Quasigroups and Related Systems **30** (2022), 329 – 338
https://doi.org/10.56415/qrs.v30.28

On the finite loop algebra $F[M(C_p^m \rtimes C_2, 2)]$

Swati Sidana

Abstract. Let $G = C_p^m \rtimes C_2$ be a generalized dihedral group for an odd prime p and a natural number m, $L = M(G, 2)$ be the $RA2$ loop obtained from G and F be a finite field of characteristic 2. For the loop algebra $F[L]$, we determine the Jacobson radical $J(F[L])$ of $F[L]$ and the Wedderburn decomposition of $F[L]/J(F[L])$. The structure of $1 + J(F[L])$ is also determined.

1. Introduction

The problem of determining the structure of the unit loop of the loop ring is of great interest to many authors. Goodaire in [4], Jespers and Leal in [5] determined the unit loops of integral loop rings of RA loops. Ferraz, Goodaire and Milies [3] studied some classes of semisimple loop algebras of RA loops over finite fields. Sidana and Sharma have characterized the structure of the unit loops of the finite loop algebras of many RA and $RA2$ loops in [7, 8, 9]. In [1], Chein and Goodaire studied the loops whose loop rings over the field of characteristic 2 are alternative. In this paper, we study the structure of the unit loop of the loop algebra $F[L]$ of $RA2$ loop $L = M(G, 2)$ obtained from the group

$$G = C_p^m \rtimes C_2 = \langle a_1, a_2, \ldots, a_m, b \mid a_i^p, b^2, a_i a_j a_i^{-1} a_j^{-1}, b a_i b a_i,\ i, j = 1, 2, \ldots, m\rangle,$$

p an odd prime and m a natural number, over the finite field F of characteristic 2 which contains a primitive p^{th} root of unity. The structure of $1 + J(F[L])$ is also determined.

Following is the main theorem of this paper.

Theorem 1.1. *Let p be an odd prime, $m \in \mathbb{N}$, F be a finite field with $|F| = 2^n$ containing a primitive p^{th} root of unity and $L = M(C_p^m \rtimes C_2, 2)$.*

2010 Mathematics Subject Classification: 20N05, 17D05
Keywords: Loop algebra, $RA2$ loop, Zorn's algebra, unit loop, general linear loop.

Then

$$\mathcal{U}(F[L]/J(F[L])) \cong F^* \times GLL(2,F)^{\frac{p^m-1}{2}}$$

and $1 + J(F[L]) \cong C_2^{3n}$, *an elementary abelian* 2*-group of order* 2^{3n}.

Throughout the paper, p is an odd prime, F denotes the finite field of characteristic 2 containing a primitive p^{th} root of unity, $F^* = F\backslash\{0\}$, C_m the cyclic group of order m, $\Phi_n(x)$ the n^{th} cyclotomic polynomial and ξ_p a primitive p^{th} root of unity.

2. Preliminaries

A loop L is said to be a *Moufang Loop* if it satisfies any of the following three equivalent identities:

$$\begin{aligned} ((xy)x)z &= x(y(xz)), && \text{the left Moufang identity,}\\ ((xy)z)y &= x(y(zy)), && \text{the right Moufang identity,}\\ (xy)(zx) &= (x(yz))x, && \text{the middle Moufang identity} \end{aligned}$$

for all $x, y, z \in L$.

Let G be a non-abelian group, $g_0 \in \mathcal{Z}(G)$, the center of G and $g \mapsto g^*$ be an involution of G such that $g_0^* = g_0$ and $gg^* \in \mathcal{Z}(G)$ for every $g \in G$. For an indeterminate u, let $L = G \,\dot\cup\, Gu$ and extend the binary operation from G to L by the rules

$$g(hu) = (hg)u, \quad (gu)h = (gh^*)u, \quad (gu)(hu) = g_0h^*g, \quad \text{for all } g, h \in G.$$

The loop L so constructed is a Moufang loop denoted by $M(G, *, g_0)$ and its order is twice the order of the group G. If the involution '$*$' is the inverse map on G and $g_0 = 1$, the identity element of G, then $M(G, -1, 1)$ is denoted as $M(G, 2)$.

