pseudo symmetric ideals of T such that $I = X \cap Y \cap Z$ then $I \subseteq X, I \subseteq Y$ and $I \subseteq Z$. If X, Y and Z properly contain I, then according to hypothesis $t \in X, t \in Y$ and $t \in Z$. Thus $t \in X \cap Y \cap Z = I$. Which is contradiction to the fact that $t \notin I$. Therefore either $I = X$ or $I = Y$ or $I = Z$. This shows that I is an irreducible pseudo symmetric ideal of T. □

Theorem 3.7. *Any proper pseudo symmetric ideal* I *of* T *is the intersection of all irreducible pseudo symmetric ideals of* T *containing it.*

Proof. Let I be a pseudo symmetric ideal of T. Let $\{I_i\}_{i \in \Delta}$ (where Δ is any indexing set) be the collection of all irreducible pseudo symmetric ideals of T containing I. Then $I \subseteq \bigcap_{i \in \Delta} I_i$. If this inclusion is proper that is $I \subsetneq \bigcap_{i \in \Delta} I_i$ then there exists $a(\neq 0) \in \bigcap_{i \in \Delta} I_i$ such that $a \notin I$. This implies $a \in I_i$ for all $i \in \Delta$. As

$a \notin I$, then by Theorem 3.6, there exists an irreducible pseudo symmetric ideal say H of T such that $I \subseteq H$ and $a \notin H$. This is a contradiction to $a \in I_i$ for all $i \in \Delta$. Thus $\bigcap_{i\in\Delta} I_i \subseteq I$. Therefore $I = \bigcap_{i\in\Delta} I_i$. □

Theorem 3.8. *A proper pseudo symmetric ideal I of T is prime pseudo symmetric ideal if and only if it is semiprime pseudo symmetric ideal and irreducible pseudo symmetric ideal of T.*

Proof. Suppose I is a prime pseudo symmetric ideal of T. As every prime pseudo symmetric ideal is semiprime pseudo symmetric ideal. So, I is a semiprime pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be any three pseudo symmetric ideals of T such that $I_1 \cap I_2 \cap I_3 = I$. Now $I_1I_2I_3 \subseteq I_1TT \subseteq I_1, I_1I_2I_3 \subseteq TI_2T \subseteq I_2$ and $I_1I_2I_3 \subseteq TTI_3 \subseteq I_3 \Rightarrow I_1I_2I_3 \subseteq I_1 \cap I_2 \cap I_3 = I \Rightarrow I_1I_2I_3 \subseteq I$. Since I is a prime pseudo symmetric ideal of T. Therefore $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. But $I \subseteq I_1, I \subseteq I_2$ and $I \subseteq I_3$ (since $I_1 \cap I_2 \cap I_3 = I$). Thus either $I = I_1$ or $I = I_2$ or $I = I_3$. This shows that I is an irreducible pseudo symmetric ideal of T.

Conversely, suppose that I is both semiprime pseudo symmetric ideal and irreducible pseudo symmetric ideal of T. If I_1, I_2 and I_3 are three pseudo symmetric ideals of T such that $I_1I_2I_3 \subseteq I$. Then $(I_1 \cap I_2 \cap I_3)^3 = [(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)] \subseteq I_1I_2I_3 \subseteq I$. So by the semiprimeness of I, $(I_1 \cap I_2 \cap I_3) \subseteq I$. Thus $(I_1 \cap I_2 \cap I_3) \cup I = I$. This implies that $\{(I_1 \cap I_2) \cup I\} \cap (I_3 \cup I) = I \Rightarrow \{(I_1 \cup I) \cap (I_2 \cup I)\} \cap (I_3 \cup I) = I \Rightarrow (I_1 \cup I) \cap (I_2 \cup I) \cap (I_3 \cup I) = I$. As I is an irreducible pseudo symmetric ideal, so either $(I_1 \cup I) = I$ or $(I_2 \cup I) = I$ or $(I_3 \cup I) = I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. Hence I is a prime pseudo symmetric ideal of T. □

Theorem 3.9. *The following statements for a partially ordered ternary semigroup T with identity are equivalent,*

