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On the intersection ideal graph of semigroups

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Abstract. The intersection ideal graph $\Gamma(S)$ of a semigroup S is a simple undirected graph whose vertices are all nontrivial left ideals of S and two distinct left ideals I, J are adjacent if and only if their intersection is nontrivial. In this paper, we investigate the connectedness of $\Gamma(S)$. We show that if $\Gamma(S)$ is connected, then the diameter of $\Gamma(S)$ is at most two. Further, we classify the semigroups S in terms of their ideals such that the diameter of $\Gamma(S)$ is two. We obtain the domination number, independence number, girth and the strong metric dimension of $\Gamma(S)$. We have also investigated the completeness, planarity and perfectness of $\Gamma(S)$. We show that if S is a completely simple semigroup, then $\Gamma(S)$ is weakly perfect. Moreover, in this article, we give an upper bound of the chromatic number of $\Gamma(S)$. Finally, if S is the union of n minimal left ideals, then we obtain the metric dimension and the automorphism group of $\Gamma(S)$.

1. Introduction

Literature is abound with numerous remarkable results concerning a number of constructions of graphs from rings, semigroups or groups and their applications, including automata theory, see for instance [1, 11, 19, 28, 29, 30, 31, 36, 43, 45] and references therein. The intersection graph of a semigroup was introduced by Bosák [10] in 1964. The *intersection subsemigroup graph* $\Gamma(S)$ of S is a simple undirected graph whose vertex set is the collection of proper subsemigroups of S and two distinct vertices A, B are adjacent if and only if $A \cap B \neq \emptyset$. In [10], it was shown that if S is a nondenumerable semigroup or a periodic semigroup with more than two elements, then the graph $\Gamma(S)$ is connected. Bosák then raised the following open problem: Does there exists a semigroup with more than two elements whose graph

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is disconnected? Y. F. Lin [33], answered the problem posed by Bosák, in a negative manner and proved that every semigroup with more than two elements has a connected graph. Also, B. Pondělíček [37] proved that the diameter of a semigroup with more than two elements does not exceed three.

Inspired by the work of J. Bosák , Csákány and Pollák [16] studied the intersection graphs of groups and showed that there is an edge between two proper subgroups if they have at least two elements common. Further, Zelinka [46], continued the work for finite abelian groups. R. Shen [40], characterized all finite groups whose intersection graphs are disconnected. This solves the problem posed in [16]. The groups whose intersection graphs of normal subgroups are connected, complete, forests or bipartite are classified in [21]. Tamizh *et al.* [41], continued the seminal paper of Csákány and Pollák to introduce the subgroup intersection graph of a finite group G. Further, in [34], it was shown that the diameter of the intersection graph of a finite non-abelian simple group has an upper bound 28. Shahsavari *et al.* [39] have studied the structure of the automorphism group of this graph. The intersection graph on cyclic subgroups of a group has been studied in [18]. Further, Kayacan *et al.* [27] studied the conjecture given in [46], that two (noncyclic) finite abelian groups with isomorphic intersection graphs are isomorphic. In [25], finite solvable groups whose intersection graphs are not 2-connected and finite nilpotent groups whose intersection graphs are not 3-connected are classified. Further, the dominating sets of the intersection graph of finite groups have been investigated in [26].

Recently, Chakrabarty et al. [12] introduced the notion of intersection ideal graph of rings. The *intersection ideal graph* $\Gamma(R)$ of a ring R is an undirected simple graph whose vertex set is the collection of nontrivial left ideals of R and two distinct vertices I, J are adjacent if and only if $I \cap J \neq \{0\}$. They characterized the rings R for which the graph $\Gamma(R)$ is connected and obtained several necessary and sufficient conditions on a ring R such that $\Gamma(R)$ is complete. Jafari *et al.* [20] studied planarity of the intersection ideal graphs $\Gamma(R)$ of a commutative ring R with unity. The domination number of $\Gamma(R)$ has been obtained in [22]. Akbari *et al.* [5] classified all rings whose intersection graphs of ideals are not connected and also determined all rings whose clique number is finite. The intersection graphs of ideals of the direct product of rings have been discussed in [24]. Pucanović *et al.* [38] classified all graphs of genus two that are intersection graphs of ideals of some commutative rings and obtained some lower bounds for the genus of the intersection graph of ideals of a non local commutative ring.

Das [17] characterized the positive integer n for which the intersection graph of ideals of $\mathbb{Z}_n$ is perfect. The intersection graph of submodules of a module has been studied in [6, 7, 44]. Moreover, we refer the reader to [8] and references therein for the graded case. The intersection graph on algebraic structures has also been studied in [2, 3, 4, 23, 32, 43].

It is pertinent as well as interesting to associate graphs to ideals of a semigroup as ideals gives a lot of information about the structure of semigroups. Motivated with the work of [5, 12], in this paper, we consider the intersection ideal graph associated with semigroups. The *intersection ideal graph* $\Gamma(S)$ of a semigroup S is an undirected simple graph whose vertex set is nontrivial left ideals of S and two distinct nontrivial left ideals I, J are adjacent if and only if their intersection is nontrivial. The paper is arranged as follows. In Section 2, we state necessary fundamental notions and recall some necessary results. Section 3 comprises the results concerning the connectedness of the intersection ideal graph of an arbitrary semigroup. In Section 4, we study various graph invariants of $\Gamma(S)$ viz. girth, dominance number, independence number and clique number etc. Further, if S is the union of n minimal left ideals then the automorphism group of $\Gamma(S)$ is obtained.

2. Preliminaries

In this section, first we recall necessary definitions and results of semigroup theory from [15]. A *semigroup* S is a non-empty set together with an associative binary operation on S. The Green's $\mathcal{L}$-relation on a semigroup S defined as $x \; \mathcal{L} \; y \Longleftrightarrow S^1x = S^1y$ where $S^1x = Sx \cup \{x\}$. The $\mathcal{L}$-class of an element $a \in S$ is denoted by L_a. A non-empty subset I of S is said to be a *left [right] ideal* if $SI \subseteq I[IS \subseteq I]$ and an *ideal* of S if $SIS \subseteq I$. Union of two left [right] ideals of S is again a left [right] ideal of S. A left ideal I is *maximal* if it does not contained in any nontrivial left ideal of S. If S has a unique maximal left ideal then it contains every nontrivial left ideal of S. A left ideal I of S is *minimal* if it does not properly contain any left ideal of S. It is well known that every non-zero element of a minimal left ideal of S is in same $\mathcal{L}$-class. If S has a minimal left ideal then every nontrivial left ideal contains at least one minimal left ideal. If A is any left ideal of S other than I, then either $I \subset A$ or $I \cap A = \emptyset$. Thus we have the following remark.

Remark 2.1. Any two different minimal left ideals of a semigroup S are disjoint.

