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General form of the automorphism group of bicyclic graphs

Somayeh Madani and *Ali Reza Ashrafi*

Abstract. In 1869, Jordan proved that the set $\mathcal{T}$ of all finite groups that can be represented as the automorphism group of a tree is containing the trivial group, it is closed under taken the direct product of groups of lower orders in $\mathcal{T}$, and wreath product of a member of $\mathcal{T}$ and the symmetric group on n symbols is again an element of $\mathcal{T}$. The aim of this paper is to continue this work and another works by Klavík and Zeman in 2017 to present a class $\mathcal{S}$ of finite groups for which the automorphism group of each bicyclic graph is a member of $\mathcal{S}$ and this class is minimal with this property.

1. Basic definitions

The aim of this section is to provide some introductory materials that will be kept throughout. All graphs are assumed to be undirected, simple and finite. The set of all vertices and edges of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A rooted graph is a graph in which one vertex is distinguished as the root. If the graph Γ is containing n vertices and m edges, then the cyclomatic number of Γ is defined as $c = m - n + 1$. If $c = 0, 1, 2$ then Γ is called a tree, a unicyclic and a bicyclic graph, respectively.

Suppose G and H are groups and H acts on a set X. Define:

$$\{(h; f) \mid f : X \longrightarrow G \ \&\ h \in H\} \ \ ; \ \ (h_1; f_1)(h_2; f_2) = (h_1h_2; f_1^{h_2} f_2),$$

where $f_1^{h_2}(x) = f_1(x^{h_2})$. This product defines a group structure and the resulting group is called the wreath product of G with H, denoted by $G \wr H$. The wreath product is an important tool to describe the automorphism

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Ali Reza Ashrafi died a tragic death on January 9, 2023.

group of graphs. Let the connected components of a graph Γ consist of n_1 copies of G_1, n_2 copies of G_2, ..., n_r copies of G_r, where G_1, ..., G_r are pairwise non-isomorphic. Then by a well-known result of Jordan [?] $\mathrm{Aut}(G) \cong (\mathrm{Aut}(G_1) \wr S_{n_1}) \times \cdots \times (\mathrm{Aut}(G_r) \wr S_{n_r})$.

Suppose G_1, G_2 and G_3 are graphs with disjoint vertex sets and $v_1 \in V(G_1)$, $w_1 \in V(G_2)$, $v_2 \in V(G_1 \cup G_2) \setminus \{v_1, w_1\}$ and $w_2 \in V(G_3)$. The union $G_1 \cup G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graph union of more than two graphs can be defined inductively. Following Došlić [?], the splice $S(G_1, G_2; v_1, w_1)$ is defined by identifying the vertices v_1 and w_1 in $G_1 \cup G_2$. In a similar way, $S(G_1, G_2, G_3; v_1, w_1; v_2, w_2) = S(S(G_1, G_2; v_1, w_1), G_3; v_2, w_2)$ and we can define the splice of more than two graphs with respect to a parent graph by an inductive method. The link $L(G_1, G_2; v, w)$ is defined by adding an edge to the union graph $G_1 \cup G_2$ connecting the vertices v and w. The link of more than two graphs can be defined similar to the splice.

Suppose Γ is a simple and undirected graph and $u, v \in V(\Gamma)$. The distance between u and v is defined as the length of a shortest path connecting these vertices. The eccentricity $\varepsilon(v)$ is defined to be the greatest distance between v and any other vertices of Γ. The center of Γ is the set of all vertices with minimum eccentricity, i.e the set of all vertices u such that the greatest distance $d(u, v)$ to other vertices v is minimal.

All calculations of this paper are done with the aid of GAP [?] and Mathematica [?]. We refer to [?, ?] for basic definitions and notations not presented here.

2. Backgrounds

Suppose $\mathcal{C}$ is a class of graphs and $\mathrm{Aut}(\mathcal{C})$ denotes the set of all groups that can be presented as the automorphism group of a member in $\mathcal{C}$. If C is the class of all trees then $\mathrm{Aut}(\mathcal{C})$ is denoted by $\mathcal{T}$. By a result of Jordan [?], $\mathcal{T}$ is the class of all finite groups that can be defined inductively as follows:

1. $\{1\} \in \mathcal{T}$;
2. if $G_1, G_2 \in \mathcal{T}$, then $G_1 \times G_2 \in \mathcal{T}$;
3. if $G \in \mathcal{T}$ and $n \geqslant 2$, then $G \wr S_n \in \mathcal{T}$.

One of the most interesting results after Jordan is a result of Babai [?]. To state this result, we assume that X and Y are graphs and $f : V(X) \to$

$V(Y)$ is a mapping between vertex sets of X and Y. The function f is called a contraction if (i) $y_1y_2 \in E(Y)$ if and only if $y_1 \neq y_2$ and there is an edge $x_1x_2 \in E(X)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$; (ii) for any $y \in V(Y)$, the induced subgraph of X on $f^{-1}(y)$ is connected. In the mentioned paper, Babai proved that if $\mathcal{C}$ is a class of finite graphs with this property that $\mathcal{C}$ is closed under contraction and forming subgraphs, and if every finite group occurs as the automorphism group of a graph in $\mathcal{C}$, then $\mathrm{Aut}(\mathcal{C})$ contains all finite graphs up to isomorphism. In another paper [?], he proved that if Γ is planar, then the group $\mathrm{Aut}(\Gamma)$ has a subnormal chain $\mathrm{Aut}(\Gamma) \rhd Y_1 \rhd Y_2 \rhd \cdots \rhd Y_m = 1$.

Set $\mathcal{I} = \{\ \mathrm{Aut}(U) \mid U \textit{ is an interval graph}\}$. Klavík and Zeman [?] proved that $\mathcal{T} = \mathcal{I}$. They also obtained some interesting relation between the set of automorphism groups of some known classes of graphs. We encourage the interested readers to consult [?] for more information on this problem.

The aim of this paper is to continue the interesting works of Klavík and Zeman by computing the automorphism group of bicyclic graphs. In an exact phrase, if $\mathcal{S}$ denotes the set of all groups in the form of $\mathrm{Aut}(G)$ with bicyclic graph G then the set $\mathcal{S}$ will be determined in general.

3. Main results

The aim of this section is to compute the automorphism group of an arbitrary bicyclic graph. To do this, we define:

$$\mathcal{B}_1 = \{C \times (D \wr (Z_2 \times Z_2)) \mid C, D \in \mathcal{T}\},$$

$$\mathcal{B}_2 = \{C \times [(D \times D \times D \times D \times H \times H \times K \times K) \rtimes (Z_2 \times Z_2)] \mid C, D, H, K \in \mathcal{T}\},$$

and $\mathcal{S} = \mathcal{T} \cup \mathcal{B}_1 \cup \mathcal{B}_2$. In this section, it will be proved that $\mathcal{S}$ is the set of all groups in the form of $\mathrm{Aut}(\Delta)$, when Δ is a bicyclic graph.

Suppose T is a tree, G is a group, X is a set and $u \in V(T)$. A branch at u in T is a maximal subtree containing u as an endpoint, see [?, p. 35]. If the group G acts on X and $x \in X$ then G_x denotes the stabilizer subgroup of G at the point x. An asymmetric graph is one with trivial automorphism group.

The following simple lemma will be useful in our calculations.

Lemma 3.1. *Suppose T is a tree, $G = \mathrm{Aut}(T)$ and $v \in V(T)$. Then $G_v \in \mathcal{T}$.*

Proof. Choose an asymmetric tree Λ containing a pendent vertex w such that the degree of the unique vertex u adjacent to w is different from all vertices of T. Define $T' = S(T, \Lambda; v, w)$. Now it is easy to see that $\mathrm{Aut}(T') \cong G_v$ and so $G_v \in \mathcal{T}$. □

Suppose Δ is an arbitrary bicyclic graph. Then the graph Δ has one of the following forms:

There are two cycles in Δ with at least one common edge.
There are two cycles in Δ without common edges and common vertices.
There are two cycles in Δ with a common vertex and without common edges.

A bicyclic graph H is said to be of type i ($i = 1, 2, 3$) if H satisfies the condition i.

Suppose Δ is a graph and T_1, T_2 are two subgraphs of Δ such that T_1, T_2 are trees and $v_1 \in V(T_1)$, $v_2 \in V(T_2)$ are vertices of a cycle in Δ. We say these trees satisfy the condition ($\star$) if and only if (T_1, T_2) and ($\mathrm{Aut}(T_1)_{v_1}$, $\mathrm{Aut}(T_2)_{v_2}$) are pairs of isomorphic graphs.