- (i) *T is semisimple.*
- (ii) *For every pseudo symmetric ideals I_1, I_2, I_3 of $T, I_1 \cap I_2 \cap I_3 = [I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2]$.*
- (iii) *Each proper pseudo symmetric ideal of T is semiprime.*
- (iv) *Each proper pseudo symmetric ideal of T is the intersection of all irreducible semiprime pseudo symmetric ideals of T which contain it.*
- (v) *Each proper pseudo symmetric ideal of T is the intersection of all prime pseudo symmetric ideals of T which contain it.*

Proof. $(i) \Rightarrow (ii)$: Suppose T is semisimple, i.e. every ideal of T is idempotent. Let I_1, I_2 and I_3 be three pseudo symmetric ideals of T. Then $I_1 \cap I_2 \cap I_3$ is a pseudo symmetric ideal of T. Now from $(i), I_1 \cap I_2 \cap I_3 = (I_1 \cap I_2 \cap I_3)^3 =$

$[(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)] \subseteq I_1I_2I_3$. Similarly $I_1 \cap I_2 \cap I_3 \subseteq I_2I_3I_1$ and $I_1 \cap I_2 \cap I_3 \subseteq I_3I_1I_2$. This proves that

$$I_1 \cap I_2 \cap I_3 \subseteq [I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2]. \qquad (A)$$

Since I_1, I_2 and I_3 are pseudo symmetric ideals of T then $I_1I_2I_3 \subseteq I_1TT \subseteq I_1, I_1I_2I_3 \subseteq TI_2T \subseteq I_2$ and $I_1I_2I_3 \subseteq TTI_3 \subseteq I_3 \Rightarrow I_1I_2I_3 \subseteq I_1 \cap I_2 \cap I_3$. Similarly $I_2I_3I_1 \subseteq I_1 \cap I_2 \cap I_3$ and $I_3I_1I_2 \subseteq I_1 \cap I_2 \cap I_3$. This proves that

$$[I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2] \subseteq I_1 \cap I_2 \cap I_3. \qquad (B)$$

From (A) and (B) we get $I_1 \cap I_2 \cap I_3 = [I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2]$.

$(ii) \Rightarrow (i)$: Let I be a pseudo symmetric ideal of T. Then from $(ii), I = I \cap I \cap I \subseteq [III] \cap [III] \cap [III] = I^3 \cap I^3 \cap I^3 = I^3 \Rightarrow I = I^3$. This shows that every pseudo symmetric ideal of T is idempotent. Therefore T is semisimple.

$(i) \Rightarrow (iii)$: Suppose T is semisimple, i.e. every ideal of T is idempotent. Let I be a pseudo symmetric ideal of T. Let P be a pseudo symmetric ideal of T such that $P^3 \subseteq I$. Since every ideal of T is idempotent, so $P^3 = P$. Thus $P \subseteq I$. This shows that I is semiprime pseudo symmetric ideal of T. Hence each pseudo symmetric ideal of T is semiprime.

$(iii) \Rightarrow (iv)$: Suppose that each proper pseudo symmetric ideal of T is semiprime. By Theorem 3.7, any proper pseudo symmetric ideal I of T is the intersection of all irreducible pseudo symmetric ideals of T containing it. By (iii), every proper pseudo symmetric ideal of T is the intersection of all irreducible semiprime pseudo symmetric ideals of T which containing it.

$(iv) \Rightarrow (v)$: Suppose that each proper pseudo symmetric ideal of T is the intersection of all irreducible semiprime pseudo symmetric ideals of T which contain it. By Theorem 3.8, every irreducible semiprime pseudo symmetric ideals of T is a prime pseudo symmetric ideal of T.Thus each proper pseudo symmetric ideal of T is the intersection of all prime pseudo symmetric ideals of T which contain it.