Remark 2.2. Let S be the union of n minimal left ideals. Then each nontrivial left ideal is the union of these minimal left ideals.

The following lemma is useful in the sequel and we shall use this without referring to it explicitly.

Lemma 2.3 ([9, Lemma 2.2])**.** *A left ideal K of S is maximal if and only if $S \setminus K$ is an $\mathcal{L}$-class.*

A semigroup S is said to be *simple* if it has no proper ideal. Let $\mathcal{E}$ be the set of idempotents of a semigroup S. If $e, f \in \mathcal{E}$, we define $e \leqslant f$ to mean $ef = fe = e$. Recall that a semigroup S is called *completely simple* if S is simple and contains a primitive idempotent. By *primitive idempotent* we mean an idempotent which is minimal within the set of all idempotents under the relation $\leqslant$.

Lemma 2.4 ([15, Corollary 2.49])**.** *A completely simple semigroup is the union of its minimal left* (*right*) *ideals.*

We also require the following graph theoretic notions [42]. A *graph* Γ is a pair $\Gamma = (V, E)$, where $V = V(\Gamma)$ and $E = E(\Gamma)$ are the set of vertices and edges of Γ, respectively. We say that two different vertices u, v are *adjacent*, denoted by $u \sim v$ or (u, v), if there is an edge between u and v. We write $u \nsim v$, if there is no edge between u and v. The *distance* between two vertices u, v in Γ is the number of edges in a shortest path connecting them and it is denoted by $d(u, v)$. If there is no path between u and v, we say that the distance between u and v is *infinity* and we write as $d(u, v) = \infty$. The diameter $diam(\Gamma)$ of Γ is the greatest distance between any pair of vertices. The *degree* of the vertex v in Γ is the number of edges incident to v and it is denoted by $deg(v)$. A *cycle* is a closed walk with distinct vertices except for the initial and end vertex, which is equal and a cycle of length n is denoted by C_n. The *girth* of Γ is the length of its shortest cycle and is denoted by $g(\Gamma)$. A subset X of $V(\Gamma)$ is said to be *independent* if no two vertices of X are adjacent. The *independence number* of Γ is the cardinality of the largest independent set and it is denoted by $\alpha(\Gamma)$. A graph Γ is *bipartite* if $V(\Gamma)$ is the union of two disjoint independent set. It is well known that a graph is bipartite if and only if it has no odd cycle [42, Theorem 1.2.18]. A connected graph Γ is Eulerian if and only if the degree of every vertex is even [42, Theorem 1.2.26]. A *subgraph* of Γ is a graph Γ' such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A subgraph Γ' of Γ is called an *induced subgraph* by the elements of $V(\Gamma') \subseteq V(\Gamma)$ if for $u, v \in V(\Gamma')$, we have $u \sim v$ in Γ' if and only if $u \sim v$ in Γ. The *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. A *clique* in Γ is a set of pairwise adjacent vertices. The *clique number* of Γ is the size of the maximum clique in Γ and it is denoted by $\omega(\Gamma)$. It is well known that $\omega(\Gamma) \leqslant \chi(\Gamma)$ (see [42]). A graph Γ is *weakly perfect* if $\omega(\Gamma) = \chi(\Gamma)$. A graph Γ is *perfect* if $\omega(\Gamma') = \chi(\Gamma')$ for every induced subgraph Γ' of Γ. Recall that the *complement* $\overline{\Gamma}$ of Γ is a graph with the same vertex set as Γ and distinct vertices u, v are adjacent in $\overline{\Gamma}$ if they are not adjacent in Γ. A subgraph Γ' of Γ is called a *hole* if Γ' is a cycle as an induced subgraph, and Γ' is called an *antihole* of Γ if $\overline{\Gamma'}$ is a hole in $\overline{\Gamma}$.

Theorem 2.5. [14] *A finite graph Γ is perfect if and only if it does not contain a hole or antihole of odd length at least 5.*

A subset D of $V(\Gamma)$ is said to be a dominating set if any vertex in $V(\Gamma) \setminus D$ is adjacent to at least one vertex in D. If D contains only one vertex, then that vertex is called dominating vertex. The *domination number* $\gamma(\Gamma)$ of Γ is the minimum size of a dominating set in Γ. A graph Γ is said to be planar if it can be drawn on a plane without any crossing of its edges. In Γ, a vertex z resolves a pair of distinct vertices x and y if $d(x,z) \neq d(y,z)$. A resolving set of Γ is a subset $R \subseteq V(\Gamma)$ such that every pair of distinct vertices of Γ is resolved by some vertex in R. The metric dimension of Γ, denoted by $\beta(\Gamma)$, is the minimum cardinality of a resolving set of Γ. For vertices u and v in a graph Γ, we say that z *strongly resolves* u and v if there exists a shortest path from z to u containing v, or a shortest path from z to v containing u. A subset U of $V(\Gamma)$ is a *strong resolving set* of Γ if every pair of vertices of Γ is strongly resolved by some vertex of U. The least cardinality of a strong resolving set of Γ is called the *strong metric dimension* of Γ and is denoted by $\mathrm{sdim}(\Gamma)$. For vertices u and v in a graph Γ, we write $u \equiv v$ if $N[u] = N[v]$. Notice that that $\equiv$ is an equivalence relation on $V(\Gamma)$. We denote by $\widehat{v}$ the $\equiv$-class containing a vertex v of Γ. Consider a graph $\widehat{\Gamma}$ whose vertex set is the set of all $\equiv$-classes, and vertices $\widehat{u}$ and $\widehat{v}$ are adjacent if u and v are adjacent in Γ. This graph is well-defined because in Γ, $w \sim v$ for all $w \in \widehat{u}$ if and only if $u \sim v$. We observe that $\widehat{\Gamma}$ is isomorphic to the subgraph $\mathcal{R}_\Gamma$ of Γ induced by a set of vertices consisting of exactly one element from each $\equiv$-class. Subsequently, we have the following result of [35] with $\omega(\mathcal{R}_\Gamma)$ replaced by $\omega(\widehat{\Gamma})$.

Theorem 2.6 ([35, Theorem 2.2]). *For any graph* Γ *with diameter* 2, $\operatorname{sdim}(\Gamma) = |V(\Gamma)| - \omega(\widehat{\Gamma})$.

3. Connectivity of the intersection ideal graph $\Gamma(S)$

In this section, we investigate the connectedness of $\Gamma(S)$. We show that $diam(\Gamma(S)) \leqslant 2$ if it is connected. Also, we classify the semigroups, in terms of their left ideals, such that the diameter of $\Gamma(S)$ is two.