A loop whose loop ring in characteristic 2 is alternative but not associative is known as $RA2$ *loop.*

Theorem 2.1. [1, Theorem 5.4] *The loop* $M(G, -1, g_0)$ *is an* $RA2$ *loop if and only if either* $G = Dih(A)$ *is the generalized dihedral group of some abelian group* A *of exponent* > 2*, or* G *is a non-abelian group of exponent* 4 *having exactly* 2 *squares.*

The Zorn's vector matrix algebra is an 8-dimensional alternative algebra and is a generalization of the matrix algebra over an associative ring. For any commutative and associative ring R (with unity), let R^3 denotes the set of ordered triples over R. Consider the set of 2×2 matrices of the form $\begin{bmatrix} a & x \\ y & b \end{bmatrix}$, where $a, b \in R$ and $x, y \in R^3$ with the usual addition

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix} + \begin{bmatrix} c & z \\ w & d \end{bmatrix} = \begin{bmatrix} a+c & x+z \\ y+w & b+d \end{bmatrix}$$

and the multiplication defined by

$$\begin{bmatrix} a & x \\ y & b \end{bmatrix} \begin{bmatrix} c & z \\ w & d \end{bmatrix} = \begin{bmatrix} ac + x \cdot w & az + dx - y \times w \\ cy + bw + x \times z & bd + y \cdot z \end{bmatrix},$$

where $\cdot$ and $\times$ denote the dot product and the cross product respectively in R^3. By this construction, we obtain an alternative algebra called as *Zorn's vector matrix algebra* denoted by $\mathfrak{Z}(R)$.

The loop of the invertible elements of the Zorn's vector matrix algebra,

$$GLL(2, R) = \{A \in \mathfrak{Z}(R) \mid \det A \text{ is a unit in } R\}$$

is a Moufang loop called the *General Linear Loop*. This loop is a generalization of the General Linear group for associative algebras.

For any abelian group A, the *generalized dihedral group* of A is the semidirect product of A and C_2, with C_2 acting on A by inverting the elements and is written as $Dih(A) = A \rtimes C_2$.

If G is a non-abelian group with a faithful two dimensional matrix representation, then we can find a matrix representation of Moufang loop $M(G, 2)$ with the help of the following remark.

Remark 2.2. [10, §2.3] Let G be a non-abelian group with a faithful, two-dimensional representation over a commutative ring R with identity. That is, there exists an embedding $\phi : G \to GL(2, R)$. If we choose two orthogonal unit vectors v, w in R^3 such that $||v \times w|| = 1$ and consider the map $\psi : GL(2, R) \to \mathfrak{Z}(R)$ defined as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & bv \\ cv & d \end{bmatrix}$. Then $\psi\phi : G \to \mathfrak{Z}(R)$ and $u \mapsto \begin{bmatrix} 0 & w \\ w & 0 \end{bmatrix}$ give the matrix representation of L.

The following lemma will be used repeatedly in this article.

Lemma 2.3. *Let p be an odd prime and ξ_p be a primitive p^{th} root of unity. If $\xi_p, \xi_p^2, \ldots, \xi_p^{p-1}$ are the roots of a polynomial $f(x) = a_{p-1}x^{p-1} + a_{p-2}x^{p-2} + \ldots + a_1 x + a_0$ over F, then the coefficients of $f(x)$ are all the same, that is, $a_{p-1} = a_{p-2} = \ldots = a_1 = a_0 = a(say)$.*

Proof. Since the factor $1 + x + x^2 + \ldots + x^{p-1}$ of p^{th} cyclotomic polynomial $\Phi_p(x)$ divides $f(x)$, therefore all the coefficients of the polynomial $f(x)$ must be the same. □

An element $a \in R$ is said to be *quasiregular* if there exists $b \in R$ such that $a + b = ab = ba$ and b is called the *quasi-inverse* of a. An ideal is said to be *quasiregular ideal* if all its elements are quasiregular elements. The *Jacobson radical* $J(R)$ of an alternative ring R is the largest quasiregular ideal of R. If the ring R has unity, this ideal is also the intersection of all the maximal left ideals of R. Let θ be an onto ring homomorphism from a ring R_1 to a ring R_2. Then $\theta(J(R_1)) \subseteq J(R_2)$.