$(v) \Rightarrow (i)$: Suppose each proper pseudo symmetric ideal of T is the intersection of all prime pseudo symmetric ideals of T which contain it. Let I be a pseudo symmetric ideal of T. Therefore it is the intersection of all prime pseudo symmetric ideals of T which contain it. Since intersection of prime pseudo symmetric ideals is a semiprime pseudo symmetric ideal, so I is a semiprime pseudo symmetric ideal. As $I^3 \subseteq I^3 \Rightarrow I \subseteq I^3$ but $I^3 \subseteq I$ always. This shows that $I = I^3$. Thus T is semisimple. □

Theorem 3.10. *The following statements for a partially ordered ternary semigroup T with identity are equivalent.*

(i) *T is semisimple.*

(ii) *For every pseudo symmetric ideals I_1, I_2, I_3 of $T, I_1 \cap I_2 \cap I_3 = [I_1I_2I_3]$.*

(*iii*) *Each proper pseudo symmetric ideal of T is semiprime.*

(*iv*) *Each proper pseudo symmetric ideal of T is the intersection of all irreducible semiprime pseudo symmetric ideals of T which contain it.*

(*v*) *Each proper pseudo symmetric ideal of T is the intersection of all prime pseudo symmetric ideals of T which contain it.*

Proof. Analogous to the proof of the Theorem 3.9. □

Theorem 3.11. *If every pseudo symmetric ideal of T is strongly prime pseudo symmetric ideal of T then each pseudo symmetric ideal of T is idempotent and the set of pseudo symmetric ideals of T is totally ordered by set inclusion.*

Proof. Suppose that, every pseudo symmetric ideal of T is strongly prime pseudo symmetric ideal then every pseudo symmetric ideal of T is semiprime pseudo symmetric ideal of T. Thus by Theorem 3.9, every pseudo symmetric ideal of T is idempotent. Now we prove that the set of pseudo symmetric ideals of T is totally ordered by set inclusion. Let I_1 and I_2 be any two pseudo symmetric ideals of T. By Theorem 3.9, $I_1 \cap I_2 = I_1 \cap I_2 \cap T = [I_1 I_2 T] \cap [I_2 T I_1] \cap [T I_1 I_2]$. Therefore $[I_1 I_2 T] \cap [I_2 T I_1] \cap [T I_1 I_2] \subseteq I_1 \cap I_2$. Since every pseudo symmetric ideal of T is strongly prime pseudo symmetric ideal, therefore $I_1 \cap I_2$ is strongly prime pseudo symmetric ideal. Hence either $I_1 \subseteq I_1 \cap I_2$ or $I_2 \subseteq I_1 \cap I_2$ or $T \subseteq I_1 \cap I_2$. Now, if $I_1 \subseteq I_1 \cap I_2$ then $I_1 \subseteq I_2$; if $I_2 \subseteq I_1 \cap I_2$ then $I_2 \subseteq I_1$; if $T \subseteq I_1 \cap I_2$ then $T = I_1 = I_2$. Thus the set of pseudo symmetric ideals of T is totally ordered by set inclusion. □

Theorem 3.12. *If the set of pseudo symmetric ideals of T is totally ordered by set inclusion then every pseudo symmetric ideal of T is idempotent if and only if each pseudo symmetric ideal of T is a prime pseudo symmetric ideal of T.*

Proof. Suppose every pseudo symmetric ideal of T is idempotent. Let I, X, Y and Z be pseudo symmetric ideals of T such that $XYZ \subseteq I$. As every pseudo symmetric ideal of T is idempotent so, $X \cap Y \cap Z$ is idempotent. Then $X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq XYZ \subseteq I$. Therefore $X \cap Y \cap Z \subseteq I$. As the set of all pseudo symmetric ideal of T is totally ordered by set inclusion, therefore for pseudo symmetric ideals X, Y, Z of T, we have the following six possibilities,

1) $X \subseteq Y \subseteq Z$, 2) $X \subseteq Z \subseteq Y$, 3) $Y \subseteq X \subseteq Z$
4) $Y \subseteq Z \subseteq X$, 5) $Z \subseteq X \subseteq Y$, 6) $Z \subseteq Y \subseteq X$.

In such cases, we have respectively,

1) $X \cap Y \cap Z = X$, 2) $X \cap Y \cap Z = X$, 3) $X \cap Y \cap Z = Y$,
4) $X \cap Y \cap Z = Y$, 5) $X \cap Y \cap Z = Z$, 6) $X \cap Y \cap Z = Z$.

Therefore $X \cap Y \cap Z = X$ or $X \cap Y \cap Z = Y$ or $X \cap Y \cap Z = Z$. Thus from $X \cap Y \cap Z \subseteq I$, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. This shows that I is a prime pseudo symmetric ideal of T.