Lemma 3.2. *Suppose Δ is a bicyclic graph depicted in Figure* **??** *and all pairs of elements in each set $\{T_1, T_2, T_3, T_4\}$, $\{\Theta_5, \Theta_6\}$ and $\{\Upsilon_1, \Upsilon_2\}$ satisfy the condition ($\star$). We also assume that $D = (\mathrm{Aut}(T_1))_a \cong (\mathrm{Aut}(T_2))_b \cong (\mathrm{Aut}(T_3))_c \cong (\mathrm{Aut}(T_4))_d$, $K = (\mathrm{Aut}(\Upsilon_1))_u \cong (\mathrm{Aut}(\Upsilon_2))_v$ and $H = (\mathrm{Aut}(\Theta_5))_{e_1} \cong (\mathrm{Aut}(\Theta_6))_{e_2}$. Then,*

$\mathrm{Aut}(\Delta) \cong (D \times D \times D \times D \times H \times H \times K \times K) \rtimes_\phi (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

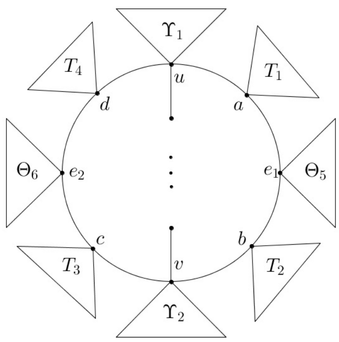


Figure 1: The bicyclic graph of Lemma **??**.

Proof. Define $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(D \times D \times D \times D \times H \times H \times K \times K)$ by $\phi(0,0) = I$ and $\phi(0,1), \phi(1,1)$, $\phi(1,1)$ are defined as follows:

$$\phi(0,1) = \psi_1 : (\alpha, \beta, \gamma, \delta, \lambda, \lambda', \sigma, \sigma') \mapsto (\gamma, \delta, \alpha, \beta, \lambda, \lambda', \sigma', \sigma)$$
$$\phi(1,0) = \psi_2 : (\alpha, \beta, \gamma, \delta, \lambda, \lambda', \sigma, \sigma') \mapsto (\beta, \alpha, \delta, \gamma, \lambda', \lambda, \sigma, \sigma')$$
$$\phi(1,1) = \psi_3 : (\alpha, \beta, \gamma, \delta, \lambda, \lambda', \sigma, \sigma') \mapsto (\beta, \alpha, \delta, \gamma, \lambda', \lambda, \sigma', \sigma)$$

where $\alpha \in \mathrm{Aut}(T_1)$, $\beta \in \mathrm{Aut}(T_2)$, $\gamma \in \mathrm{Aut}(T_3)$, $\delta \in \mathrm{Aut}(T_4)$, $\lambda \in \mathrm{Aut}(\Upsilon_1)$, $\lambda' \in \mathrm{Aut}(\Upsilon_2)$, $\sigma \in \mathrm{Aut}(\Theta_5)$ and $\sigma' \in \mathrm{Aut}(\Theta_6)$. Moreover, we assume that $V(T_i) = \{t_1^i, \cdots, t_m^i\}$, $V(\Theta_j) = \{s_1^j, \cdots, s_p^j\}$ and $V(\Upsilon_k) = \{r_1^k, \cdots, r_q^k\}$, where $1 \leqslant i \leqslant 4$, $j = 5, 6$ and $k = 1, 2$.

There are three paths connecting vertices u and v. These paths have the vertex sets $V(P_1) = \{v, a, e_1, b, u\}$, $V(P_2) = \{u, d, e_2, c, v\}$ and

$$V(P_3) = \begin{cases} \{v, u_{11}, u_{12}, \ldots, u_{1t}, u_{21}, u_{22}, \ldots, u_{2t}, u\} & 2 \nmid l(P_3) \\ \{v, u_{11}, u_{12}, \ldots, u_{1t}, z, u_{21}, u_{22}, \ldots, u_{2t}, u\} & 2 \mid l(P_3) \end{cases}.$$

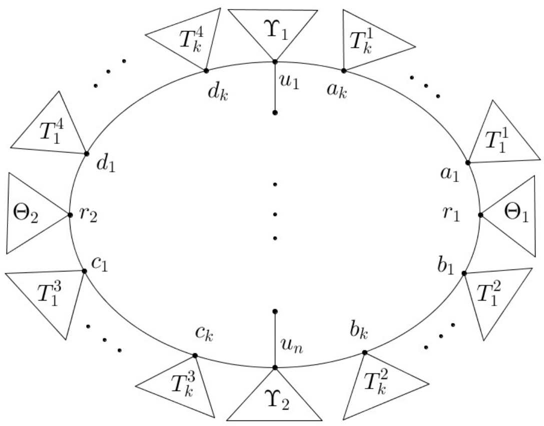


Figure 2: A general bicyclic graph of the first type.

Suppose $O_1 = V(\Delta) \backslash (V(T_1) \cup V(T_2) \cup V(T_3) \cup V(T_4) \cup V(\Theta_5) \cup V(\Theta_6))$, $O_2 = V(P_3) \setminus \{u, v\}$, $\sigma = (23)(14)(56)$ and $\tau = (12)(34)$. For $i = 1, 2, 3, 4$, $j = 5, 6$ and $k = 1, 2$, we define two permutations f_1 and f_2 on $V(\Delta)$ and eight sets V_i, U_j and V_k as follows:

$$f_1 = \begin{pmatrix} s_l^j & t_l^k & x \\ s_l^{\sigma(j)} & t_l^{\sigma(k)} & x \end{pmatrix}; \quad x \in O_1,$$
$$f_2 = \begin{pmatrix} u_{lt} & r_l^j & t_l^k & s_l^j \\ u_{\tau(l)t} & r_l^{\tau(j)} & t_l^{\tau(k)} & s_l^j \end{pmatrix},$$
$$V_i = \{f \in Aut(\Delta) \mid \forall x \in V(T_i), f(x) = x \ \&\ f(t_1^i) = t_1^i\}; \quad i = 1, 2, 3, 4,$$
$$U_j = \{f \in Aut(\Delta) \mid \forall x \in V(\Theta_j), f(x) = x \ \&\ f(s_1^j) = s_1^j\}; \quad j = 5, 6,$$
$$G_k = \{f \in Aut(\Delta) \mid \forall x \in V(\Upsilon_k), f(x) = x \ \&\ f(r_1^k) = r_1^k\}; \quad k = 1, 2.$$

It is easy to see that f_1, f_2 are involutions in $\mathrm{Aut}(\Delta)$. Define $L = \{I, f_1\}$ and $M = \{I, f_2\}$. Note that $|ML| \leqslant 4$ and so the group ML is abelian. This proves that $ML = LM \cong L \times M$. We are now ready to prove that $V_1V_2V_3V_4U_5U_6G_1G_2 \leq \mathrm{Aut}(\Delta)$. If $f \in V_i$ and $g \in G_k$, $1 \leqslant i \leqslant 4$ and $k = 1, 2$, then

$$fog(x) = \begin{cases} f(x) & x \in T_i \\ g(x) & x \in \Upsilon_j \\ x & otherwise \end{cases} = gof(x).$$

Thus elements of V_i and G_k are commute to each other and so V_iG_k is a subgroup of $\mathrm{Aut}(\Delta)$. A similar argument shows that each element of A commutes with each elements of B such that

$$A, B \in \Gamma_1 = \{V_1, V_2, V_3, V_4, U_5, U_6, G_1, G_2\}.$$

This proves that $\Gamma_2 = V_1V_2V_3V_4U_5U_6G_1G_2$ is a subgroup of $\mathrm{Aut}(\Delta)$ and since each element of Γ_1 is a normal subgroup of Γ_2, $V_1V_2V_3V_4U_5U_6G_1G_2 \simeq V_1 \times V_2 \times V_3 \times V_4 \times U_5 \times U_6 \times G_1 \times G_2$. On the other hand, $V_i \simeq (\mathrm{Aut}(T_i))_{t_1^i}$, $U_j \simeq (\mathrm{Aut}(\Theta_j))_{s_1^j}$ and $G_k \simeq (\mathrm{Aut}(\Upsilon_k))_{r_1^k}$.