3. Irreducible matrix representations of $C_p^m \rtimes C_2$

In this section, we determine the irreducible and inequivalent representations of the group $C_p^m \rtimes C_2$ over F induced from the irreducible representations of its subgroup C_p^m over F. In [6, §3], the irreducible and inequivalent representations of $C_p^2 \rtimes C_2$ over F have been discussed. Here we extend this to $C_p^m \rtimes C_2$. Since $H = C_p^m$ is an abelian group, therefore, all the irreducible representations of H are of degree 1.

For $1 \leqslant k \leqslant m$, $0 \leqslant i_k \leqslant p-1$, let

$$\rho_{(i_1, i_2, \ldots, i_m)} : H \to F$$

be defined by

$$a_k \mapsto \xi_p^{i_k}.$$

Using [2, Ch 1, §10], we get the induced representations of G as

$$\theta_{(i_1, i_2, \ldots, i_m)} : G \to M(2, F)$$

defined by

$$a_k \mapsto \begin{bmatrix} \xi_p^{i_k} & 0 \\ 0 & \xi_p^{-i_k} \end{bmatrix}, \quad b \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ for all } 0 \leqslant i_k \leqslant p-1, 1 \leqslant k \leqslant m.$$

All these representations of G need not be irreducible and inequivalent.

For each $(i_1, i_2, \ldots, i_m) \in \{0, 1, \ldots, p-1\}$, the representation $\theta_{(i_1,i_2,\ldots,i_m)}$ is similar to the representation $\theta_{(-i_1,-i_2,\ldots,-i_m)}$. Also it is clear that the representation $\theta_{(0,0,\ldots,0)}$ is not irreducible. Thus, for each $1 \leqslant k \leqslant m$, if we define

$$\mathcal{J}_k^m = \left\{ (i_1, i_2, \ldots, i_m) \;\middle|\; \begin{array}{ll} 1 \leqslant i_j \leqslant \frac{p-1}{2}, & \text{if } j = k \\ 0 \leqslant i_j \leqslant p-1, & \text{if } j < k \\ i_j = 0, & \text{if } j > k \end{array} \right\}$$

and

$$S^m = \{(i_1, i_2, \ldots, i_m) \mid (i_1, i_2, \ldots, i_m) \in \mathcal{J}_k^m, \;\; 1 \leqslant k \leqslant m\},$$

then the representations $\theta_{(i_1,i_2,\ldots,i_m)}$ for all $(i_1, i_2, \ldots, i_m) \in S^m$ are irreducible and inequivalent over F.

Hence the total number of 2-degree irreducible and inequivalent representations of G are

$$\frac{p-1}{2} + p.\frac{p-1}{2} + p^2.\frac{p-1}{2} + \ldots + p^{m-1}.\frac{p-1}{2} = \frac{p^m - 1}{2}.$$

4. The unit loop $\mathcal{U}(F[L]/J(F[L]))$ for $L = M(C_p^m \rtimes C_2, 2)$

In this section, we determine the Wedderburn decomposition of $F[L]/J(F[L])$ for $L = M(C_p^m \rtimes C_2, 2)$ and prove the main theorem. Consider the following loop homomorphisms:

1. $\phi_0 : L \to F^*$ defined by

$$a_j \mapsto 1, \;\; \forall\, j = 1, 2, \ldots, m, \qquad b \mapsto 1, \qquad u \mapsto 1.$$

2. For each $(i_1, i_2, \ldots, i_m) \in S^m$, define

$$\phi_{(i_1,i_2,\ldots,i_m)} : L \to GLL(2, F)$$

by

$$a_j \mapsto \begin{bmatrix} \xi_p^{i_j} & (0,0,0) \\ (0,0,0) & \xi_p^{-i_j} \end{bmatrix} \text{ for all } j = 1, 2, \ldots, m,$$

$$b \mapsto \begin{bmatrix} 0 & (0,1,0) \\ (0,1,0) & 0 \end{bmatrix}, \qquad u \mapsto \begin{bmatrix} 0 & (0,0,1) \\ (0,0,1) & 0 \end{bmatrix}.$$

Then

$$T_m : L \to F^* \times (GLL(2,F))^{\frac{p^m-1}{2}}$$

defined as

$$T_m := \phi_0 \ \times \prod_{(i_1,i_2,\ldots,i_m)\in S^m} \phi_{(i_1,i_2,\ldots,i_m)}$$

is a well defined loop homomorphism.