Conversely, suppose that every pseudo symmetric ideal of T is a prime pseudo symmetric ideal of T. Since the set of pseudo symmetric ideals of T is totally ordered by set inclusion, therefore the notions of primeness and strongly primeness coincide. Hence by Theorem 3.11, every pseudo symmetric ideal of T is idempotent. □

Theorem 3.13. *If each pseudo symmetric ideal of T is idempotent then a pseudo symmetric ideal I of T is strongly irreducible pseudo symmetric ideal if and only if I is strongly prime pseudo symmetric ideal of T.*

Proof. Suppose that I is strongly irreducible pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be three pseudo symmetric ideals of T such that $I_1I_2I_3 \cap I_2I_3I_1 \cap I_3I_1I_2 \subseteq I$. Then $I_1 \cap I_2 \cap I_3$ is a pseudo symmetric ideals of T. Since each pseudo symmetric ideal of T is idempotent, $I_1 \cap I_2 \cap I_3 = (I_1 \cap I_2 \cap I_3)^3 = [(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)] \subseteq I_1I_2I_3$. Similarly $I_1 \cap I_2 \cap I_3 \subseteq I_2I_3I_1$ and $I_1 \cap I_2 \cap I_3 \subseteq I_3I_1I_2$. This proves that $I_1 \cap I_2 \cap I_3 \subseteq I_1I_2I_3 \cap I_2I_3I_1 \cap I_3I_1I_2 \subseteq I$. Therefore $I_1 \cap I_2 \cap I_3 \subseteq I$. Since I is strongly irreducible pseudo symmetric ideal of T. So, either $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. This shows that I is strongly prime pseudo symmetric ideal of T.

Conversely, suppose that I is strongly prime pseudo symmetric ideal of T and $I_1 \cap I_2 \cap I_3 \subseteq I$ where I_1, I_2, I_3 are three pseudo symmetric ideals of T. Then $I_1I_2I_3 \subseteq I_1TT \subseteq I_1, I_1I_2I_3 \subseteq TI_2T \subseteq I_2$ and $I_1I_2I_3 \subseteq TTI_3 \subseteq I_3 \Rightarrow I_1I_2I_3 \subseteq I_1 \cap I_2 \cap I_3$. Similarly, $I_2I_3I_1 \subseteq I_1 \cap I_2 \cap I_3$ and $I_3I_1I_2 \subseteq I_1 \cap I_2 \cap I_3$. This proves that $I_1I_2I_3 \cap I_2I_3I_1 \cap I_3I_1I_2 \subseteq I_1 \cap I_2 \cap I_3 \subseteq I$. Since I is strongly prime pseudo symmetric ideal of T. So, either $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. This shows that I is strongly irreducible pseudo symmetric ideal of T. □

Theorem 3.14. *For a partially ordered ternary semigroup T, the following statements are equivalent.*

- (i) *The set of pseudo symmetric ideals of T is totally ordered by set inclusion.*
- (ii) *Each proper pseudo symmetric ideal of T is strongly irreducible pseudo symmetric ideal.*
- (iii) *Each proper pseudo symmetric ideal of T is irreducible pseudo symmetric ideal.*

Proof. $(i) \Rightarrow (ii)$: Let I be a proper pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be the three pseudo symmetric ideals of T such that

$$I_1 \cap I_2 \cap I_3 \subseteq I. \qquad (A)$$

As the set of all pseudo symmetric ideal of T is totally ordered by set inclusion, therefore for pseudo symmetric ideals I_1, I_2, I_3 of T, we get the following six possibilities,

1) $I_1 \subseteq I_2 \subseteq I_3$, 2) $I_1 \subseteq I_3 \subseteq I_2$, 3) $I_2 \subseteq I_1 \subseteq I_3$
4) $I_2 \subseteq I_3 \subseteq I_1$, 5) $I_3 \subseteq I_1 \subseteq I_2$, 6) $I_3 \subseteq I_2 \subseteq I_1$.