We claim that if $\xi \in L$, $\zeta \in M$, and $\varrho \in \mathrm{Aut}(\Delta)$, then $\zeta^{-1}\xi^{-1}\varrho(x) = \varrho\zeta^{-1}\xi^{-1}(x)$. To prove, we know that $\varrho(\Theta_j) \in \{\Theta_j, \Theta_{\sigma(j)}\}$, $\varrho(\Upsilon_k) \in \{\Upsilon_k, \Upsilon_{\tau(k)}\}$ and $\varrho(T_i) \in \{T_i, T_{\sigma(i)}, T_{\tau(i)}, T_{\sigma\tau(i)}\}$. If $\xi = \zeta = I$ then our claim is clear. We have three other cases as follows:

(a) $\xi = f_1 \in L$ and $\zeta = f_2 \in M$. Then $\varrho(\Theta_j) = \Theta_{\sigma(j)}$, $\varrho(\Upsilon_k) = \Upsilon_{\tau(k)}$ and $\varrho(T_i) = T_{\sigma\tau(i)}$. It is enough to show that $f_2^{-1}f_1^{-1}\varrho(x) = \varrho(f_2^{-1}f_1^{-1}(x))$.

If $x = s_l^j \in \Theta_j$ and $\varrho(s_l^j) = s_{l'}^{j'}$, then

$$f_2^{-1}(f_1^{-1}\varrho(s_l^j)) = f_2^{-1}(s_{l'}^{\sigma(j')}) = s_{l'}^{\sigma(j')} = s_{l'}^j$$
$$\varrho(f_2^{-1}f_1^{-1}(s_l^j)) = \varrho(s_l^{\sigma(j)}) = s_{l'}^j.$$

If $x = r_l^k \in \Upsilon_k$ and $\varrho(r_l^k) = r_{l'}^{k'}$, then

$$f_2^{-1}(f_1^{-1}\varrho(r_l^k)) = f_2^{-1}(r_{l'}^{\tau(k')}) = r_{l'}^{\tau(k')} = r_{l'}^k$$
$$\varrho(f_2^{-1}f_1^{-1}(r_l^k)) = \varrho(r_l^{\tau(k)}) = r_{l'}^k.$$

If $x = t_l^i \in T_i$ and $\varrho(t_l^i) = t_{l'}^{i'}$, then

$$f_2^{-1}(f_1^{-1}\varrho(t_l^i)) = f_2^{-1}(t_{l'}^{\sigma(i')}) = t_{l'}^{\tau\sigma(i')} = t_{l'}^i$$
$$\varrho(f_2^{-1}f_1^{-1}(t_l^i)) = \varrho(t_l^{\tau\sigma(i)}) = t_{l'}^{\sigma\tau\tau\sigma(i)} = t_{l'}^i.$$

This completes the proof of this case.

(b) Suppose $\xi = f_1 \in L$ and $\zeta = I \in M$. Then $\varrho(\Theta_j) = \Theta_{\sigma(j)}$, $\varrho(\Upsilon_k) = \Upsilon_k$ and $\varrho(T_i) = T_{\sigma(i)}$. It is enough to show that $f_1^{-1}\varrho(x) = \varrho(f_1^{-1}(x))$. If $x = s_l^j \in \Theta_j$ and $\varrho(s_l^j) = s_{l'}^{j'}$, then

$$f_1^{-1}\varrho(s_l^j) = s_{l'}^{\sigma(j')} = s_{l'}^{\sigma(j')} = s_{l'}^j$$
$$\varrho(f_1^{-1}(s_l^j)) = \varrho(s_l^{\sigma(j)}) = s_{l'}^j.$$

If $x = r_l^k \in \Upsilon_k$ and $\varrho(r_l^k) = r_{l'}^{k'}$, then

$$f_1^{-1}\varrho(r_l^k) = f_1(r_{l'}^{\sigma(k)}) = r_{l'}^k$$
$$\varrho(f_1^{-1}(r_l^k)) = \varrho(r_l^{\sigma(k)}) = r_{l'}^k.$$

If $x = t_l^i \in T_i$ and $\varrho(t_l^i) = t_{l'}^{i'}$, then

$$f_1^{-1}\varrho(t_l^i) = f_1(t_{l'}^{\sigma(i')}) = t_{l'}^{\sigma(i')} = t_{l'}^i$$
$$\varrho(f_1^{-1}(t_l^i)) = \varrho(t_l^{\sigma(i)}) = t_{l'}^{\sigma\sigma(i)} = t_{l'}^i.$$

This completes the proof of this case.

(c) Suppose that $\xi = I \in L$ and $\zeta = f_2 \in M$. Then $\varrho(\Theta_j) = \Theta_j$, $\varrho(\Upsilon_k) = \Upsilon_{\tau(k)}$ and $\varrho(T_i) = T_{\tau(i)}$. If $x = s_l^j \in \Theta_j$ and $\varrho(s_l^j) = s_{l'}^{j'}$, then

$$\begin{aligned} f_2^{-1}(f_1^{-1}\varrho(s_l^j)) &= s_{l'}^j \\ \varrho(f_2^{-1}f_1^{-1}(s_l^j)) &= \varrho(s_l^j) = s_{l'}^j. \end{aligned}$$

If $x = r_l^k \in \Upsilon_k$ and $\varrho(r_l^k) = r_{l'}^{k'}$, then

$$\begin{aligned} f_2^{-1}(f_1^{-1}\varrho(r_l^k)) &= f_2^{-1}(r_{l'}^{\tau(k')}) = r_{l'}^{\tau(k')} = r_{l'}^k \\ \varrho(f_2^{-1}f_1^{-1}(r_l^k)) &= \varrho(r_l^{\tau(k)}) = r_{l'}^k. \end{aligned}$$

If $x = t_l^i \in T_i$ and $\varrho(t_l^i) = t_{l'}^{i'}$, then

$$\begin{aligned} f_2^{-1}(f_1^{-1}\varrho(t_l^i)) &= f_2^{-1}(t_{l'}^{i'}) = t_{l'}^{\tau(i')} = t_{l'}^i \\ \varrho(f_2^{-1}f_1^{-1}(t_l^i)) &= \varrho(t_l^{\tau(i)}) = t_{l'}^{\tau\tau(i)} = t_{l'}^i. \end{aligned}$$

Therefore, the conditions of this case also lead to our desired result.

Now we show that every $\varrho \in \mathrm{Aut}(\Delta)$ can be written in the form $\phi_1\phi_2\phi_3\phi_4h_5h_6g_1g_2\xi\zeta$ in such a way that $(\phi_i, g_k, h_j, \xi, \zeta) \in V_i \times G_k \times U_j \times L \times M$, where $1 \leqslant i \leqslant 4$, $k = 1, 2$, $j = 5, 6$, $\xi \in L$ and $\zeta \in M$. Define:

$$\phi_i(x) = \begin{cases} \zeta^{-1}\xi^{-1}\varrho(x) = \varrho\zeta^{-1}\xi^{-1}(x) & x = t_l^i \in T_i \setminus \{t_1^i\} \\ x & otherwise \end{cases} \in V_i,$$
$$h_j(x) = \begin{cases} \zeta^{-1}\xi^{-1}\varrho(x) = \varrho\zeta^{-1}\xi^{-1}(x) & x = s_l^j \in \Theta_j \setminus \{s_1^j\} \\ x & otherwise \end{cases} \in U_j,$$
$$g_k(x) = \begin{cases} \zeta^{-1}\xi^{-1}\varrho(x) = \varrho\zeta^{-1}\xi^{-1}(x) & x = r_l^k \in \Upsilon_k \setminus \{r_1^k\} \\ x & otherwise \end{cases} \in G_k.$$

We are ready to prove that ϕ_i is an automorphism. To prove ϕ_i is one to one, we assume that $x, x' \in V(\Delta)$ with $x \neq x'$ are arbitrary. We have to show that $\phi_i(x) \neq \phi_i(x')$. To do this, the following two cases will be considered:

(i) $x, x' \in T_i$. Since ϱ, ξ, ζ are permutations of $V(\Delta)$, $\phi_i(x) = \varrho\zeta^{-1}\xi^{-1}(x) \neq \varrho\zeta^{-1}\xi^{-1}(x') = \phi_i(x')$, as desired.

(ii) $x \in T_i, x' \notin T_i$. If $\varrho(T_i) = T_i$, then will have the case (i) and there is nothing to prove. Suppose that $\varrho(T_i) \neq T_i$. This implies that $\varrho(T_i) \in \{T_{\sigma(i)}, T_{\tau(i)}, T_{\sigma\tau(i)}\}$. If $x' \in \varrho(T_i)$, $\xi \in L$ and $\zeta \in M$ then $\phi_i(x') = x' \notin T_i$ and $\phi_i(x) = \zeta^{-1}\xi^{-1}\varrho(x) \in T_i$ and so $\phi_i(x) \neq \phi_i(x')$, as desired.

Next we prove that ϕ_i is homomorphism. To do this, we assume that u and v are adjacent in Δ. Then one of the following cases will be occurred:

1. If $u, v \in V(T_i)$, then $\phi_i(uv) = \zeta^{-1}\xi^{-1}\varrho(uv) = \zeta^{-1}\xi^{-1}\varrho(u)\zeta^{-1}\xi^{-1}\varrho(v)$ $= \phi_i(u)\phi_i(v) \in E(\Delta)$. Since ϱ, ζ and ξ are automorphism, they preserve adjacency in Δ and so ϕ_i has the same property.