Let $\phi^*_{(i_1,i_2,\ldots,i_m)} : F[L] \to \mathfrak{Z}(F)$ be the loop algebra homomorphism obtained by extending $\phi_{(i_1,i_2,\ldots,i_m)}$ linearly over F and

$$T^*_m : F[L] \to F \bigoplus (\mathfrak{Z}(F))^{\frac{p^m-1}{2}}$$

be defined as

$$T^*_m := \phi^*_0 \ \bigoplus \ \underset{(i_1,i_2,\ldots,i_m)\in S^m}{\oplus} \phi^*_{(i_1,i_2,\ldots,i_m)}.$$

Now we shall calculate the kernel of T^*_m.

Let

$$\begin{aligned} X_m &= \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\cdots\sum_{i_m=0}^{p-1} \alpha_{i_1,i_2,\ldots,i_m} a_1^{i_1} a_2^{i_2}\ldots a_m^{i_m} \\ &+ \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\cdots\sum_{i_m=0}^{p-1} \beta_{i_1,i_2,\ldots,i_m} a_1^{i_1} a_2^{i_2}\ldots a_m^{i_m} b \\ &+ \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\cdots\sum_{i_m=0}^{p-1} \gamma_{i_1,i_2,\ldots,i_m} a_1^{i_1} a_2^{i_2}\ldots a_m^{i_m} u \\ &+ \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\cdots\sum_{i_m=0}^{p-1} \delta_{i_1,i_2,\ldots,i_m} a_1^{i_1} a_2^{i_2}\ldots a_m^{i_m} bu \\ &= \ X_{m1} + X_{m2} + X_{m3} + X_{m4} \in Ker\ T^*_m. \end{aligned}$$

For $(i_1, i_2, \ldots, i_m) \in \mathcal{J}^m_k$, on applying $\phi^*_{(i_1,i_2,\ldots,i_m)}$ on X_m, we get

$$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{m1}) = \begin{bmatrix} Y_{11} & (0,0,0) \\ (0,0,0) & Y_{12} \end{bmatrix},$$

$$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{m2}) = \begin{bmatrix} 0 & (0,Y_{21},0) \\ (0,Y_{22},0) & 0 \end{bmatrix},$$

$$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{m3}) = \begin{bmatrix} 0 & (0,0,Y_{31}) \\ (0,0,Y_{32}) & 0 \end{bmatrix},$$

$$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{m4}) = \begin{bmatrix} 0 & (Y_{41},0,0) \\ (Y_{42},0,0) & 0 \end{bmatrix}$$

for some $Y_{11}, Y_{12}, Y_{21}, Y_{22}, Y_{31}, Y_{32}, Y_{41}$ and $Y_{42} \in F$.

That is,

$$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_m) = \begin{bmatrix} Y_{11} & (Y_{41},Y_{21},Y_{31}) \\ (Y_{42},Y_{22},Y_{32}) & Y_{12} \end{bmatrix}.$$

Thus $\phi^*_{(i_1,i_2,\ldots,i_m)}(X_m) = 0$ gives that $Y_{11} = Y_{12} = Y_{21} = Y_{22} = Y_{31} = Y_{32} = Y_{41} = Y_{42} = 0$. This means that for all $(i_1, i_2, \ldots, i_m) \in \mathcal{J}_k^m$,

$\phi^*_{(i_1,i_2,\ldots,i_m)}(X_m) = 0$ implies that $\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{mj}) = 0$ for all $j = 1,2,3,4$.