Theorem 3.1. *The intersection ideal graph* $\Gamma(S)$ *is disconnected if and only if* S *contains at least two minimal left ideals and every nontrivial left ideal of* S *is minimal as well as maximal.*

Proof. First suppose that $\Gamma(S)$ is not connected. Then S has at least two nontrivial left ideals I_1 and I_2. Without loss of generality, assume that $I_1 \in C_1$ and $I_2 \in C_2$, where C_1 and C_2 are distinct components of $\Gamma(S)$. If I_1 is not minimal then there exists at least one nontrivial left ideal I_k of S such that $I_k \subset I_1$ so that their intersection is nontrivial. Therefore, $I_1 \sim I_k$. Now if the intersection of I_2 and I_k is nontrivial then $I_1 \sim I_k \sim I_2$, a contradiction. Therefore the intersection of I_2 and I_k is trivial. If $I_2 \cup I_k \neq S$ then $I_1 \sim I_2 \cup I_k \sim I_2$, a contradiction. Thus, $I_k \cup I_2 = S$. It follows that $I_1 \sim I_2$, again a contradiction. Thus I_1 is minimal. Similarly, we get I_2 is minimal.

Further assume that I_1 is not maximal. Then there exists a nontrivial left ideal I_k of S such that $I_1 \subset I_k$ so that $I_1 \sim I_k$. If $I_1 \cup I_2 \neq S$ then $I_1 \sim I_1 \cup I_2 \sim I_2$, a contradiction to the fact that $\Gamma(S)$ is disconnected. It follows that $I_1 \cup I_2 = S$ so that the intersection of I_k and I_2 is nontrivial. Thus we have $I_1 \sim I_k \sim I_2$, a contradiction. Hence I_1 is maximal. Similarly, we observe that I_2 is maximal. The converse follows from the Remark 2.1. □

Corollary 3.2. *If the graph* $\Gamma(S)$ *is disconnected then it is a null graph (i.e. it has no edge).*

Theorem 3.3. *The intersection ideal graph* $\Gamma(S)$ *is disconnected if and only if* S *is the union of exactly two minimal left ideals.*

Proof. First note that the inclusion ideal graph $\mathcal{I}n(S)$ (see [9]) is a spanning subgraph of $\Gamma(S)$. Thus, the result follows from Lemma 3.3, Theorem 3.4 of [9] and Theorem 3.1. $\square$

Theorem 3.4. *If the intersection ideal graph* $\Gamma(S)$ *is connected then we have* $diam(\Gamma(S)) \leqslant 2$.

Proof. Let I_1, I_2 be two nontrivial left ideals of S. If $I_1 \sim I_2$ then $d(I_1, I_2) = 1$. If $I_1 \nsim I_2$ i.e. $I_1 \cap I_2$ is trivial then in the following cases we show that $d(I_1, I_2) \leqslant 2$.

Case 1. $I_1 \cup I_2 \neq S$. Then $I_1 \sim (I_1 \cup I_2) \sim I_2$ so that $d(I_1, I_2) = 2$.

Case 2. $I_1 \cup I_2 = S$. Since $\Gamma(S)$ is a connected graph, there exists a nontrivial left ideal I_k of S such that either $I_1 \cap I_k$ is nontrivial or $I_2 \cap I_k$ is nontrivial. Now we have the following subcases.

Subcase 1. $I_1 \not\subset I_k$ and $I_k \not\subset I_1$. Since $I_1 \not\subset I_k$ it follows that there exists $x \in I_k$ but $x \notin I_1$ so that $x \in I_2$. Consequently, $I_2 \cap I_k$ is nontrivial. Therefore, we get a path $I_1 \sim I_k \sim I_2$ of length two. Thus, $d(I_1, I_2) = 2$.

Subcase 2. $I_k \subset I_1$. There exists $x \in I_1$ but $x \notin I_k$. If $I_2 \cup I_k = S$ then $x \in I_2$. Thus, we get $I_1 \cap I_2$ is nontrivial, a contradiction. Consequently, $I_2 \cup I_k \neq S$. Further, we get a path $I_1 \sim (I_2 \cup I_k) \sim I_2$ of length two. Thus, $d(I_1, I_2) = 2$.

Subcase 3. $I_1 \subset I_k$. Since $I_1 \cup I_2 = S$ we get $I_k \cup I_2 = S$. Further, the intersection of I_k and I_2 is nontrivial. Consequently, $I_1 \sim I_k \sim I_2$ gives a path of length two between I_1 and I_2. Thus, $d(I_1, I_2) = 2$. Hence, $diam(\Gamma(S)) \leqslant 2$. $\square$

Lemma 3.5. *Let* S *be a semigroup having minimal left ideals. Then* $\Gamma(S)$ *is complete if and only if* S *has a unique minimal left ideal.*

Proof. Suppose that S contains a unique minimal left ideal I_1. Note that every nontrivial left ideal of S contains at least one minimal left ideal. Since I_1 is unique then it must contained in every nontrivial left ideals of S. Thus, the graph $\Gamma(S)$ is complete.

Conversely, suppose that $\Gamma(S)$ is a complete graph. On the contrary, if S has at least two minimal left ideals I_1 and I_2, then $I_1 \nsim I_2$ by Remark 2.1, a contradiction to the fact that $\Gamma(S)$ is complete. Thus S has a unique minimal left ideal. $\square$

Lemma 3.6. *The graph* $\Gamma(S)$ *is regular if and only if either* $\Gamma(S)$ *is null or a complete graph.*

Proof. First suppose that $\Gamma(S)$ is not a null graph. Suppose S has at least two minimal left ideals I_1 and I_2. Since $\Gamma(S)$ is not a null graph, we get I_1 and $I_1 \cup I_2$ form nontrivial left ideals of S and $I_1 \sim (I_1 \cup I_2)$. If J is any nontrivial left ideal of S such that $J \sim I_1$, then $J \sim (I_1 \cup I_2)$. It follows that every nontrivial left ideal of S which is adjacent with I_1 is also adjacent with $(I_1 \cup I_2)$ and $I_2 \sim I_1 \cup I_2$ but $I_2 \nsim I_1$ implies that $deg(I_1) < deg(I_1 \cup I_2)$, a contradiction. Therefore, $\Gamma(S)$ is a complete graph. □

Next we classify the semigroups such that the diameter of intersection ideal graph $\Gamma(S)$ is two.