2. If $v \notin T_i$ and $u \in T_i$, then $u = t^i_1$ and $\phi_i(uv) = uv = \phi_i(u)\phi_i(v) \in E(\Delta)$, as desired.

3. If $u, v \notin T_i$, then $\phi_i(uv) = uv = \phi_i(u)\phi_i(v) \in E(\Delta)$.

Next, we prove that $\phi_i^{-1}(x) = \begin{cases} \xi\zeta\varrho^{-1}(x) & x = t^i_l \in T_i/\{t^i_1\} \\ x & otherwise \end{cases}$ is also graph homomorphism. To do this, we assume that u and v are adjacent vertices in Δ. Then, one of the following three cases can be occurred:

(I) $u, v \in V(T_i)$. Since ξ, ζ and ϱ are automorphism, $\phi_i^{-1}(uv) = \xi\zeta\varrho^{-1}(uv)$ $= \xi\zeta\varrho^{-1}(u)\xi\zeta\varrho^{-1}(v) = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$, as desired.

(II) $u \in T_i$ and $v \notin T_i$. In this case, $u = t^i_1$ and $\phi_i^{-1}(uv) = uv = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$.

(IV) $u, v \notin T_i$. As similar argument as above shows that $\phi_i^{-1}(uv) = uv = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$.

To complete the proof, we note that

$$\begin{array}{rcl} \phi_1\phi_2\phi_3\phi_4h_5h_6g_1g_2\xi\zeta(t^i_k) & = & \varrho\zeta^{-1}\xi^{-1}\xi\zeta(t^i_k) = \varrho(t^i_k) \\ \phi_1\phi_2\phi_3\phi_4h_5h_6g_1g_2\xi\zeta(s^j_k) & = & \varrho\zeta^{-1}\xi^{-1}\xi\zeta(s^j_k) = \varrho(s^j_k) \\ \phi_1\phi_2\phi_3\phi_4h_5h_6g_1g_2\xi\zeta(r^l_k) & = & \varrho\zeta^{-1}\xi^{-1}\xi\zeta(r^l_k) = \varrho(r^l_k). \end{array}$$

This completes the proof. □

Define the functions $\xi_1, \xi_2 : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$\xi_1(n) = \left\{ \begin{array}{ll} \frac{n}{2} & 2 \mid n \\ \frac{n-1}{2} & 2 \nmid n \end{array} \right. \quad \text{and} \quad \xi_2(n) = \left\{ \begin{array}{ll} \frac{n}{2}+1 & 2 \mid n \\ \frac{n+3}{2} & 2 \nmid n \end{array} \right. .$$

Corollary 3.3. *Let Δ be an arbitrary bicyclic graph of the first type depicted in Figure* **??** *and $T_j^i \cong T_j^r$, for each i, j, r such that $1 \leqslant j \leqslant k$ and $1 \leqslant i, r \leqslant 4$. Then,*

$$\begin{aligned}\mathrm{Aut}(\Delta) = &(\mathrm{Aut}(T_1^1))_{a_1} \times \cdots \times (\mathrm{Aut}(T_k^1))_{a_k} \times (\mathrm{Aut}(T_1^1))_{b_1} \times \cdots \\ &\times (\mathrm{Aut}(T_k^2))_{b_k} \times (\mathrm{Aut}(\Upsilon_1))_{u_1} \times (\mathrm{Aut}(\Upsilon_2))_{u_n} \times ('(T_1^3))_{c_1} \times \cdots \\ &\times (\mathrm{Aut}(T_k^3))_{c_k} \times (\mathrm{Aut}(T_1^4))_{d_1} \times \cdots \times (\mathrm{Aut}(T_k^4))_{d_k} \times (\mathrm{Aut}(\Theta_1))_{r_1} \\ &\times (\mathrm{Aut}(\Theta_2))_{r_2} \rtimes_\phi \mathbb{Z}_2 \times \mathbb{Z}_2\end{aligned}$$

in which ϕ is a homomorphism from $\mathbb{Z}_2 \times \mathbb{Z}_2$ into $\mathcal{C}$ given by $\phi(0,0) = id$, $\phi(0,1) = \psi_1$, $\phi(1,0) = \psi_2$ and $\phi(1,1) = \psi_3$. Here, $\mathcal{C}$, ψ_1, ψ_2 and ψ_3 are defined as follows:

$$\begin{aligned}&\mathcal{C} = \mathrm{Aut}((\mathrm{Aut}(T_1^1))_{a_1} \times \cdots \times (\mathrm{Aut}(T_k^1))_{a_k} \times \cdots \times (\mathrm{Aut}(\Upsilon_1))_{u_1} \times (\mathrm{Aut}(\Upsilon_2))_{u_n}),\\ &\psi_1 = (\alpha_1, \delta_1)...(\alpha_k, \delta_k)(\beta_1, \gamma_1)...(\beta_k, \gamma_k)(\mu_1, \mu_2)(u_1, u_n), (u_2, u_{n-1}) \cdots (u_{\xi_1(n)} u_{\xi_2(n)}),\\ &\psi_2 = (\alpha_1, \beta_1) \cdots (\alpha_k, \beta_k)(\gamma_1, \delta_1) \cdots (\gamma_k, \delta_k)(\epsilon_1, \epsilon_2)(u_1, u_n)(u_2, u_{n-1}) \cdots (u_{\xi_1(n)} u_{\xi_2(n)}),\\ &\psi_3 = (\alpha_1, \gamma_1) \cdots (\alpha_k, \gamma_k)(\beta_1, \delta_1) \cdots (\delta_k, \beta_k)(\epsilon_1, \epsilon_2)(\mu_1, \mu_2)(u_1, u_n) \cdots (u_{\xi_1(n)} u_{\xi_2(n)}).\end{aligned}$$

Proof. The induced subgraph of $\cup_{i=1}^k V(T_i^j)$ is denoted by Λ^j, $1 \leqslant j \leqslant 4$. By assumption $\Lambda^1 \cong \Lambda^2 \cong \Lambda^3 \cong \Lambda^4$ and all of them satisfy the condition $(\star)$. Apply Lemma **??**, we have:

$$\begin{aligned}\mathrm{Aut}(\Delta) = &\Big[(\mathrm{Aut}(\Lambda^1)_{\{a_1,\cdots,a_k\}} \times \mathrm{Aut}(\Lambda^2)_{\{b_1,\cdots,b_k\}} \times \mathrm{Aut}(\Lambda^3)_{\{c_1,\cdots,c_k\}}\Big]\\ &\times \Big[\mathrm{Aut}(\Lambda^4)_{\{d_1,\cdots,d_k\}} \times \mathrm{Aut}(\Theta_1)_{r_1} \times \mathrm{Aut}(\Theta_2)_{r_2} \times \mathrm{Aut}(\Upsilon_1)_{u_1} \times \mathrm{Aut}(\Upsilon_2)_{u_n}\Big]\\ &\rtimes_\phi \mathbb{Z}_2 \times \mathbb{Z}_2,\end{aligned}$$

proving the result. □

Lemma 3.4. *Suppose T_1, T_2, ..., T_6 are trees such that $T_1 \cong T_2$, $T_3 \cong T_4$, $T_5 \cong T_6$, $G_1 = \mathrm{Aut}(T_1)_{a_1} \cong \mathrm{Aut}(T_2)_{a_2}$, $G_2 = \mathrm{Aut}(T_3)_{a_3} \cong \mathrm{Aut}(T_4)_{a_4}$ and $G_3 = \mathrm{Aut}(T_5)_{a_5} \cong \mathrm{Aut}(T_6)_{a_6}$, see Figure* **??**. *Then,*

$$\mathrm{Aut}(\Delta) = (G_1 \times G_2 \times G_3) \wr \mathbb{Z}_2.$$

Proof. Suppose $V(T_i) = \{t_1^i, \cdots, t_{k_i}^i\}$, $a_i = t_1^i$ and define $\sigma = (1\,2)(3\,4)(5\,6)$, $f_1 = (t_j^i\ t_j^{\sigma(i)})$, $L = \{1, f_1\}$ and $U_i = \{\alpha \in \mathrm{Aut}(\Delta) \mid \alpha(x) = x;\ x \notin T_i\ \&\ \alpha(t_1^i) = t_1^i\}$, $1 \leqslant i \leqslant 6$. Obviously, L and U_i, $1 \leqslant i \leqslant 6$, are subgroups of $\mathrm{Aut}(\Delta)$. It is easy to see that the mapping $\psi_i : \mathrm{Aut}(T_i)_{a_i} \longrightarrow V_i$