Firstly, consider $\phi^*_{(i_1,i_2,\ldots,i_m)}(X_{m1}) = 0$. For a fixed $(i_1, i_2, \ldots, i_m) \in \mathcal{J}_k^m$, define

$$\mathcal{A}_k^m = \left\{ (j_1, j_2, \ldots, j_m) \;\middle|\; \begin{array}{ll} j_l \in \{i_l, 0\}, & \text{if } 1 \leqslant l < k \\ j_k = i_k, & \\ i_j = 0, & \text{if } l > k \end{array} \right\}.$$

Let us start with $k = m$, for $(j_1, j_2, \ldots, j_m) \in \mathcal{A}_m^m$, $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m1}) = 0$ and using Lemma 2.3, we get that

$$\alpha_{i_1,i_2,\ldots,i_{m-1},i_m} = \alpha_{i_1,i_2,\ldots,i_{m-1}}(say) \text{ for all } i_1, \ldots, i_m = 0,1,\ldots,m.$$

Then $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m1}) = 0$ for $(j_1, j_2, \ldots, j_m) \in \mathcal{A}_{m-1}^m$ gives

$$\alpha_{i_1,i_2,\ldots,i_{m-2},i_{m-1}} = \alpha_{i_1,i_2,\ldots,i_{m-2}}(say) \text{ for all } i_1, \ldots, i_{m-1} = 0,1,\ldots,m.$$

Continuing the same process, $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m1}) = 0$ for $(j_1, \ldots, j_m) \in \mathcal{A}_2^m$, implies that $\alpha_{i_1,i_2} = \alpha_{i_1}$(say) for all $i_1, i_2 = 0,1,\ldots,m$.

Finally, $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m1}) = 0$ for $(j_1, j_2, \ldots, j_m) \in \mathcal{A}_1^m$ gives that $\alpha_{i_1} = \alpha$(say) for all $i_1 = 0,1,\ldots,m$. Hence $\alpha_{i_1,i_2,\ldots,i_m} = \alpha$ for all $i_1, i_2, \ldots, i_m = 0,1,\ldots,m$. By repeating the same procedure for $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m2}) = 0$, $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m3}) = 0$ and for $\phi^*_{(j_1,j_2,\ldots,j_m)}(X_{m4}) = 0$, we get $\beta_{i_1,i_2,\ldots,i_m} = \beta$, $\gamma_{i_1,i_2,\ldots,i_m} = \gamma$ and $\delta_{i_1,i_2,\ldots,i_m} = \delta$ for all $i_1, i_2, \ldots, i_m = 0,1,\ldots,m$.

Next, $\phi_0^*(X_m) = 0$ implies that $\alpha + \beta + \gamma + \delta = 0$. Thus

$$X_m = \beta\left(\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m} + \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m}b\right)$$
$$+\gamma\left(\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m} + \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m}u\right)$$
$$+\delta\left(\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m} + \sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m}bu\right)$$
$$= \beta f_{m1} + \gamma f_{m2} + \delta f_{m3}.$$

We have few observations to note, which will be used here:

In the group $G = C_p^m \rtimes C_2$, $(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})^2 = p(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})$ and $(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})b = b(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})$, since $a_i^k b = b a_i^{-k}$ (a presenting relator of G).

Further, the definition of the loop gives $ug = g^{-1}u$, which implies $(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})u = u(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})$.

Also we can write

$\sum_{i_1=0}^{p-1}\sum_{i_2=0}^{p-1}\dots\sum_{i_m=0}^{p-1} a_1^{i_1}a_2^{i_2}\dots a_m^{i_m} = \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})$.

Consequently, we have $f_{m1} = \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1}) + \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})b$. This gives

$f_{m1}^2 = 2\prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1}) + 2\prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})b$ $= 0$, since the characteristic of F is 2. Similarly, we can prove $f_{m2}^2 = 0$, and $f_{m3}^2 = 0$.

Also for $1 \leqslant r, s \leqslant 3$, f_{mr} and f_{ms} commute, as

$$f_{mr}f_{ms} = \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1}) + \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})b$$
$$+\prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})u + \prod_{i=1}^{m}(a_i^0 + a_i^1 + a_i^2 + \ldots + a_i^{p-1})bu$$
$$= \sum_{l\in L} l.$$

It follows that every element of $ker\ T_m^*$ is a nilpotent element of nilpotency index 2 and hence is quasiregular with quasi-inverse as itself. Thus $ker\ T_m^*$ is a quasiregular ideal of $F[L]$, which implies that $ker\ T_m^* \subseteq J(F[L])$.