In such cases, we have respectively,

1) $I_1 \cap I_2 \cap I_3 = I_1$, 2) $I_1 \cap I_2 \cap I_3 = I_1$, 3) $I_1 \cap I_2 \cap I_3 = I_2$,
4) $I_1 \cap I_2 \cap I_3 = I_2$, 5) $I_1 \cap I_2 \cap I_3 = I_3$, 6) $I_1 \cap I_2 \cap I_3 = I_3$.

Therefore $I_1 \cap I_2 \cap I_3 = I_1$ or $I_1 \cap I_2 \cap I_3 = I_2$ or $I_1 \cap I_2 \cap I_3 = I_3$. Thus from (A), either $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. This shows that I is a strongly irreducible pseudo symmetric ideal of T.

$(ii) \Rightarrow (iii)$: Let I be a proper pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be the three pseudo symmetric ideals of T such that $I_1 \cap I_2 \cap I_3 = I$. Then $I \subseteq I_1, I \subseteq I_2, I \subseteq I_3$. On the other hand by hypothesis we have $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. Thus $I_1 = I$ or $I_2 = I$ or $I_3 = I$. Thus I is an irreducible pseudo symmetric ideal of T. This shows that each proper pseudo symmetric ideal of T is irreducible pseudo symmetric ideal of T.

$(iii) \Rightarrow (i)$: Suppose that each proper pseudo symmetric ideal of T is irreducible pseudo symmetric ideal of T. Let I_1, I_2 be two pseudo symmetric ideals of T then $I_1 \cap I_2$ is a pseudo symmetric ideal of T. Therefore from $(iii), I_1 \cap I_2$ is an irreducible pseudo symmetric ideal of T or $I_1 \cap I_2 = T$. As $I_1 \cap I_2 \cap T = I_1 \cap I_2$, the irreducibility of $I_1 \cap I_2$ implies that either $I_1 = I_1 \cap I_2$ or $I_2 = I_1 \cap I_2$ or $T = I_1 \cap I_2$. Therefore either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$ or $I_1 = I_2 = T$. Hence the set of pseudo symmetric ideals of T is totally ordered by set inclusion. □

Theorem 3.15. *A proper pseudo symmetric ideal I of T is a strongly irreducible semiprime pseudo symmetric ideal if and only if it is a strongly prime pseudo symmetric ideal of T.*

Proof. Let I be a strongly irreducible semiprime pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be the three pseudo symmetric ideals of T such that $[I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2] \subseteq I$. Now, $(I_1 \cap I_2 \cap I_3)^3 = [(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)(I_1 \cap I_2 \cap I_3)] \subseteq I_1I_2I_3$. Similarly $(I_1 \cap I_2 \cap I_3)^3 \subseteq I_2I_3I_1$ and $(I_1 \cap I_2 \cap I_3)^3 \subseteq I_3I_1I_2$. Thus $(I_1 \cap I_2 \cap I_3)^3 \subseteq [I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2] \subseteq I$. As $I_1 \cap I_2 \cap I_3$ is pseudo symmetric ideal of T and I is a semiprime pseudo symmetric ideal of T, therefore $I_1 \cap I_2 \cap I_3 \subseteq I$. Since I is a strongly irreducible pseudo symmetric ideal of T, therefore $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. This shows that I strongly prime pseudo symmetric ideal of T.

Conversely suppose that I is a strongly prime pseudo symmetric ideal of T. As every strongly prime pseudo symmetric ideal of T is a prime pseudo symmetric ideal of T and every prime pseudo symmetric ideal of T is a semiprime pseudo symmetric ideal of T. Hence I is a semiprime pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be the three pseudo symmetric ideals of T such that $I_1 \cap I_2 \cap I_3 \subseteq I$. Now $I_1I_2I_3 \subseteq I_1TT \subseteq I_1, I_1I_2I_3 \subseteq TI_2T \subseteq I_2$ and $I_1I_2I_3 \subseteq TTI_3 \subseteq I_3 \Rightarrow I_1I_2I_3 \subseteq I_1 \cap I_2 \cap I_3$. Similarly $I_2I_3I_1 \subseteq I_1 \cap I_2 \cap I_3$ and $I_3I_1I_2 \subseteq I_1 \cap I_2 \cap I_3$. Then $[I_1I_2I_3] \cap [I_2I_3I_1] \cap [I_3I_1I_2] \subseteq I_1 \cap I_2 \cap I_3 \subseteq I$. Since I is a strongly prime pseudo symmetric ideal of T. Therefore $I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. This shows that I strongly irreducible pseudo symmetric ideal of T. □