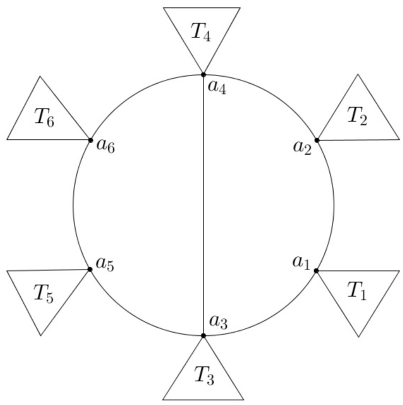


Figure 3: The bicyclic graph of Lemma ??.

given by $\psi_i(\alpha) = \alpha'$ is an isomorphism in which $\alpha'(x) = x$, when $x = a_i$ or $x \notin T_i$, and $\alpha'(x) = \alpha(x)$, otherwise. Note that for each $f \in U_i$ and $h \in U_j$, $1 \leqslant i \neq j \leqslant 6$, $fh = hf$. This implies that $U = U_1U_2 \cdots U_6$ is a subgroup of $\mathrm{Aut}(\Delta)$ and each subgroup U_i, $1 \leqslant i \leqslant 6$, is a normal subgroup of U. Since $U_i \cap U_1 \cdots U_{i-1}U_{i+1} \cdots U_6 = \{id\}$, $U_1U_2 \cdots U_6 \cong U_1 \times \cdots \times U_6$.

To complete the proof, we show that $\mathrm{Aut}(\Delta) = (U_1U_1 \cdots U_6) \cdot L$. To do this, we choose an arbitrary automorphism α in $\mathrm{Aut}(\Delta)$. Suppose $\alpha \in \mathrm{Aut}(\Delta)$ and $\xi \in L$ are arbitrary. We first show that $\alpha\xi^{-1}(x) = \xi^{-1}\alpha(x)$. If $\xi = I$ then obviously this equation is true. If $\xi = f$, then $\alpha(T_i) = T_{\sigma(i)}$. Assume that $\alpha(t_l^i) = t_{l'}^{\sigma(i)}$. It is enough to show that $\alpha f^{-1}(x) = f^{-1}\alpha(x)$. To do this, we note that $\alpha f^{-1}(t_l^i) = \alpha(t_l^{\sigma(i)}) = t_{l'}^i$ and $f^{-1}\alpha(t_l^i) = f^{-1}(t_{l'}^{\sigma(i)})$ $= t_{l'}^i$. Define:

$$\phi_i(x) = \begin{cases} \alpha\xi^{-1}(x) = \xi^{-1}\alpha(x) & x = t_k^i \in T_i \setminus \{t_1^i\} \\ x & otherwise \end{cases}.$$

We claim that ϕ_i is an automorphism of Δ. To prove ϕ_i is one to one, we assume that $x \neq x'$. We have two cases as follows:

(I) $x, x' \in T_i$. Since α and ξ are automorphism, $\alpha\xi^{-1}(x) \neq \alpha\xi^{-1}(x')$, as desired.

(II) $x \in T_i$ and $x' \notin T_i$. If $\alpha(T_i) = T_i$, then our we will have the case (I). We assume that $\alpha(T_i) \neq T_i$. Then $\alpha(T_i) = T_{\sigma(i)}$. If $x' \in \alpha(T_i)$

and $\xi \in L$, then $\phi_i(x') = x' \notin T_i$ and $\phi_i(x) = \xi^{-1}\alpha(x) \in T_i$ and so $\phi_i(x') \neq \phi_i(x)$. If $x' \notin \alpha(T_i)$, then $\phi_i(x') = x' \notin T_i$ and $\phi_i(x) = \xi^{-1}\alpha(x) \in T_i$. Again $\phi_i(x') \neq \phi_i(x)$, as desired.

We are now ready to prove that ϕ_i is homomorphism. To see this, we assume that u and v are adjacent vertices of Δ. Suppose $u, v \in V(T_i)$. Then, $\phi_i(uv) = \xi^{-1}\alpha(uv) = \xi^{-1}\alpha(u)\xi^{-1}\alpha(v)$. Since both of ξ and α are automorphism, $\phi_i(uv) = \phi_i(u)\phi_i(v) \in E(\Delta)$, as desired. If $u \in T_i$ and $v \notin T_i$, then $u = t_1^i$ and $\phi_i(uv) = uv = \phi_i(u)\phi_i(v) \in E(\Delta)$, and if $u, v \notin T_i$, then $\phi_i(uv) = uv = \phi_i(u)\phi_i(v) \in E(\Delta)$. This proves that ϕ_i is homomorphism. Next, we prove that

$$\phi_i^{-1}(x) = \begin{cases} \xi\alpha^{-1}(x) & x = t_k^i \in T_i \setminus \{t_1^i\} \\ x & otherwise \end{cases}$$

is also a homomorphism. Choose adjacent vertices $u, v \in V(\Delta)$. Suppose $u, v \in V(T_i)$. Then, $\phi_i^{-1}(uv) = \xi\alpha^{-1}(uv) = \xi\alpha^{-1}(u)\xi\alpha^{-1}(v)$ and since both of ξ and α are automorphism, $\phi_i^{-1}(uv) = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$. We now assume that $u \in T_i$ and $v \notin T_i$. Then, $u = t_1^i$ and we have $\phi_i^{-1}(uv) = uv = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$. If $u, v \notin T_i$, then $\phi_i^{-1}(uv) = uv = \phi_i^{-1}(u)\phi_i^{-1}(v) \in E(\Delta)$.

Hence $\mathrm{Aut}(\Delta) = U_1U_2\cdots U_6 \cdot L$. Since $|L| = 2$ and $U_1U_2\cdots U_6 \cong U_1 \times U_2 \times \cdots \times U_6$, $\mathrm{Aut}(\Delta) = (U_1 \times U_2 \times \cdots \times U_6) \rtimes L \cong (G_1 \times G_2 \times G_3) \rtimes Z_2$, proving the lemma. □

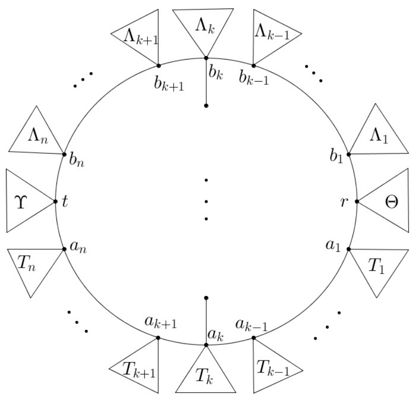


Figure 4: The graph $\overline{\mho}$ in Corollary ??.

Corollary 3.5. *Suppose* $G_i = (\mathrm{Aut}(T_i))_{a_i} = (\mathrm{Aut}(\Lambda_i))_{b_i}$, $1 \leqslant i \leqslant n$, $H = (\mathrm{Aut}(\Upsilon))_t$ *and* $K = (\mathrm{Aut}(\Theta))_r$, *where* t *and* r *are shown in the graph* $\overline{\mho}$ *depicted in Figure* **??**. *Then,* $\mathrm{Aut}(\Delta) = H \times K \times (G_1 \times G_2 \times \cdots \times G_n) \wr \mathbb{Z}_2$.

Lemma 3.6. *Suppose* $G_i = (\mathrm{Aut}(T_i))_{a_i} = (\mathrm{Aut}(\Lambda_i))_{b_i}$, $H = (\mathrm{Aut}(\Upsilon))_t$ *and* $K = (\mathrm{Aut}(\Theta))_r$, *see Figure* **??**. *Then,* $\mathrm{Aut}(\Delta) = H \times K \times (G_1 \times G_2 \times G_3) \wr \mathbb{Z}_2$.

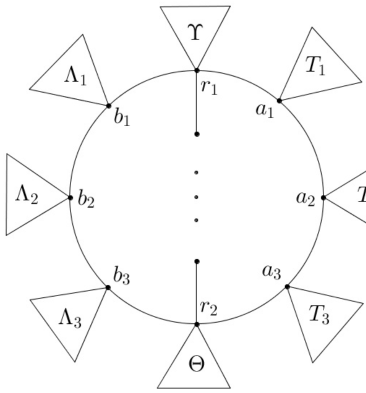


Figure 5: A figure for Lemma **??**.

Proof. The proof is similar to the proof of Lemma **??** and so we omit it. □

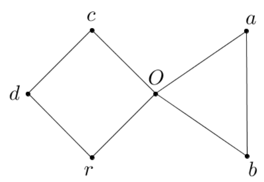
Note that if a given tree T has a central vertex v then for each automorphism $\alpha \in \mathrm{Aut}(T)$, $\alpha(v) = v$. For other type of trees, we will have the following definition.