Theorem 3.7. *Let S be a semigroup having minimal left ideals. Then for a connected graph $\Gamma(S)$, we have $diam(\Gamma(S)) = 2$ if and only if S has at least two minimal left ideals.*

Proof. Suppose that $diam(\Gamma(S)) = 2$. Assume that I_1 is the only minimal left ideal of S. Since I_1 is a unique minimal left ideal, we have $I_1 \subset K$, for any nontrivial left ideal K of S. Therefore, for any nontrivial left ideals J and K, we have $I_1 \subset (J \cap K)$. Consequently, $d(J, K) = 1$ for any $J, K \in V(\Gamma(S))$. Therefore S has at least two minimal left ideals. Conversely, suppose that S has at least two minimal left ideals I_1 and I_2. Then by Remark 2.1, we have $I_1 \nsim I_2$. Consequently, by Theorem 3.4, $d(I_1, I_2) = 2$. Thus, $diam(\Gamma(S)) = 2$. □

4. Invariants of $\Gamma(S)$

In this section, first we obtain the girth of $\Gamma(S)$. Then we discuss planarity and perfectness of $\Gamma(S)$. Also we classify the semigroup S such that $\Gamma(S)$ is bipartite, star graph and tree, respectively. Further, we investigate the other graph invariants viz. clique number, independence number and strong metric dimension of $\Gamma(S)$.

Theorem 4.1. *Let S be a semigroup such that $\Gamma(S)$ contains a cycle. Then $g(\Gamma(S)) = 3$.*

Proof. If $\Gamma(S)$ is disconnected or a tree, then clearly $g(\Gamma(S)) = \infty$. Suppose that the semigroup S has n minimal left ideals. Now we prove the result by observing the following cases.

Case 1. $n = 0$. If S has no nontrivial left ideals then there is nothing to prove. Otherwise, there exists a chain of nontrivial left ideals of S such that $I_1 \supset I_2 \supset \cdots \supset I_k \supset \cdots$. Thus, $g(\Gamma(S)) = 3$.

Case 2. $n = 1$. Suppose that I_1 is the only minimal left ideal of S. Since I_1 is a unique minimal left ideal, we obtain $I_1 \subset K$, for any nontrivial left ideal K of S. Therefore, for any nontrivial left ideals I and J, we get $I_1 \subset I \cap J \neq \emptyset$. If S has at least three nontrivial left ideals, then $g(\Gamma(S)) = 3$. Otherwise, $g(\Gamma(S)) = \infty$.
Case 3. $n = 2$. Let I_1, I_2 be two minimal left ideals of S. If $I_1 \cup I_2 = S$ then by Theorem 3.3 and Corollary 3.2, $g(\Gamma(S)) = \infty$. If $I_1 \cup I_2 \neq S$, then $J = I_1 \cup I_2$ is a nontrivial left ideal of S. Suppose I_1, I_2 and J are the only nontrivial left ideals of S. Then $I_1 \sim J \sim I_2$ and so $g(\Gamma(S)) = \infty$. Further, assume that S has a nontrivial left ideal K other than I_1, I_2 and J. Since I_1, I_2 are minimal left ideals of S, we have either $I_1 \subset K$ or $I_2 \subset K$. Without loss of generality, assume that $I_1 \subset K$. Then $I_1 \sim K \sim J \sim I_1$. It follows that $g(\Gamma(S)) = 3$.
Case 4. $n \geqslant 3$. Let I_1, I_2, I_3 be the minimal left ideals of S. Then we have a cycle $(I_1 \cup I_2) \sim (I_2 \cup I_3) \sim (I_1 \cup I_3) \sim (I_1 \cup I_2)$ of length 3. Thus, $g(\Gamma(S)) = 3$. □

Let $\mathrm{Min}(S)$ $(\mathrm{Max}(S))$ be the set of all minimal (maximal) left ideals of S. By a nontrivial left ideal $I_{i_1 i_2 \cdots i_k}$, we mean $I_{i_1} \cup I_{i_2} \cup \cdots \cup I_{i_k}$, where $I_{i_1}, I_{i_2}, \cdots, I_{i_k} \in \mathrm{Min}(S)$.

Theorem 4.2. *Let $\Gamma(S)$ be the intersection ideal graph of S. Then the following statements hold:*

(i) *If $\Gamma(S)$ is planar then $|\mathrm{Min}(S)| \leqslant 3$.*

(ii) *Let S be a semigroup such that it is a union of n minimal left ideals. Then $\Gamma(S)$ is planar if and only if $n \leqslant 3$.*

Proof. (i) Suppose that $|\mathrm{Min}(S)| = 4$ with $\mathrm{Min}(S) = \{I_1, I_2, I_3, I_4\}$. Then note that the subgraph induced by the vertices $I_1, I_{12}, I_{123}, I_{14}$ and I_{124} is isomorphic to K_5. Thus, $\Gamma(S)$ is nonplanar.

(ii) The proof for $\Gamma(S)$ is nonplanar for $n \geqslant 4$ follows from part (i). If $n = 2$ then by Corollary 3.2 and Theorem 3.3, $\Gamma(S)$ is planar. For $n = 3$, $\Gamma(S)$ is planar as shown in Figure 1. □

Theorem 4.3. *Let $\Gamma(S)$ be the intersection ideal graph of S. Then the following statements hold:*

(i) *If $\Gamma(S)$ is a perfect graph then $|\mathrm{Min}(S)| \leqslant 4$.*

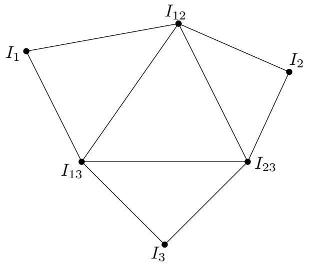

Figure 1: Planar drawing of $\Gamma(S)$ for $S = I_{123}$.

(ii) *Let S be the union of n minimal left ideals. Then $\Gamma(S)$ is perfect if and only if $n \leqslant 4$.*

Proof. (i) Suppose that $|\mathrm{Min}(S)| = 5$ with $\mathrm{Min}(S) = \{I_1, I_2, I_3, I_4, I_5\}$. Note that $I_{12} \sim I_{23} \sim I_{34} \sim I_{45} \sim I_{15} \sim I_{12}$ induces a cycle of length 5. Then by Theorem 2.5, $\Gamma(S)$ is not perfect.

