Quasigroups and Related Systems **30** (2022), 307 – 315
https://doi.org/10.56415/qrs.v30.26

The relationship between EQ algebras and equality algebras

Akbar Paad

Abstract. It is proved that every involutive equivalential equality algebra $(E, \wedge, \sim, 1)$, is an involutive residualted lattice EQ-algebra, which operation $\otimes$ is defined by $x \otimes y = (x \to y')'$. Moreover, it is showen that by an involutive residualted lattice EQ-algebra we have an involutive equivalential equality algebra.

1. Introduction

Fuzzy type theory (FTT) has been developed by Novák as a fuzzy logic of higher order, the fuzzy version of the classical type theory of the classical logic of higher order. BL-algebras, MTL-algebras, MV-algebras are the best known classes of residuated lattices [4, 5] and since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra [7] by Novák and De Baets was introduced. EQ-algebras generalize the residuated lattices that have three binary operations meet, multiplication, fuzzy equality and a unit element. If the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original product we still obtain an EQ-algebra, and this fact might make it difficult to obtain certain algebraic results. For this reason, equality algebras were introduced by Jeni [6], which the motivation cames from EQ-algebras [7]. These algebras are assumed for a possible algebraic semantics of fuzzy type theory. It was proved [1, 6], that any equality algebra has a corresponding BCK-meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra. Since equality algebras could also be candidates for a possible algebraic semantics for fuzzy type theory, their study is highly motivated. In [9], by considering the notion of equality algebra, it is shown that there are relations among equality algebras

2010 Mathematics Subject Classification: 08A72, 06E99, 03G25.
Keywords: EQ-algebra, Equality algebra

and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. Specially, it was proved that every good EQ-algebra is equality algebra but the converse is open problem which means how multiplication operation, $\otimes$, on equality algebra $(E, \wedge, \sim, 1)$ should be defined such that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra?

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1. (cf. [6]) An algebra $(E, \wedge, \sim, 1)$ of the type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions, for all $x, y, z \in E$:
$(E1)$ $(E, \wedge, 1)$ is a meet-semilattice with top element 1,
$(E2)$ $x \sim y = y \sim x$,
$(E3)$ $x \sim x = 1$,
$(E4)$ $x \sim 1 = x$,
$(E5)$ $x \leqslant y \leqslant z$ implies $x \sim z \leqslant y \sim z$ and $x \sim z \leqslant x \sim y$,
$(E6)$ $x \sim y \leqslant (x \wedge z) \sim (y \wedge z)$,
$(E7)$ $x \sim y \leqslant (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet* (infimum) and $\sim$ is an equality operation. We write $x \leqslant y$ if and only if $x \wedge y = x$, for all $x, y \in E$. Also, other two operations are defined, called *implication* and *equivalence operation*, respectively:

$$x \to y = x \sim (x \wedge y). \tag{I}$$

$$x \leftrightarrow y = (x \to y) \wedge (y \to x). \tag{II}$$

An equality algebra $(E, \sim, \wedge, 1)$ is bounded if there exists an element $0 \in E$ such that $0 \leq x$, for all $x \in E$. In a bounded equality algebra E, we define the negation " $'$ " on E by, $x' = x \to 0 = x \sim 0$, for all $x \in E$. If $x'' = x$, for all $x \in E$, then the bounded equality algebra E is called involutive. A lattice equality algebra is an equality algebra which is a lattice. Equality algebra E (and as well as its equality operation $\sim$) called *equivalential*, if $\sim$ coincides with the equivalence operation of a suitably chosen equality algebra.

Theorem 2.2. (cf. [6]) *An equality algebra $(E, \sim, \wedge, 1)$ is equivalential if and only if for all $x, y \in E$, $x \sim y = (x \sim (x \wedge y)) \wedge (y \sim (x \wedge y))$.*

Proposition 2.3. (cf. [6]) *Let $(E, \wedge, \sim, 1)$ be an equality algebra. Then the following properties hold, for all $x, y, z \in E$:*
(i) $x \to y = 1$ *if and only if* $x \leqslant y$,
(ii) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(iii) $x \leqslant (x \sim y) \sim y$,
(iv) $x \leqslant y$ *implies* $y \to z \leqslant x \to z$, $z \to x \leqslant z \to y$,
(v) $x \sim y \leqslant x \leftrightarrow y \leqslant x \to y$,
(vi) $x \to (y \to z) = y \to (x \to z)$.
(vii) $x \to y \leqslant (y \to z) \to (x \to z.)$