Definition 3.7. Suppose $G = \mathrm{Aut}(T) \in \mathcal{T}$, $Fix(G) = \emptyset$ and u, v are central vertices of T. It is well-known that $uv \in E(T)$. Add the vertex u_T in the middle of uv, join vertices u, v with u_T and add another vertex v_T together with the edge $u_T v_T$ to construct a new tree $\overline{T}$.

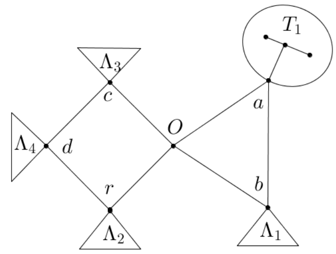
Remark 3.8. By Definition **??**, $V(\overline{T}) = V(T) \cup \{u_T, v_T\}$ and $E(\overline{T}) = (E(T) \setminus \{uv\}) \cup \{uu_T, u_T v, u_T v_T\}$. Also, it is easy to see that $\mathrm{Aut}(T) \cong \mathrm{Aut}(\overline{T})$.

Theorem 3.9. *Every member of* $\mathcal{S}$ *is isomorphic to the automorphism group of a bicyclic graph.*

Proof. Suppose W is an arbitrary element of $\mathcal{S} = \mathcal{T} \cup \mathcal{B}_1 \cup \mathcal{B}_2$. We first assume that $W \in \mathcal{T}$ and write W=Aut(T_1), where T_1 is a tree. There are two different cases that $Fix(T_1) \neq \emptyset$ or $Fix(T_1) = \emptyset$.



The graph $\mho_1$.



The graph Ω_1.

Figure 6: Two graphs presented in the proof of Theorem **??**.

1. $Fix(T_1) \neq \emptyset$. Suppose $v \in Fix(T_1)$. In the graph $\mho_1$ in Figure **??**, we choose the cycles C_4 and C_5 containing a common vertex O together with four non-isomorphic asymmetric trees Λ_1, Λ_2, Λ_3 and Λ_4 containing vertices v_1, v_2, v_3 and v_4, respectively. We also assume that $\Lambda_i \ncong T_1$, for each natural number i in $\{1, 2, 3, 4\}$. Unify mutually the vertices $(v, a), (v_1, b), (v_2, r), (v_3, c)$ and (v_4, d) to construct a bicyclic graph Ω_1. Then $\mathrm{Aut}(\Omega_1) \cong (\mathrm{Aut}(T_1))_v \cong W_v = W$.

2. $Fix(T_1) = \emptyset$. In this case the center of T_1 is containing a unique edge ℓ. Add a vertex into the edge ℓ and connect it to the vertex a, see Figure **??**. In a similar way as in Case (1), we choose asymmetric trees Λ_1, Λ_2, Λ_3 and Λ_4 and form a graph Ω_1 as in depicted in the Figure **??**. From this figure one can be easily seen that $\mathrm{Aut}(\Omega_1) = \mathrm{Aut}(T_1) = W$.

Next we assume that $W \in \mathcal{B}_1$. Then there are two trees T_2 and Ψ_1 such that $W \cong C_1 \times [D_2 \wr (\mathbb{Z}_2 \times \mathbb{Z}_2)]$, where $C_1 = \mathrm{Aut}(\Psi_1)$ and $D_2 = \mathrm{Aut}(T_2)$. Define the graph Ω_2 as follows:

(I) $Fix(T_2) \neq \emptyset$ *and* $Fix(\Psi_1) \neq \emptyset$. Choose $w_{T_2} \in Fix(T_2)$ and $w_{\Psi_1} \in Fix(\Psi_1)$. Suppose ${T_2}^1$, ${T_2}^2$, ${T_2}^3$, ${T_2}^4$, ${T_2}^5$, ${T_2}^6$, ${T_2}^7$ and ${T_2}^8$ are eight isomorphic copies of T_2 in which the image of $w_{T_2} \in V(T_2)$ under these isomorphisms are $w_{T_2^1}$, $w_{T_2^2}$, $w_{T_2^3}$, $w_{T_2^4}$, $w_{T_2^5}$, $w_{T_2^6}$, $w_{T_2^7}$ and $w_{T_2^8}$, respectively. Define $\Omega_2 = S(\mho_2,\, T_2^1,\, T_2^2, T_2^3, T_2^4,\, T_2^5, T_2^6,\, T_2^7, T_2^8, \Psi_1; a_1, w_{T_2^1}; a_2, w_{T_2^2}; a_3, w_{T_2^3}; a_4, w_{T_4^4}; b_1, w_{T_2^5}; b_2, w_{T_2^6}; b_3, w_{T_2^7}; b_4, w_{T_4^8}; c, w_{\Psi_1})$.

(II) $Fix(T_2) = \emptyset$ *and* $Fix(\Psi_1) \neq \emptyset$. Consider the vertex $w_{\Psi_1} \in Fix(\Psi_1)$ and the tree $\overline{T_2}$ in Definition **??**. We also assume that $\overline{T_2}^1$, $\overline{T_2}^2$, $\overline{T_2}^3$, $\overline{T_2}^4$, $\overline{T_2}^5$, $\overline{T_2}^6$, $\overline{T_2}^7$ and $\overline{T_2}^8$ are eight isomorphic copies of $\overline{T_2}$ and $v_{T_2^1}$, $v_{T_2^2}$, $v_{T_2^3}$, $v_{T_2^4}$, $v_{T_2^5}$, $v_{T_2^6}$, $v_{T_2^7}$ and $v_{T_2^8}$ are the image of v_{T_2} under isomorphisms between $\overline{T_2}$ and trees $\overline{T_2}^1$, $\overline{T_2}^2$, $\overline{T_2}^3$, $\overline{T_2}^4$, $\overline{T_2}^5$, $\overline{T_2}^6$, $\overline{T_2}^7$ and $\overline{T_2}^8$, respectively. Define $\Omega_2 = S(\mho_2, \overline{T}^1_2, \overline{T}^2_2, \overline{T}^3_2, \overline{T}^4_2, \overline{T}^5_2, \overline{T}^6_2, \overline{T}^7_2, \overline{T}^8_2, \Psi_1; a_1, v_{T_2^1}; a_2, v_{T_2^2}; a_3, v_{T_2^3}; a_4, v_{T_4^4}; b_1, v_{T_2^5}; b_2, v_{T_2^6}; b_3, v_{T_2^7}; b_4, v_{T_4^8}; c, w_{\Psi_1})$.

(III) $Fix(D_2) \neq \emptyset$ *and* $Fix(C_1) = \emptyset$. Suppose $w_{T_2} \in Fix(D_2)$, $\overline{\Psi}_1$ and v_{Ψ_1} are those defined in Definition **??**. Define $\Omega_2 = S(\mho_2, T_2^1, T_2^2, T_2^3, T_2^4, T_2^5, T_2^6, T_2^7, T_2^8, \overline{\Psi}_1; a_1, w_{T_2^1}; a_2, w_{T_2^2}; a_3, w_{T_2^3}; a_4, w_{T_4^4}; b_1, w_{T_2^5}; b_2, w_{T_2^6}; b_3, w_{T_2^7}; b_4, w_{T_4^8}; c, v_{\Psi_1})$.

(IV) $Fix(D_2) = \emptyset$ *and* $Fix(C_1) = \emptyset$. Consider the tree $\overline{T_2}$ as in Definition **??** together with eight isomorphic copies of this tree named as $\overline{T_2}^1$, $\overline{T_2}^2$, $\overline{T_2}^3$, $\overline{T_2}^4$, $\overline{T_2}^5$, $\overline{T_2}^6$, $\overline{T_2}^7$ and $\overline{T_2}^8$. Furthermore, we choose the vertices $v_{T_2^1}$, $v_{T_2^2}$, $v_{T_2^3}$, $v_{T_2^4}$, $v_{T_2^5}$, $v_{T_2^6}$, $v_{T_2^7}$ and $v_{T_2^8}$ to be the image of v_{T_2} under appropriate isomorphism between $\overline{T_2}$ and $\overline{T_2}^1$, $\overline{T_2}^2$, $\overline{T_2}^3$, $\overline{T_2}^4$, $\overline{T_2}^5$, $\overline{T_2}^6$, $\overline{T_2}^7$ and $\overline{T_2}^8$, respectively. We also assume that the tree $\overline{\Psi}_1$ and the vertex v_{Ψ_1} are according to Definition **??**. Define $\Omega_2 = S(\mho_2, \overline{T}^1_2, \overline{T}^2_2, \overline{T}^3_2, \overline{T}^4_2, \overline{T}^5_2, \overline{T}^6_2, \overline{T}^7_2, \overline{T}^8_2, \overline{\Psi}_1; a_1, v_{T_2^1}; a_2, v_{T_2^2}; a_3, v_{T_2^3}; a_4, v_{T_4^4}; b_1, v_{T_2^5}; b_2, v_{T_2^6}; b_3, v_{T_2^7}; b_4, v_{T_4^8}; c, v_{\Psi_1})$.