(ii) The proof for $\Gamma(S)$ is not a perfect graph for $n \geqslant 5$ follows from part (i). If $n = 2$ then by Corollary 3.2 and Theorem 3.1, $\Gamma(S)$ is disconnected. Thus, being a null graph, $\Gamma(S)$ is perfect. For $n \in \{3, 4\}$, we show that $\Gamma(S)$ does not contain a hole or an antihole of odd length at least five (cf. Theorem 2.5). If $n = 3$, $\Gamma(S)$ is perfect as shown in Figure 1. If $n = 4$ then from Figure 2, note that $\Gamma(S)$ does not contain a hole or an antihole of odd length at least five. □

Theorem 4.4. *Let S be a semigroup having minimal left ideals such that $V(\Gamma(S)) > 1$. Then the following conditions are equivalent:*

(i) $\Gamma(S)$ *is a star graph.*

(ii) $\Gamma(S)$ *is a tree.*

(iii) $\Gamma(S)$ *is bipartite.*

(iv) *Either S has exactly three nontrivial left ideals I_1, I_2 and I_{12} such that I_1 and I_2 are minimal or S has two nontrivial left ideals I_1, I_2 such that $I_1 \subset I_2$.*

Proof. We prove (ii), (iii) ⇒ (iv). The proof of remaining parts is straightforward. Suppose $\Gamma(S)$ is a tree. Then clearly $|\mathrm{Min}(S)| \leqslant 2$. Otherwise, for minimal left ideals I_1, I_2, I_3 we have $I_{12} \sim I_{13} \sim I_{23} \sim I_{12}$ a cycle, a

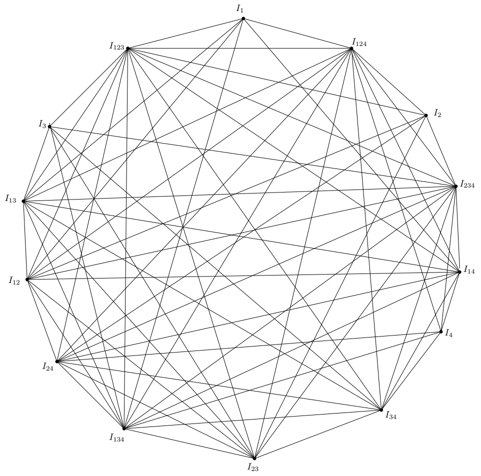

Figure 2: The intersection graph $\Gamma(S)$ for $S = I_{1234}$.

contradiction. Suppose that $|\mathrm{Min}(S)| = 1$. Let I_1 be the unique minimal left ideal of S. Consequently, I_1 is contained in all the other nontrivial left ideals of S. If S has at least three nontrivial left ideals then we get a cycle, a contradiction. Thus $|V(\Gamma(S))| = 2$. Now we assume that $|\mathrm{Min}(S)| = 2$. Let I_1, I_2 be two minimal left ideals of S. Let $S = I_{12}$. Then by Corollary 3.2 and Theorem 3.3, $\Gamma(S)$ is disconnected so is not a tree. Thus $S \neq I_{12}$. Then $J = I_{12}$ is a nontrivial left ideal of S. Suppose S has a nontrivial left ideal K other than I_1, I_2 and J. Without loss of generality, assume that $I_1 \subset K$ then we get a cycle $I_1 \sim I_{12} \sim K \sim I_1$, a contradiction. Thus, for $S \neq I_{12}$, we have $V(\Gamma(S)) = \{I_1, I_2, I_{12}\}$.

(iii) $\Rightarrow$ (iv). If $\Gamma(S)$ is bipartite then we have $|\mathrm{Min}(S)| \leqslant 2$. In the similar lines of the work discussed above, (iv) holds. □

Theorem 4.5. *If S is the union of n minimal left ideals, then $\gamma(\Gamma(S)) = 2$. Otherwise, $\gamma(\Gamma(S)) = 1$.*

Proof. Suppose that S is the union of n minimal left ideals, that is, $S = I_{12\cdots n}$. Note that there is no dominating vertex in $\Gamma(S)$ so that $\gamma(\Gamma(S)) \geqslant 2$. Now we show that $D = \{I_1, I_{23\cdots n}\}$ is a dominating set. Since S is the union of n minimal left ideals so any nontrivial left ideal of S is the union of these minimal left ideals (cf. Remark 2.2). Let $J \in V(\Gamma(S)) \setminus D$ be any nontrivial left ideal of S. Then J is a union of k minimal left ideals of S, where $1 \leqslant k \leqslant n-1$. If $I_1 \subset J$, then we are done. Otherwise, J

must be the union of $I_2, I_3, \dots, I_n$. It follows that the intersection of J and $I_{23\cdots n}$ is nontrivial. Consequently, $J \sim I_{23\cdots n}$. Thus D is a dominating set. Further, suppose that $S \neq I_{12\cdots n}$. It follows that $J = I_{12\cdots n}$ is a nontrivial left ideal of S. It is well known that every nontrivial left ideal of S contains at least one minimal left ideal. Consequently, for any nontrivial left ideal K of S, we have $J \cap K$ is nontrivial. Thus, J is a dominating vertex. Hence, $\gamma(\Gamma(S)) = 1$. This completes the proof. □

Theorem 4.6. *Let S be a semigroup with n minimal left ideals. Then the independence number of $\Gamma(S)$ is n.*

Proof. Let $\mathrm{Min}(S) = \{I_{i_1} : i_1 \in [n]\}$ be the set of all minimal left ideals of S. Then, by Remark 2.1, $\mathrm{Min}(S)$ is an independent set of $\Gamma(S)$. It follows that $\alpha(\Gamma(S)) \geqslant n$. Now we prove that for any arbitrary independent set U, we have $|U| \leqslant n$. Assume that $I \in V(\Gamma(S))$ such that $I \in U$. Since every nontrivial left ideal contains at least one minimal left ideal. Without loss of generality, assume that $I_{i_1 i_2 \cdots i_k} \subseteq I$ for some $i_1, i_2, \cdots, i_k \in [n]$. Then note that $|U| \leqslant n - k + 1$. Otherwise, there exist at least two nontrivial left ideals which are adjacent, a contradiction. Consequently, we have $|U| \leqslant n$. Thus, $\alpha(\Gamma(S)) = n$. □

Lemma 4.7. *Let S be a semigroup with n $(\geqslant 3)$ minimal left ideals. Then there exists a clique in $\Gamma(S)$ of size n.*

Proof. Let $I_1, I_2, \dots, I_n$ be n minimal left ideals. Consider $\mathcal{C} = \{I_{i_1 i_2 \cdots i_{n-1}} : i_1, i_2, \dots, i_{n-1} \in [n]\}$. Clearly, $|\mathcal{C}| = n$. Notice that for any $J, K \in \mathcal{C}$, we have $J \cap K$ is nontrivial so that $J \sim K$. Thus, $\mathcal{C}$ becomes a clique of size n. □

Theorem 4.8. *Let S be a semigroup with $n(> 1)$ minimal left ideals. Then $\omega(\Gamma(S)) = n$ if and only if one of the following holds:*

- (i) *S is the union of exactly three minimal left ideals.*
- (ii) *S has only two minimal left ideals I_1 and I_2 and a unique maximal left ideal I_{12}.*

Proof. First suppose that $\omega(\Gamma(S)) = n$. Assume that S has $n(\geqslant 4)$ minimal left ideals, namely $I_1, I_2, \dots, I_n$. Then $\mathcal{C} = \{I_{i_1 i_2 \cdots i_{n-1}}, I_{i_1 i_2} : i_1, i_2, \dots, i_n \in [n]\}$ forms a clique of size greater than n of $\Gamma(S)$. It follows that $\omega(\Gamma(S)) > n$. If $n = 3$, assume that $S \neq I_{123}$. Then $\mathcal{C} = \{I_{12}, I_{13}, I_{23}, I_{123}\}$ forms a clique of size four of $\Gamma(S)$. It follows that $S = I_{123}$. For $n = 2$, we have either $S = I_{12}$ or $S \neq I_{12}$. For $S = I_{12}$, by Corollary 3.2 and by Theorem 3.3, $\Gamma(S)$ is disconnected. Thus, $\omega(\Gamma(S)) < n$. Thus $S \neq I_{12}$. If S has a nontrivial left ideal $K \notin \{I_1, I_2, I_{12}\}$ then we get a clique of size three. Therefore, I_{12} is a unique maximal left ideal. Converse follows trivially. $\square$

Lemma 4.9. *If* $\Gamma(S)$ *is connected then* $\mathrm{Max}(S)$ *forms a clique of* $\Gamma(S)$.