Definition 2.4. (cf. [7]) An *EQ-algebra* is an algebra $(E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms:
$(EQ1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $x \leqslant y$ if and only
if $x \wedge y = x$,
$(EQ2)$ $(E, \otimes, 1)$ is a commutative monoid and $\otimes$ is isotone with respect to $\leq$,
$(EQ3)$ $x \sim x = 1$ (reflexivity axiom),
$(EQ4)$ $((x \wedge y) \sim z) \otimes (s \sim x) \leqslant z \sim (s \wedge y)$ (substitution axiom),
$(EQ5)$ $(x \sim y) \otimes (s \sim t) \leqslant (x \sim s) \sim (y \sim t)$ (congruence axiom),
$(EQ6)$ $(x \wedge y \wedge z) \sim x \leqslant (x \wedge y) \sim x$ (monotonicity axiom),
$(EQ7)$ $x \otimes y \leqslant x \sim y$ (boundedness axiom),
For all $s, t, x, y, z \in E$.

Let E be an EQ-algebra. Then for all $x, y \in E$, we put

$$x \to y = (x \wedge y) \sim x, \qquad \tilde{x} = x \sim 1.$$

The derived operation $\to$ is called *implication.* If an EQ-algebra E contains a bottom element 0, then we may define the unary operation $\neg$ on E by $\neg x = x \sim 0 = x \to 0$.

Definition 2.5. (cf. [7]) Let E be an EQ-algebra. We say that it is
(i) *good*, if $\tilde{x} = x$ for all $x \in E$.
(ii) *residuated*, if $(x \otimes y) \wedge z = x \otimes y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$ for all
$x, y, z \in E$
(iii) *envolutive* (*IEQ-algebra*), if $\neg\neg x = x$, for all $x \in E$.
(iv) *lattice-ordered EQ-algebra* if it has a lattice reduct.

(v) *lattice EQ-algebra* (*lEQ-algebra*) if it is a lattice-ordered EQ-algebra in which the following substitution axiom holds forall $x, y, z, w \in E$:

$$((x \vee y) \sim z) \otimes (w \sim x) \leqslant ((w \vee y) \sim z)$$

Proposition 2.6. (cf. [3]) *For an EQ-algebra E the following are equivalent:*
(i) *E is residuated,*
(ii) *E is good and $x \leqslant y \to (x \otimes y)$ holds for all $x, y \in E$.*

Proposition 2.7. (cf. [2, 7]) *Let E be an EQ-algebra. Then for any $x, y, z \in E$:*
(i) *$x = 1 \to x$ and $x \to (y \to z) = y \to (x \to z)$, where E is residuated.*
(ii) *$x \otimes y \leqslant z$ if and only if $x \leqslant y \to z$, where E is good.*

Theorem 2.8. (cf. [7]) *Any lEQ-algebra E is a good, spanned and separated lattice EQ-algebra.*

Definition 2.9. (cf. [8]) A *residuated lattice* is an algebra $(E, \vee, \wedge, \otimes, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following axioms:
(i) $(E, \vee, \wedge, 0, 1)$ is a bounded lattice,
(ii) $(E, \otimes, 1)$ is a commutative monoid,
(iii) $x \otimes y \leqslant z$ if and only if $x \leqslant y \to z$, for any $x, y, z \in E$.

Theorem 2.10. (cf. [9]) *The algebraic structure $(E, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice if and only if*
$(RL1)$ *$(E, \vee, \wedge, 0, 1)$ is a bounded lattice,*
$(RL2)$ *$(E, \to, 1)$satisfies $x = 1 \to x$ and $x \to (y \to z) = y \to (x \to z)$,*
$(RL3)$ *$x \otimes y \leqslant z$ if and only if $x \leqslant y \to z$, for any $x, y, z \in E$.*

Theorem 2.11. (cf. [9]) *For any residuated lattice $\mathcal{E} = (E, \vee, \wedge, \to, 0, 1)$, the structure $\psi(\mathcal{E}) = (E, \vee, \wedge, \leftrightarrow, 0, 1)$ is a bounded lattice equality algebra, where $\leftrightarrow$ denotes the equivalence operation of E. Moreover, the implication of $\psi(\mathcal{E})$ coincides with $\leftrightarrow$, that is, $x \to y = x \leftrightarrow (x \wedge y)$.*

3. Relation between algebras

Theorem 3.1. (cf. [9]) *Every good EQ-algebra $(E, \wedge, \sim, \otimes, 1)$ is an equality algebra.*

Open problem. Under what suitable conditions the converse of Theorem 3.1 is correct? Which means how multiplication operation, $\otimes$, on equality algebra $(E, \wedge, \sim, 1)$ should be defined such that $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra?