Theorem 3.16. *Every maximal pseudo symmetric ideal I of T is irreducible pseudo symmetric ideal of T.*

Proof. Let I be a maximal pseudo symmetric ideal of T. Suppose I is not irreducible pseudo symmetric ideal of T. i.e. for any three pseudo symmetric ideals I_1, I_2 and I_3 of T such that $I_1 \cap I_2 \cap I_3 = I \Rightarrow I_1 \neq I, I_2 \neq I$ and $I_3 \neq I \Rightarrow I \subset I_1 \subset T, I \subset I_2 \subset T, I \subset I_3 \subset T$. Which is contradiction to I be a maximal pseudo symmetric ideal of T. Hence I is irreducible pseudo symmetric ideal of T. □

Theorem 3.17. *If I is a prime pseudo symmetric ideal of T then I is a strongly irreducible pseudo symmetric ideal of T.*

Proof. Suppose I is a prime pseudo symmetric ideal of T. Let I_1, I_2 and I_3 be the three pseudo symmetric ideals of T such that $I_1 \cap I_2 \cap I_3 \subseteq I$. Now, $I_1I_2I_3 \subseteq I_1TT \subseteq I_1, I_1I_2I_3 \subseteq TI_2T \subseteq I_2$ and $I_1I_2I_3 \subseteq TTI_3 \subseteq I_3 \Rightarrow I_1I_2I_3 \subseteq I_1 \cap I_2 \cap I_3 \subseteq I$. Then $I_1I_2I_3 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ or $I_3 \subseteq I$. Since I is a prime pseudo symmetric ideal. Hence I is a strongly irreducible pseudo symmetric ideal of T. □

4. Topological space of pseudo symmetric ideals

In this section, we describe the topological space of strongly irreducible pseudo symmetric ideals of T.

Let $\mathcal{I}$ be the family of all pseudo symmetric ideals of T and $\mathcal{S}$ be the family of all strongly irreducible pseudo symmetric ideals of T. For each $I \in \mathcal{I}$, we define $\Theta_I = \{J \in \mathcal{S} : I \nsubseteq J\}$

Theorem 4.1. *If X and Y are the two pseudo symmetric ideals of T such that $X \subseteq Y$ then $\Theta_X \subseteq \Theta_Y$.*

Proof. Let X, Y be the two pseudo symmetric ideals of T such that $X \subseteq Y$. Let $J \in \Theta_X$ then $J \in \mathcal{S}$ such that $X \nsubseteq J$. Suppose $Y \subseteq J$. If $Y \subseteq J$ then $X \subseteq J$ (since $X \subseteq Y$); which is a contradiction to $X \nsubseteq J$. Therefore $J \in \mathcal{S}$ such that $Y \nsubseteq J$. Hence $J \in \Theta_Y$. This shows that $\Theta_X \subseteq \Theta_Y$. □

Theorem 4.2. *The following statements hold in T.*

(*i*) $\Theta_{\{0\}} = \emptyset$.

(*ii*) $\Theta_T = \mathcal{S}$.

(*iii*) $\Theta_{I_1} \cap \Theta_{I_2} = \Theta_{I_1 \cap I_2}$ *for all pseudo symmetric ideals I_1, I_2 of T.*

(*iv*) $\bigcup_{\alpha\in\Delta}\Theta_{I_\alpha} = \Theta_{\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)_{ps}}$ *for every family* $\{I_\alpha\}_{\alpha\in\Delta}$ *of pseudo symmetric ideals of* T, *where* $\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)_{ps}$ *is the pseudo symmetric ideal of* T *generated by* $\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)$ *and* Δ *is any indexing set.*

Proof. (i) As $\{0\}$ is a pseudo symmetric ideal of T and 0 belongs to every pseudo symmetric ideal of T, we get $\Theta_{\{0\}} = \{J \in \mathcal{S} : \{0\} \nsubseteq J\} = \emptyset$.