Proof. We prove the result by showing that if $J, K \in \mathrm{Max}(S)$ then $J \sim K$. Let $J \nsim K$. The maximality of J and K follows that $J \cup K = S$. By Lemma 2.3, $S \setminus J$ and $S \setminus K$ are $\mathcal{L}-$classes of S. It follows that J and K are only nontrivial left ideals of S. Thus, being a null graph $\Gamma(S)$ is disconnected, a contradiction. $\square$

Theorem 4.10. *If* K *is a maximal left ideal of* S *such that* $deg(K)$ *is finite, then* $\chi(\Gamma(S)) < \infty$.

Proof. Let J be an arbitrary nontrivial left ideal of S such that $J \nsim K$. Note that J is the minimal left ideal of S. On the contrary, suppose that J is not a minimal left ideal of S. Then there exists a nontrivial left ideal J' of S such that $J' \subset J$. Since K is the maximal left ideal of S, we get $J' \cup K = S$. It follows that the intersection of J and K is nontrivial, a contradiction. By Remark 2.1, we can color all the vertices which are not adjacent with K with one color. Since $deg(K)$ is finite, we have $\chi(\Gamma(S)) < \infty$. $\square$

Proposition 4.11. *If* S *is the union of* n *minimal left ideals, then* $\omega(\Gamma(S)) = \chi(\Gamma(S)) = 2^{n-1} - 1$. *Moreover,* $\Gamma(S)$ *is weakly perfect.*

Proof. First note that S has $2^n - 2$ nontrivial left ideals and every nontrivial left ideal of S is of the form $I_{i_1 i_2 \cdots i_k}$ and $1 \leqslant k \leqslant n-1$ (cf. Remark 2.2). If n is odd then consider $\mathcal{C} = \{I_{j_1 j_2 \cdots j_t} : \lceil \frac{n}{2} \rceil \leqslant t \leqslant n-1\}$. Note that $\mathcal{C}$ forms a clique of size $2^{n-1} - 1$. We may now suppose that n is even. Consider $\mathcal{C}_1 = \{I_{j_1 j_2 \cdots j_t} : \frac{n}{2} + 1 \leqslant t \leqslant n-1\}$. Notice that $\mathcal{C}_1$ forms a clique. Further, observe that $\mathcal{C}^{'} = \{I_{i_1 i_2 \cdots i_{\frac{n}{2}}} : i_1, i_2, \ldots, i_{\frac{n}{2}} \in [n]\}$ do not form a clique because for $j_1, j_2, \ldots, j_{\frac{n}{2}} \in [n] \setminus \{i_1, i_2, \ldots, i_{\frac{n}{2}}\}$, $I_{i_1 i_2 \cdots i_{\frac{n}{2}}} \nsim I_{j_1 j_2 \cdots j_{\frac{n}{2}}}$. However, $\mathcal{C}^{''} = \{I_{i_1 i_2 \cdots i_{\frac{n}{2}}} \in \mathcal{C}^{'} \setminus \{I_{j_1 j_2 \cdots j_{\frac{n}{2}}}\} : j_1, j_2, \ldots, j_{\frac{n}{2}} \notin \{i_1, i_2, \ldots, i_{\frac{n}{2}}\}\}$ forms a clique of size $\frac{|\mathcal{C}^{'}|}{2}$. Further note that the set $\mathcal{C}_1 \cup \mathcal{C}^{''}$ also forms a clique of size $2^{n-1} - 1$. Thus, $\omega(\Gamma(S)) \geqslant 2^{n-1} - 1$. To complete the proof, we show that $\chi(\Gamma(S)) \leqslant 2^{n-1} - 1$. For $I = I_{i_1 i_2 \cdots i_k}$ and $J = I_{j_1 j_2 \cdots j_{n-k}}$, where $i_1, i_2, \ldots, i_k \in [n] \setminus \{j_1, j_2, \ldots, j_{n-k}\}$ we have $I \cap J$ is trivial. Consequently, we can color these vertices with same color so that we can cover all the vertices with $2^{n-1}-1$ colors. Thus $\chi(\Gamma(S)) \leqslant 2^{n-1}-1$. Hence $\omega(\Gamma(S)) = \chi(\Gamma(S)) = 2^{n-1}-1$. $\square$

Corollary 4.12. *Let S be a completely simple semigroup. Then the graph $\Gamma(S)$ is weakly perfect.*

In order to find the upper bound of the chromatic number of $\Gamma(S)$, where S is an arbitrary semigroup, first we define

$$\begin{aligned}
&X_1 = \{I \in V(\Gamma(S)) : I_{i_1 i_2 \cdots i_n} \subseteq I\},\\
&X_2 = \{I \in V(\Gamma(S)) : I \subset I_{i_1 i_2 \cdots i_n} \text{ and } I \neq I_{i_1 i_2 \cdots i_n}\},\\
&X_3 = V(\Gamma(S)) \setminus (X_1 \cup X_2).
\end{aligned}$$

Let $\widetilde{\mathrm{Min}(I)}$ be the set of all minimal left ideals contained in I . Further define a relation ρ on X_3 such that

$$J \ \ \rho \ \ K \Longleftrightarrow \widetilde{\mathrm{Min}(J)} = \widetilde{\mathrm{Min}(K)}.$$

Note that ρ is an equivalence relation.

Theorem 4.13. *Let S be a semigroup with n minimal left ideals and $\chi(\Gamma(S)) < \infty$. Then*

$$\chi(\Gamma(S)) \leqslant |X_1| + (2^{n-1}-1) + (2^{n-1}-1)m,$$

where $m = \max\{|C(I)| : C(I)$ is an equivalence class of $\rho\}$.