In the following, by adding some conditions to an equality algebra, we answer to this open problem as follow:

Theorem 3.2. *Let $(E, \wedge, \sim, 1)$ be an involutive equivalential equality algebra. Then $(E, \wedge, \sim, \otimes, 1)$ is an involutive residualted lattice EQ-algebra, which operation $\otimes$ is defined by $x \otimes y = (x \to y')'$.*

Proof. Let $(E, \wedge, \sim, 1)$ be an involutive equivalential equality algebra. Then $(E, \wedge, 1)$ is a meet semilattice with top element 1 and so $(EQ1)$ holds. For $x, y \in E$, we define $x \otimes y = (x \to y')'$ and we prove that $(E, \otimes, 1)$ is a commutative monoid and $\otimes$ is isotone with respect to $\leqslant$. By Proposition 2.3(vi), for $x, y \in E$, we have

$$x \otimes y = (x \to y')' = (x \to (y \to 0))' = (y \to (x \to 0))' = (y \to x')' = y \otimes x.$$

Hence, operation $\otimes$ is commutative.

Let $x, y, z \in E$. Then by Proposition 2.3(vi), we have

$$\begin{aligned} x \otimes (y \otimes z) &= (x \to (y \otimes z)'))' = (x \to (y \to z')'')' = (x \to (y \to z'))' \\ &= (x \to (y \to (z \to 0)))' = (x \to (z \to (y \to 0)))' \\ &= (z \to (x \to (y \to 0)))' = (z \to (x \to (y \to 0)''))' \\ &= (z \to (x \otimes y)')' = z \otimes (x \otimes y) = (x \otimes y) \otimes z. \end{aligned}$$

Hence, operation $\otimes$ is associative. Now, let $x \leqslant y$. Then by Proposition 2.3(iv), $y' = y \to 0 \leqslant x \to 0 = x'$ and so $z \to y' \leqslant z \to x'$. Hence, $x \otimes z = z \otimes x = (z \to x')' \leqslant (z \to y')' = z \otimes y = y \otimes z$. Thus, the operation $\otimes$ is isotone respect to $\leqslant$. Moreover, $x \otimes 1 = (x \to 1')' = x'' = x$ and so $(E, \otimes, 1)$ is a commutative monoid which proves the $(EQ2)$. Since by $(E3)$, $x \sim x = 1$, for any $x \in E$, we conclude that $(EQ3)$. In the follow, we prove $x \otimes y \leqslant z$ if and only if $x \leqslant y \to z$, for any $x, y, z \in E$. Since E is involutive and by Proposition 2.3(i) and (iv), for any $x, y, z \in E$, we have $x \otimes y \leqslant z$ if and only if $(x \to y')' \leqslant z$ if and only if $z' \leqslant (x \to y')''$ if and only if $z' \leqslant x \to y'$ if and only if $x \leqslant z' \to y'$ if and only if $x \leqslant y \to z''$ if and only if $x \leqslant y \to z$. Now, we prove $(EQ4)$. Let $x, y, z, w \in E$. Then

$$((x \wedge y) \sim z) \otimes (w \sim x) \leqslant z \sim (w \wedge y)$$

if and only if

$$(((x \wedge y) \sim z) \rightarrow (w \sim x)')' \leqslant z \sim (w \wedge y)$$

if and only if

$$(z \sim (w \wedge y))' \leqslant (((x \wedge y) \sim z) \rightarrow (w \sim x)')''$$

if and only if

$$(z \sim (w \wedge y))' \leqslant ((x \wedge y) \sim z) \rightarrow (w \sim x)'$$

if and only if

$$(z \sim (w \wedge y))' \otimes ((x \wedge y) \sim z) \leqslant (w \sim x)'$$

if and only if

$$(w \sim x)'' \leqslant ((z \sim (w \wedge y))' \otimes ((x \wedge y) \sim z))'$$

if and only if

$$(w \sim x) \leqslant ((z \sim (w \wedge y))' \otimes ((x \wedge y) \sim z))'$$

if and only if

$$(w \sim x) \leqslant ((z \sim (w \wedge y))' \rightarrow ((x \wedge y) \sim z)')''$$

if and only if

$$(w \sim x) \leqslant (z \sim (w \wedge y))' \rightarrow ((x \wedge y) \sim z)'.$$