In all cases, $\mathrm{Aut}(\Omega_2) = C_1 \times (D_2 \wr (\mathbb{Z}_2 \times \mathbb{Z}_2))$, as desired.

Finally, we assume that $W \in \mathcal{B}_2$. Set $W = C_2 \times (D_3 \times D_3 \times D_3 \times D_3 \times H \times H \times K \times K \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2))$. Hence, there are trees T_3, Λ, Υ and Ψ_2 such that $D_3 = \mathrm{Aut}(T_3)$, $H = \mathrm{Aut}(\Lambda)$, $K = \mathrm{Aut}(\Upsilon)$ and $C_2 = \mathrm{Aut}(\Psi_2)$. Set $FE = \{Fix(D_3), Fix(H), Fix(K), Fix(C_2)\}$. There are sixteen cases that for which one, two, three or all elements of FE are empty.

If $Fix(D_3) = \emptyset$, then we apply Definition **??** to construct four copies $\overline{T_3}^1$, $\overline{T_3}^2$, $\overline{T_3}^3$ and $\overline{T_3}^4$ of $\overline{T}_3$ together with four vertices $v_{T_3^1} \in V(\overline{T_3}^1)$, $v_{T_3^2} \in V(\overline{T_3}^2)$, $v_{T_3^3} \in V(\overline{T_3}^3)$ and $v_{T_3^4} \in V(\overline{T_3}^4)$ in which these four vertices are images of the vertex v_{T_3} under some appropriate isomorphisms from $\overline{T}_3$ onto $\overline{T_3}^1$, $\overline{T_3}^2$, $\overline{T_3}^3$ and $\overline{T_3}^4$, respectively. If $Fix(D_3) \neq \emptyset$, then we will consider four copies T_3^1, T_3^2, T_3^3 and T_3^4 of T_3 together with four vertices $w_{T_3^1} \in V({T_3}^1)$, $w_{T_3^2} \in V(T_3^2)$, $w_{T_3^3} \in V({T_3}^3)$ and $w_{T_3^4} \in V(T_3^4)$ in which these four vertices are images of the vertex $w_{T_3} \in Fix(D_3)$ under some appropriate isomorphisms from T_3 onto ${T_3}^1$, ${T_3}^2$, ${T_3}^3$ and ${T_3}^4$, respectively.

If $Fix(H) = \emptyset$, then we construct two copies $\overline{\Theta}^1$ and $\overline{\Theta}^2$ of $\overline{\Theta}$ together with two vertices $v_{\Theta^1} \in V(\overline{\Theta}^1)$, $v_{\Theta^2} \in V(\overline{\Theta}^2)$ in which these two vertices are images of the vertex $v_\Theta \in V(\overline{\Theta})$ under some appropriate isomorphisms from $\overline{\Theta}$ onto $\overline{\Theta}^1$ and $\overline{\Theta}^2$, respectively. If $Fix(H) \neq \emptyset$, then we consider two copies Θ^1 and Θ^2 of Θ together with two vertices $w_{\Theta^1} \in V(\Theta^1)$, $w_{\Theta^2} \in V(\Theta^2)$ in which these vertices are images of the vertex $w_\Theta \in Fix(H)$ under some appropriate isomorphisms from Θ onto Θ^1 and Θ^2, respectively.

If $Fix(K) = \emptyset$, then we construct two copies $\overline{\Upsilon}^1$ and $\overline{\Upsilon}^2$ of $\overline{\Upsilon}$ together with two vertices $v_{\Upsilon^1} \in V(\overline{\Upsilon}^1)$, $v_{\Upsilon^2} \in V(\overline{\Upsilon}^2)$ in which these two vertices are images of the vertex $v_\Upsilon \in V(\overline{\Upsilon})$ under some appropriate isomorphisms from $\overline{\Upsilon}$ onto $\overline{\Upsilon}^1$ and $\overline{\Upsilon}^2$, respectively. If $Fix(K) \neq \emptyset$, then we consider two copies Υ^1 and Υ^2 of Υ together with two vertices $w_{\Upsilon^1} \in V(\Upsilon^1)$, $w_{\Upsilon^2} \in V(\Upsilon^2)$ in which these vertices are images of the vertex $w_\Upsilon \in Fix(K)$ under some appropriate isomorphisms from Υ onto Υ^1 and Υ^2, respectively.

If $Fix(C_2) = \emptyset$ then we consider the vertex v_{Ψ_2} by Definition **??**, and if $Fix(C_2) \neq \emptyset$ then we choose $w_{\Psi_2} \in Fix(C_2)$. Next we define Set $\Omega_3 = S(\mho_2, \widehat{T_3^1}, \widehat{T_3^2}, \widehat{T_3^2}, \widehat{T_3^4}, \widehat{\Theta^1}, \widehat{\Theta^2}, \widehat{\Upsilon^1}, \widehat{\Upsilon^2}, \widehat{\Psi_2}; a_1, e_{T_3^1}; a_2, e_{T_3^2}; a_3, e_{T_3^3}; a_4, e_{T_3^4}; b_1, e_{\Theta^1}; b_2, e_{\Theta^2}; b_3, e_{\Upsilon^1}; b_4, e_{\Upsilon^2}; c, e_{\Psi_2})$. Here, for each tree L,

$$\widehat{L} = \begin{cases} \overline{L} & Fix(L) = \emptyset \\ L & Fix(L) \neq \emptyset \end{cases} \text{ and } e_L = \begin{cases} v_L & Fix(L) = \emptyset \\ w_L & Fix(L) \neq \emptyset \end{cases}.$$

By our construction, $\mathrm{Aut}(\Omega_3) = C_2 \times (D_3 \times D_3 \times D_3 \times D_3 \times H \times H \times K \times K \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2))$ which completes the proof. □

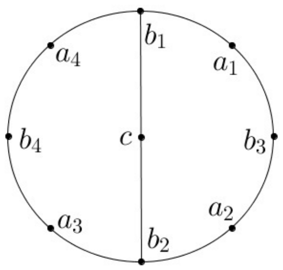


Figure 7: The graph $\mho_2$.

Theorem 3.10. *The automorphism group of every bicyclic graphs is a member of $\mathcal{S}$.*

Proof. There are three different types of bicyclic graphs as follows:

(I) *There are two cycles in the graph with some common edges.* The result follows from Lemmas **??**, **??**, **??** and Corollaries **??**, **??**.

(II) *There are two cycles with a common vertex.* Suppose these two cycles have lengths m and n, respectively. There are two different cases that the length of two cycles are equal or they have different lengths. We first assume that two cycles have the same length, i.e. $m = n$. From Figure **??**, one can see that $m = \gamma + \gamma' + 2$ and $n = \delta + \delta' + 2$. There are five separate cases for the case that $m = n$ as follows:

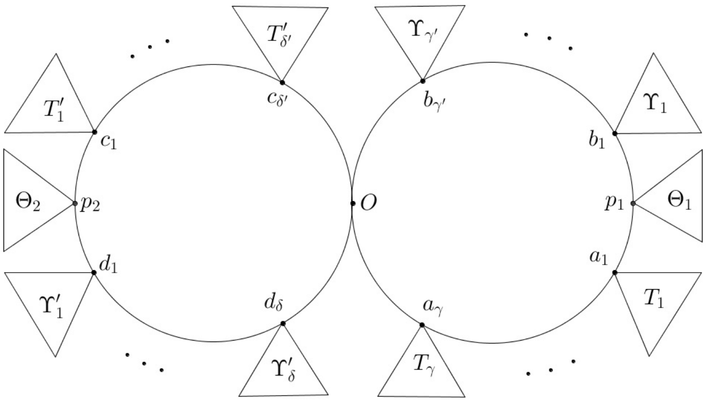


Figure 8: The general case of a bicyclic graph when the cycles have a common vertex.

(M1) *We have two cycles without trees attached to the vertices.* Suppose Δ is a bicyclic graph constructed from two cycles with a common vertex such that all vertices other than the common vertex have degree 2, see Figure **??** for details. Therefore, $\mathrm{Aut}(\Delta) = \mathbb{Z}_2 \wr \mathbb{Z}_2 \in \mathcal{T} \subseteq \mathcal{S}$.