Proof. Note that for any $I, J \in X_1$, we have $I \sim J$. Since every nontrivial left ideal contains at least one minimal left ideal, consequently each element of X_1 is a dominating vertex of $\Gamma(S)$. Therefore, we need at least $|X_1|$ colors in any coloring of $\Gamma(S)$. By proof of Proposition 4.11, we can color all the vertices of X_2 with at least $2^{n-1}-1$ colors so that we need at least $2^{n-1}-1+|X_1|$ colors to color $X_1 \cup X_2$.

To prove our result we need to show that the vertices of X_3 can be colored by using $(2^{n-1}-1)m$ colors. Now let $J, K \in X_3$ such that $I_{i_1 i_2 \cdots i_k} \subset J$ and $I_{j_1 j_2 \cdots j_t} \subset K$. Note that $J \cap K$ is nontrivial if and only if $I_{i_1 i_2 \cdots i_k} \cap I_{j_1 j_2 \cdots j_t}$ is nontrivial. It follows that $J \sim K$ in $\Gamma(S)$ if and only if either $I_{i_1 i_2 \cdots i_k} = I_{j_1 j_2 \cdots j_t}$ or $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$.

Note that the equivalence class of $I \in X_3$ under ρ is $C(I) = \{J \in X_3 : \widetilde{\mathrm{Min}(I)} = \widetilde{\mathrm{Min}(J)}\}$. Since $\chi(\Gamma(S)) < \infty$ we get $|C(I)| < \infty$. Consequently, $|C(I)| \leqslant m$. Observe that $C(I)$ forms a clique, we require maximum m

colors to color each class under ρ. Note that $J \in C(J)$ and $K \in C(K)$ such that $J \sim K$ if and only if $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$ in $\Gamma(S)$. Consequently, we can color the vertices in X_3 by using $(2^{n-1} - 1)m$ colors. $\square$

Theorem 4.14. *Let S be a semigroup with n minimal left ideals. Then*

$$\mathrm{sdim}(\Gamma(S)) = \begin{cases} 2^{n-1} - 1 & \text{if } S \text{ is a union of } n \text{ minimal left ideals;} \\ |X_1| + |X_3| + 2^{n-1} - 2 & \text{otherwise.} \end{cases}$$

Proof. Let $I, J \in V(\Gamma(S))$ such that $I_{i_1 i_2 \cdots i_k} \subseteq I$ and $I_{j_1 j_2 \cdots j_t} \subseteq J$. Then $I \sim J$ if and only if either $I_{i_1 i_2 \cdots i_k} = I_{j_1 j_2 \cdots j_t}$ or $I_{i_1 i_2 \cdots i_k} \sim I_{j_1 j_2 \cdots j_t}$. Define a relation ρ_1 on $V(\Gamma(S))$ such that $I \ \rho_1 \ J$ if and only if $\widetilde{\mathrm{Min}(I)} = \widetilde{\mathrm{Min}(J)}$. Clearly, ρ_1 is an equivalence relation on $V(\Gamma(S))$. Let $N[I_{i_1 i_2 \cdots i_k}] = \{I \in V(\Gamma(S)) : \widetilde{\mathrm{Min}(I)} = I_{i_1 i_2 \cdots i_k}\}$ be equivalence class containing $I_{i_1 i_2 \cdots i_k}$. If $S \neq I_{i_1 i_2 \cdots i_n}$, then by Theorem 2.6, we have $\mathcal{R}_{\Gamma(S)}$ whose vertex set $V(\mathcal{R}_{\Gamma(S)}) = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \cdots, i_k \in [n] \text{ and } 1 \leqslant k \leqslant n\}$. By using Proposition 4.11, note that $\omega(\mathcal{R}_{\Gamma(S)}) = 2^{n-1}$. Then $\mathrm{sdim}(\Gamma(S)) = |X_1| + |X_3| + 2^{n-1} - 2$. Next, if $S = I_{i_1 i_2 \cdots i_n}$, then $V(\mathcal{R}_{\Gamma(S)}) = \{I_{i_1 i_2 \cdots i_k} : i_1, i_2, \cdots, i_k \in [n] \text{ and } 1 \leqslant k \leqslant n-1\}$. By using Proposition 4.11, note that $\omega(\mathcal{R}_{\Gamma(S)}) = 2^{n-1} - 1$. Then $\mathrm{sdim}(\Gamma(S)) = 2^{n-1} - 1$. $\square$

In the rest of the section, we consider a class of those semigroups which are the union of n minimal left ideals. In particular, completely simple semigroups belongs to this class. In what follows, the semigroup S is assumed to be the union of n minimal left ideals $I_{i_1}, I_{i_2}, \ldots, I_{i_n}$ i.e. $S = I_{i_1 i_2 \cdots i_n}$. The following lemma gives the lower bound of the metric dimension of $\Gamma(S)$.

Lemma 4.15 ([13, Theorem 1])**.** *For positive integers d and m with $d < m$, define $f(m, d)$ as the least positive integer k such that $k + d^k \geqslant m$. Then for a connected graph Γ of order $m \geqslant 2$ and diameter d, the metric dimension $\beta(\Gamma) \geqslant f(m, d)$.*

Theorem 4.16. *If S is the union of n minimal left ideals, then the metric dimension of $\Gamma(S)$ is given below:*

$$\beta(\Gamma(S)) = \begin{cases} 2 & \text{if } n = 3; \\ n & \text{if } n \geqslant 4. \end{cases}$$

Proof. For $n = 3$, it is easy to observe that $\{I_{i_1}, I_{i_2}\}$ forms a minimum resolving set. If $n \geqslant 4$ then by Remark 2.2, we have $|V(\Gamma(S))| = 2^n - 2$. In view of Lemma 4.15, we get

$$n = f(2^n - 2, 2) \leqslant \beta(\Gamma(S)).$$

It is easy to observe that for $k = n-1$, $2^k + k \ngeqslant 2^n - 2$. Therefore, the least positive integer k is n for which $k + 2^k \geqslant 2^n - 2$. Thus $n \leqslant \beta(\Gamma(S))$. To obtain upper bound of $\beta(\Gamma(S))$, let $J = I_{i_1 i_2 \cdots i_k}$ and $K = I_{j_1 j_2 \cdots j_t}$ be distinct arbitrary vertices $\Gamma(S)$. Since $J \neq K$, there exists at least $I_{i_s} \in \mathrm{Min}(S)$ such that $I_{i_s} \sim J$ and $I_{i_s} \nsim K$. It follows that $d(J, I_{i_s}) \neq d(K, I_{i_s})$. Thus $\mathrm{Min}(S) = \{I_{i_1} : i_1 \in [n]\}$ forms a resolving set for $\Gamma(S)$ of size n. It follows that $\beta(\Gamma(S)) \leqslant n$. This completes our proof. □

An automorphism of a graph Γ is a permutation f on $V(\Gamma)$ with the property that, for any vertices u and v, we have $uf \sim vf$ if and only if $u \sim v$. The set $Aut(\Gamma)$ of all graph automorphisms of a graph Γ forms a group with respect to composition of mappings. The symmetric group of degree n is denoted by S_n. Now we obtain the automorphism group of $\Gamma(S)$, when S is the union of n minimal left ideal.