Now, since by $(E6)$ and Proposition 2.3(v), for any $x, y, z, w \in E$, we have

$$\begin{aligned}(w \sim x) &\leqslant (x \wedge y) \sim (w \wedge y) \\ &\leqslant ((w \wedge y) \sim z) \sim ((x \wedge y) \sim z) \\ &\leqslant ((w \wedge y) \sim z)' \sim ((x \wedge y) \sim z)' \\ &\leqslant ((w \wedge y) \sim z)' \rightarrow ((x \wedge y) \sim z)' \\ &= (z \sim (w \wedge y))' \rightarrow (z \sim (x \wedge y))'.\end{aligned}$$

Now, since the inequality $(w \sim x) \leqslant (z \sim (w \wedge y))' \rightarrow (z \sim (x \wedge y))'$, holds for any $x, y, z, w \in E$, we conclude that $((x \wedge y) \sim z) \otimes (w \sim x) \leqslant z \sim (w \wedge y)$, for any $x, y, z, w \in E$ and so $(EQ4)$ holds.

For $(EQ5)$, we must prove $(x \sim y) \otimes (s \sim t) \leqslant (x \sim s) \sim (y \sim t)$, for any $x, y, s, t \in E$. Since for any $x, y, s, t \in E$, by $(E7)$ and Proposition 2.3(v) and (vi), we have:

$$\begin{aligned}(s \sim t) &\leqslant (x \sim s) \sim (x \sim t) \leqslant (x \sim s) \to (x \sim t)\\ &\leqslant (x \sim s) \to ((x \sim y) \sim (y \sim t))\\ &\leqslant (x \sim s) \to ((x \sim y) \to (y \sim t))\\ &= (x \sim y) \to ((x \sim s) \to (y \sim t)).\end{aligned}$$

So, we conclude that $(s \sim t) \otimes (x \sim y) \leqslant (x \sim s) \to (y \sim t)$. Moreover, since by Proposition 2.3(iv) and (v), for any $x, y, s, t \in E$,

$$\begin{aligned}(s \sim t) &\leq (t \sim y) \sim (s \sim y)\\ &\leq (y \sim t) \to (y \sim s)\\ &\leq (y \sim t) \to ((x \sim y) \sim (x \sim s))\\ &\leq (y \sim t) \to ((x \sim y) \to (x \sim s))\\ &= (x \sim y) \to ((y \sim t) \to (x \sim s)).\end{aligned}$$

We conclude that $(s \sim t) \otimes (x \sim y) \leqslant (y \sim t) \to (x \sim s)$ and so we have

$$(s \sim t) \otimes (x \sim y) \leqslant ((x \sim s) \to (y \sim t)) \wedge ((y \sim t) \to (x \sim s))$$

and since E is equivalential, we get that

$$((x \sim s) \to (y \sim t)) \wedge ((y \sim t) \to (x \sim s)) = (x \sim s) \sim (y \sim t)$$

Hence,

$$(s \sim t) \otimes (x \sim y) \leqslant (x \sim s) \sim (y \sim t).$$

Therefore, $(EQ5)$ is established.

For $(EQ6)$, assume that $x, y, z \in E$. Then by $x \wedge y \wedge z \leqslant x \wedge y \leqslant x$ and $(E5)$, we get that

$$(x \wedge y \wedge z) \sim x \leqslant (x \wedge y) \sim x.$$

Hence, $(EQ6)$ holds. Finally, let $x, y \in E$. Then by Proposition 2.3(iii) and (v),

$$x \leqslant (x \sim y) \sim y = y \sim (x \sim y) \leq y \to (x \sim y).$$

Hence, $x \otimes y \leqslant x \sim y$ and so $(EQ7)$ is established. Therefore, $(E, \wedge, \sim, \otimes, 1)$ is an EQ-algebra and since $x = x^{''} = (x \to 0) \to 0 = \neg\neg x$ and by $(E4)$,

$1 \sim x = x$, for any $x \in E$, by Theorem 2.8, we conclude that $(E, \wedge, \sim, \otimes, 1)$ is an involutive good lattice EQ-algebra. Moreover, since by Proposition 2.3(iii), $x \leqslant (x \sim y) \sim y$, for any $x, y \in E$ and by $x \otimes y \leqslant x \otimes y$, we have $x \leqslant y \rightarrow (x \otimes y)$, for any $x, y \in E$, by Proposition 2.6, we conclude that $(E, \wedge, \sim, \otimes, 1)$ is a residuated EQ-algebra. Therefrore, $(E, \wedge, \sim, \otimes, 1)$ is an involutive residualted lattice EQ-algebra. □