(ii) Since T itself is a pseudo symmetric ideal of T and every strongly irreducible pseudo symmetric ideal of T being proper, i.e. $\mathcal{S}$ is the collection of all proper strongly irreducible pseudo symmetric ideals of T. Therefore $\Theta_T = \{J \in \mathcal{S} : T \nsubseteq J\} = \mathcal{S}$.

(iii) Let I_1, I_2 be two pseudo symmetric ideals of T and let $J \in \Theta_{I_1} \cap \Theta_{I_2}$ then $J \in \mathcal{S}$ such that $I_1 \nsubseteq J$ and $I_2 \nsubseteq J$. Suppose $I_1 \cap I_2 \subseteq J$. Now $I_1 \cap I_2 \cap T = I_1 \cap I_2 \subseteq J$ implies $I_1 \subseteq J$ or $I_2 \subseteq J$ or $T \subseteq J$ (since J is a strongly irreducible pseudo symmetric ideals of T). But $T \nsubseteq J$ (since J is proper). Therefore $I_1 \subseteq J$ or $I_2 \subseteq J$; which is a contradiction. Hence $I_1 \cap I_2 \nsubseteq J$. Therefore $J \in \Theta_{I_1\cap I_2}$. Thus $\Theta_{I_1} \cap \Theta_{I_2} \subseteq \Theta_{I_1\cap I_2}$. Then by Theorem 4.1, $\Theta_{I_1\cap I_2} \subseteq \Theta_{I_1}$ and $\Theta_{I_1\cap I_2} \subseteq \Theta_{I_2}$ (since $I_1 \cap I_2 \subseteq I_1, I_1 \cap I_2 \subseteq I_2$). Therefore $\Theta_{I_1\cap I_2} \subseteq \Theta_{I_1} \cap \Theta_{I_2}$.This shows that $\Theta_{I_1} \cap \Theta_{I_2} = \Theta_{I_1\cap I_2}$.

(iv) Let $\{I_\alpha\}_{\alpha\in\Delta}$ (where Δ is any indexing set.) be family of pseudo symmetric ideals of T. Then $J \in \bigcup_{\alpha\in\Delta}\Theta_{I_\alpha} \Leftrightarrow J \in \Theta_{I_\alpha}$ for some $\alpha \in \Delta \Leftrightarrow I_\alpha \nsubseteq J$ for some $\alpha \in \Delta \Leftrightarrow \left(\bigcup_{\alpha\in\Delta} I_\alpha\right) \nsubseteq J \Leftrightarrow J \in \Theta_{\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)_{ps}}$ where $\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)_{ps}$ is the pseudo symmetric ideal of T generated by $\left(\bigcup_{\alpha\in\Delta} I_\alpha\right)$. □

Corollary 4.3. *The family,* $\mathfrak{J}(\mathcal{S}) = \{\Theta_I : I \text{is a pseudo symmetric ideals of } T\}$ *forms a topology on the set* $\mathcal{S}$.

Theorem 4.4. *If* T *is partially ordered ternary semigroup with identity then* $\mathcal{S}$ *is a compact space.*

Proof. Suppose $\{\Theta_{I_k} : k \in \Delta\}$ is an open covering of $\mathcal{S}$, where Δ is an indexing set. That is $\mathcal{S} = \bigcup_{k\in\Delta}\Theta_{I_k}$. By Theorem 4.2, $\Theta_T = \mathcal{S} = \bigcup_{k\in\Delta}\Theta_{I_k} \Rightarrow \Theta_T = \Theta_{\left(\bigcup_{k\in\Delta} I_k\right)_{ps}} \Rightarrow T = \left(\bigcup_{k\in\Delta} I_k\right)_{ps}$. As $e \in T, e \in \left(\bigcup_{k\in\Delta} I_k\right)_{ps}$. Hence $e \in \left(\bigcup_{i=1}^{n} I_i\right)_{ps} \Rightarrow T = \left(\bigcup_{i=1}^{n} I_i\right)_{ps} \Rightarrow \mathcal{S} = \bigcup_{k=1}^{n}\Theta_{I_k}$. Therefore every open cover of $\mathcal{S}$ has finite subcover. Hence $\mathcal{S}$ is compact space. □

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