Lemma 4.17. *Let S be the union of n minimal left ideals and let $K = I_{i_1 i_2 \cdots i_k}$ be a nontrivial left ideal of S. Then $deg(K) = (2^k - 2) + (2^{n-k} - 2) + (2^{n-k} - 1)(2^k - 2)$.*

Proof. Let J be a nontrivial left ideal of S such that $J \sim K$. Clearly, $J \cap K$ is a nontrivial left ideal. We have the following cases:
Case 1. $J \not\subset K$ and $K \not\subset J$. Since $J \sim K$ and $K = I_{i_1 i_2 \cdots i_k}$, we obtain the number of nontrivial left ideals such that $J \not\subset K$ and $K \not\subset J$ is

$$= \left(\sum_{i=1}^{n-k} \binom{n-k}{i} \right) \left(\sum_{i=1}^{k-1} \binom{k}{i} \right) = (2^{n-k} - 1)(2^k - 2).$$

Case 2. $J \subset K$. The number of nontrivial left ideals of S which are properly contained in K is $2^k - 2$ (see proof of [9, Lemma 4.3]).
Case 3. $K \subset J$. The number of nontrivial left ideals of S properly containing K is $2^{n-k} - 2$ (see proof of [9, Lemma 4.3]). Thus, from the above cases we have the result. □

Corollary 4.18. *If S is the union of n minimal left ideals, then the graph $\Gamma(S)$ is Eulerian for $n \geqslant 3$.*

Lemma 4.19. *For $\sigma \in S_n$, let $\phi_\sigma : V(\Gamma(S)) \to V(\Gamma(S))$ defined by $\phi_\sigma(I_{i_1 i_2 \cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)}$. Then $\phi_\sigma \in Aut(\Gamma(S))$.*

Proof. It is easy to verify that ϕ_σ is a permutation on $V(\Gamma(S))$. Now we show that the map ϕ_σ preserves adjacency. Let $I_{i_1i_2\cdots i_t}$ and $I_{j_1j_2\cdots j_k}$ be arbitrary vertices of $\Gamma(S)$ such that $I_{i_1i_2\cdots i_t} \sim I_{j_1j_2\cdots j_k}$. Then $I_{i_1i_2\cdots i_t} \cap I_{j_1j_2\cdots j_k} \neq \emptyset$. Now

$$\begin{aligned} I_{i_1i_2\cdots i_t} \sim I_{j_1j_2\cdots j_k} &\iff I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_t)} \sim I_{\sigma(j_1)\sigma(j_2)\cdots\sigma(j_k)} \\ &\iff \phi_\sigma(I_{i_1i_2\cdots i_t}) \sim \phi_\sigma(I_{j_1j_2\cdots j_k}). \end{aligned}$$

Thus, $\phi_\sigma \in Aut(\Gamma(S))$. □

Proposition 4.20. *For each $f \in Aut(\Gamma(S))$, we have $f = \phi_\sigma$ for some $\sigma \in S_n$.*

Proof. In view of Lemma 4.17 and Lemma 4.19, suppose that $f(I_{i_1}) = I_{j_1}$, $f(I_{i_2}) = I_{j_2}$, …, $f(I_{i_n}) = I_{j_n}$. Consider $\sigma \in S_n$ such that $\sigma(i_1) = j_1, \sigma(i_2) = j_2, \ldots, \sigma(i_n) = j_n$. Then $\phi_\sigma(I_{i_1i_2\cdots i_k}) = I_{\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)} = I_{j_1j_2\cdots j_k}$ (cf. Lemma 4.19). Clearly, $I_{i_1} \sim I_{i_1i_2\cdots i_k}$, $I_{i_2} \sim I_{i_1i_2\cdots i_k}$, …, $I_{i_k} \sim I_{i_1i_2\cdots i_k}$. Also note that $I_{i_t} \cap I_{i_1i_2\cdots i_k}$ is trivial for $i_t \in \{i_{k+1}, i_{k+2}, \ldots, i_n\}$ where $i_{k+1}, i_{k+2}, \ldots, i_n \in [n] \backslash \{i_1, i_2, \ldots, i_k\}$. It follows that $I_{i_{k+1}} \nsim I_{i_1i_2\cdots i_k}$, $I_{i_{k+2}} \nsim I_{i_1i_2\cdots i_k}$, …, $I_{i_n} \nsim I_{i_1i_2\cdots i_k}$. Thus, $f(I_{i_1}) \sim f(I_{i_1i_2\cdots i_k})$, $f(I_{i_2}) \sim f(I_{i_1i_2\cdots i_k})$, …, $f(I_{i_k}) \sim f(I_{i_1i_2\cdots i_k})$ and $f(I_{i_{k+1}}) \nsim f(I_{i_1i_2\cdots i_k})$, $f(I_{i_{k+2}}) \nsim f(I_{i_1i_2\cdots i_k})$, …, $f(I_{i_n}) \nsim f(I_{i_1i_2\cdots i_k})$. Consequently, $I_{j_1} \subset f(I_{i_1i_2\cdots i_k})$, $I_{j_2} \subset f(I_{i_1i_2\cdots i_k})$, …, $I_{j_k} \subset f(I_{i_1i_2\cdots i_k})$ and $I_{j_{k+1}} \not\subset f(I_{i_1i_2\cdots i_k})$, $I_{j_{k+2}} \not\subset f(I_{i_1i_2\cdots i_k})$, …, $I_{j_n} \not\subset f(I_{i_1i_2\cdots i_k})$. It follows that $f(I_{i_1i_2\cdots i_k}) = I_{j_1j_2\cdots j_k} = \phi_\sigma(I_{i_1i_2\cdots i_k})$. Thus, $f = \phi_\sigma$. □

Theorem 4.21. *Let S be the union of n minimal left ideals. Then for $n \geqslant 2$, we have $Aut(\Gamma(S)) \cong S_n$. Moreover, $|Aut(\Gamma(S))| = n!$.*

Proof. In view of Lemma 4.19 and by Proposition 4.20, note that the underlying set of the automorphism group of $\Gamma(S)$ is $Aut(\Gamma(S)) = \{\phi_\sigma \,:\, \sigma \in S_n\}$, where S_n is a symmetric group of degree n. The groups $Aut(\Gamma(S))$ and S_n are isomorphic under the assignment $\phi_\sigma \mapsto \sigma$. Since all the elements in $Aut(\Gamma(S))$ are distinct, we have $|Aut(\Gamma(S))| = n!$. □

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