Theorem 3.3. *Let $(E, \wedge, \sim, \otimes, 1)$ be an involutive residualted lattice EQ-algebra. Then $(E, \vee, \wedge, \otimes, \leftrightarrow, 0, 1)$ be an involutive equivalential equality algebra.*

Proof. Let $(E, \wedge, \sim, \otimes, 1)$ be an involutive residualted lattice EQ-algebra. Then $(E, \vee, \wedge, 0, 1)$ is a bounded lattice and by Theorem 2.8, E is a good EQ-algebra and so by Proposition 2.7(i), $x = 1 \rightarrow x$ and $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, for any $x, y \in E$. Moreover, since E is a residuated *EQ*-algebra, by Proposition 2.7(ii), we get that $x \otimes y \leqslant z$ if and only if $x \leqslant y \rightarrow z$, for any $x, y, z \in E$. Hence, by Theorem 2.10, $(E, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a residuated lattice and so by Theorem 2.11, $(E, \vee, \wedge, \otimes, \leftrightarrow, 0, 1)$ is a bounded lattice equality algebra, where $\leftrightarrow$ denote the equivalence operation of E and $x \rightarrow y = x \leftrightarrow (x \wedge y)$ and since

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x) = (x \leftrightarrow (x \wedge y)) \wedge (y \leftrightarrow (y \wedge x))$$

by Theorem 2.2, we conclude that $(E, \wedge, \leftrightarrow, 1)$ is an equvalential equality algebra. Now, we prove $(E, \wedge, \leftrightarrow, 0, 1)$ is an involutive equality algebra. For $x, y \in E$, we have

$$x \leftrightarrow 0 = (x \rightarrow 0) \wedge (0 \rightarrow x) = (x \rightarrow 0) \wedge 1 = x \rightarrow 0$$

and since $(E, \wedge, \sim, \otimes, 1)$ an involutive EQ-algebra we get that

$$(x \leftrightarrow 0) \leftrightarrow 0 = (x \rightarrow 0) \leftrightarrow 0 = (x \rightarrow 0) \rightarrow 0 = x.$$

Therefore, $(E, \vee, \wedge, \otimes, \leftrightarrow, 0, 1)$ is an involutive equivalential equality algebra. □

4. Conclusion

The main result of this paper is devoted to solution of open problem which is about relation between EQ-algebras and equality algebras. In [9], it is proved that every good EQ-algebra is a equality algebra and it is asked under what suitable conditions the converse is correct? We proved that every involutive equivalential equality algebra $(E, \wedge, \sim, 1)$, is an involutive residualted lattice EQ-algebra, which operation $\otimes$ is defined by $x \otimes y = (x \to y')'$. Moreover, we showed that by an involutive residualted lattice EQ-algebra we have an involutive equivalential equality algebra.

References

[1] **L.C. Ciungu**, *Internal states on equality algebras*, Soft Comput., **19** (2015), 939 − 953.

[2] **M. El-Zekey**, *Representable good EQ-algebras*, Soft Comput., **14** (2010), 1011 − 1023.

[3] **M. El-Zekey, V. Novák, R. Mesiar**, *On good EQ-algebras*, Fuzzy Sets and Systems, **178** (2011), 1 − 23.

[4] **F. Esteva, L. Godo**, *Monoidal t-normbased logic: towards a logic for left-continuous t-norms*, Fuzzy Sets and Systems, **124** (2001) no. 3, 271 − 288.

[5] **P. Hájek**, *Metamathematics of Fuzzy Logic*, Springer Science and Business Media, 1998.

[6] **S. Jenei**, *Equality algebras, Studia Logica*, **100** (2012) no. 6, 1201 − 1209.

[7] **V. Novák, B. De Baets**, *EQ-algebras, Fuzzy Sets and Systems*, **160** (2009) no. 20, 2956 − 2978.

[8] **M. Ward, R.P. Dilworth**, *Residuated lattices, Trans. Am. Math. Soc.* **45** (1939), 335˘354.

[9] **F. Zebardast, R.A. Borzooei, M. Aaly Kologani**, *Results on Equality algebras*, Information Sciences, **381** (2017), 270 − 282.

Received Jun 20, 2021

Department of Mathematics
University of Bojnord
Bojnord
Iran
e-mail: akbar.paad@gmail.com, a.paad@ub.ac.ir