Since

$$dim_F\ (F \oplus \mathfrak{Z}(F)^{\frac{p^m-1}{2}}) = 4p^m - 3 = dim_F\ (F[L]/\ker\ T_m^*)$$

therefore, T_m^* is onto. This implies $J(F[L]) \subseteq ker\ T_m^*$. Consequently, $ker\ T_m^* = J(F[L])$. Hence

$$F[L]/J(F[L]) \cong F \oplus \mathfrak{Z}(F)^{\frac{p^m-1}{2}}$$

which further gives

$$\mathcal{U}(F[L]/J(F[L])) \cong F^* \times GLL(2,F)^{\frac{p^m-1}{2}}.$$

Consider $1 + J(F[L])$. An element h of $1 + J(F[L])$ is of the form $h = 1 + c_1 f_{m1} + c_2 f_{m2} + c_3 f_{m3}$, where $c_i's \in F$. As f_{mr} and f_{ms} commute for all $1 \leqslant r, s \leqslant 3$, we get that $1 + J(F[L])$ is a commutative loop.

Further, for all $r, s, t = 1, 2, 3$,

$$(f_{mr} f_{ms}) f_{mt} = 2 \sum_{l \in L} l = 0 \quad \text{and } f_{mr}(f_{ms} f_{mt}) = 2 \sum_{l \in L} l = 0.$$

Thus $1 + J(F[L])$ is an abelian group and $h^2 = 1$ for all $h \in 1 + J(F[L])$, which gives $1 + J(F[L]) \cong (C_2 \times C_2 \times C_2)^n$.

Acknowledgments. The author is thankful to the anonymous referees and Professor Petr Vojtěchovský for their useful comments and suggestions, which have greatly improved the presentation of the paper.

References

[1] **O. Chein and E. G. Goodaire**, *Loops whose loop rings in characteristic* 2 *are alternative*, Comm. Algebra, **18** (1990), no. 3, 659 – 688.

[2] **Ch.W. Curtis and I. Reiner**, *Methods of representation theory. Vol. I*, John Wiley & Sons, Inc., New York (1990).

[3] **R.A. Ferraz, E.G. Goodaire and C.P. Milies**, *Some classes of semisimple group* (*and loop*) *algebras over finite fields*, J. Algebra, **324** (2010), no. 12, 3457 – 3469.

[4] **E.G. Goodaire**, *Six Moufang loops of units*, Canad. J. Math., **44** (1992), no. 5, 951 – 973.

[5] **E. Jespers and G. Leal**, *A characterization of the unit loop of the integral loop ring* $\mathbf{Z}M_{16}(Q, 2)$, J. Algebra, **155** (1993), no. 1, 95 – 109.

[6] **S. Sidana**, *On units in loop algebra* $F[M(Dih(C_p^2), 2)]$, Beitr. Algebra Geom., **58** (2017), no. 4, 765 – 774.

[7] **S. Sidana and R.K. Sharma**, *Finite semisimple loop algebras of indecomposable RA loops*, Canad. Math. Bull., **58** (2015), no. 2, 363 – 373.

[8] **S. Sidana and R.K. Sharma**, *On the finite loop algebra of the smallest Moufang loop* $M(S_3, 2)$, Armenian J. Math., **8** (2016), no. 1, 68 – 76.

[9] **S. Sidana and R.K. Sharma**, *Units in finite loop algebras of RA2 loops*, Asian-Eur. J. Math., **9** (2016), no. 1, 1650026.

[10] **A.T. Wells**, *Zorn vector matrices over commutative rings and the loops arising from their construction*, Thesis (Ph.D.)–Iowa State University (2010).

Received November 22, 2021
Revised July 20, 2022

Post Graduate Department of Mathematics
Mehr Chand Mahajan DAV College for Women
Sector 36-A
Chandigarh
India 160036
E-mail: swatisidana@gmail.com