(M2) $(\mathrm{Aut}(T_i))_{a_i} \cong (\mathrm{Aut}(T'_{i'}))_{c_{i'}} \cong (\mathrm{Aut}(\Upsilon_j))_{b_j} \cong (\mathrm{Aut}(\Upsilon'_{j'}))_{d_{j'}} \cong (\mathrm{Aut}(\Theta_1)_{p_1} \cong (\mathrm{Aut}(\Theta_2)_{p_2} \cong G$, *where* $1 \leqslant i \leqslant \gamma$, $1 \leqslant i' \leqslant \gamma'$, $1 \leqslant j \leqslant \delta'$ *and* $1 \leqslant j' \leqslant \delta$. We consider the bicyclic graph Δ

in such a way that there are isomorphic rooted trees (T_1, a_1), (T_γ, a_γ), (Υ_1, b_1), $(\Upsilon_{\gamma'}, b_{\gamma'})$, (T'_1, c_1), $(T'_{\delta'}, c_{\delta'})$, (Υ'_1, d_1), $(\Upsilon'_\delta, d_\delta)$, (Θ_1, p_1), (Θ_2, p_2) attached to non-common vertices of two cycles which satisfy the condition $(\star)$, see Figure **??**. Then $\mathrm{Aut}(\Delta) \cong G \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2) \in \mathcal{T} \subseteq \mathcal{S}$.

(M3) Consider the bicyclic graph Δ with this property that $\gamma = \gamma' = \delta = \delta'$, $\mathrm{Aut}(T'_i)_{c_i} \cong \mathrm{Aut}(\Upsilon_i)_{b_i} \cong \mathrm{Aut}(\Upsilon'_i)_{d_i} \cong \mathrm{Aut}(T_i)_{a_i}$ and $\mathrm{Aut}(\Theta_1)_{p_1} \cong \mathrm{Aut}(\Theta_2)_{p_2}$. Define $G_i = (\mathrm{Aut}(T_i))_{a_i}$ and $H_i = (\mathrm{Aut}(\Theta_i))_{p_i}$. Then, $\mathrm{Aut}(\Delta) = (H_1 \times (G_1 \times \cdots G_\gamma) \wr \mathbb{Z}_2) \wr \mathbb{Z}_2 \in \mathcal{T} \in \mathcal{S}$.

(M4) Consider the graph Δ in such a way that $\delta' = \gamma'$ and $\delta = \gamma$. Moreover, we assume that $\Upsilon_i \cong T'_i$, $1 \leqslant i \leqslant \delta'$, and they satisfy the condition $(*)$, $T_j \cong \Upsilon'_j$, $1 \leqslant j \leqslant \delta$, and again these graphs satisfy the condition $(*)$. By Figure **??**, $\Upsilon_i \cong T'_i$, $T_i \cong \Upsilon'_i$ and $\Theta_1 \cong \Theta_2$. Set $K_i = (\mathrm{Aut}(\Upsilon_i))_{b_i}$. Then, $\mathrm{Aut}(\Delta) \cong (H_1 \times G_1 \times \cdots \times G_\gamma \times K_1 \times \cdots \times K_{\gamma'}) \wr \mathbb{Z}_2 \in \mathcal{S}$.

(M5) In this case, the general case of $(M1) - (M4)$ is considered into account in which we don't have isomorphisms between trees. Set $K_i = (\mathrm{Aut}(\Upsilon_i))_{b_i}$, $G_i = (\mathrm{Aut}(T_i))_{a_i}$, $G'_i = (\mathrm{Aut}(T'_i))_{c_i}$ and $K'_i = (\mathrm{Aut}(\Upsilon'_i)_{d_i}$. Then, $\mathrm{Aut}(\Delta) \cong H_1 \times H_2 \times K_1 \times \cdots K_{\gamma'} \times K'_1 \times \cdots \times K'_\gamma \times G_1 \times \cdots \times G_\gamma \times G'_1 \times \cdots \times G'_{\delta'}$.

If two cycles have different lengths then we will have three cases as follows:

(M6) If there is no tree T such that T is attached to a vertex of Δ, then $\mathrm{Aut}(\Delta) = \mathbb{Z}_2 \times \mathbb{Z}_2 \in \mathcal{S}$.

(M7) Suppose the graph Δ has this property that $\gamma = \gamma'$ and $\delta = \delta'$. Furthermore, we assume that for each i, $1 \leqslant i \leqslant \gamma$, $\Upsilon_i \cong T_i$ satisfy the condition $(*)$ and for each j, $1 \leqslant j \leqslant \delta$, $\Upsilon'_j \cong T'_j$ satisfy again $(*)$. Therefore, $\mathrm{Aut}(\Delta) \cong (G_1 \times \cdots \times G_\gamma \times G'_1 \cdots \times G'_\delta) \wr \mathbb{Z}_2 \times H_1 \times H_2 \in \mathcal{S}$.

(M8) Suppose that $\gamma = \gamma'$ and for each i, $1 \leqslant i \leqslant \gamma$, Υ_i, T_i are isomorphic and satisfy the condition $(*)$. Then it can be proved that $\mathrm{Aut}(\Delta) \cong (G_1 \times \cdots \times G_\gamma) \wr \mathbb{Z}_2 \times H_1 \times H_2 \times G'_1 \cdots \times G'_\delta \times K'_1 \cdots \times K'_\delta \in \mathcal{S}$.

(III) *Two cycles of the graph is connected to each other by a path.* In this case, there are three cases for the bicyclic graph Δ and its general form is depicted in Figure **??**. Suppose F_i^j =Aut$(\Omega_i^j)_{v_i^j}$ and E_l =Aut$(\Xi_l)_{a_l}$.

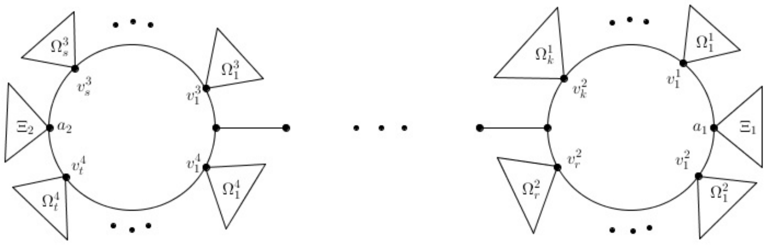


Figure 9: A figure for the proof of Theorem **??** (Case III).

(N1) *In the graph Δ, $k = r = s = t$ and for each i, Ω_i^1, Ω_i^2, Ω_i^3 and Ω_i^4 are isomorphic and satisfy the condition $(\star)$, Figure **??**. Moreover, Ξ_1 and Ξ_2 are isomorphic and satisfy again the condition* $(\star)$. By Figure **??**, one can see that $\mathrm{Aut}(\Delta) \cong (E_1 \times (F_1^1 \times \cdots \times F_k^1) \wr \mathbb{Z}_2) \wr \mathbb{Z}_2 \in \mathcal{S}$, as desired.

(N2) *In the graph Δ, $k = r$ and $s = t$ and for each i, both Ω_i^1, Ω_i^2 and Ω_i^3, Ω_i^4 are mutually isomorphic and satisfy condition $(\star)$.* In this case, by Figure **??** one can be easily seen that $\mathrm{Aut}(B) \cong E_1 \times E_2 \times (F_1^1 \times \cdots F_k^1) \wr \mathbb{Z}_2 \times (F_1^3 \times \cdots F_s^3) \wr \mathbb{Z}_2 \in \mathcal{S}$, that is our claim.

(N3) *In the graph Δ, $k = r = s = t$ and for each i, all pairs Ω_i^1, Ω_i^3; Ω_i^2, Ω_i^4 and Ξ_1, Ξ_2 are mutually isomorphic and all of them satisfy the condition $(\star)$.* Again we use the Figure **??** to prove that $\mathrm{Aut}(\Delta) \cong (E_1 \times F_1^1 \times \cdots \times F_k^1 \times F_1^2 \times \cdots \times F_r^2) \wr \mathbb{Z}_2 \in \mathcal{S}$.

Hence the result. □

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References

[1] **L. Babai**, *Automorphism groups, isomorphism, reconstruction*, Handbook of Combinatorics, Vol. 1, 2, 1447 – 1540, Elsevier Sci. B. V., Amsterdam, 1995.

[2] **L. Babai**, *Automorphism groups of graphs and edge-contraction*, Discrete Math. **306** (2006), 918 – 922.

[3] **T. Došlić**, *Splices, links and their degree-weighted Wiener polynomials*, Graph Theory Notes N. Y. **48** (2005), 47 – 55.

[4] **C. Godsil and G. Royle**, *Algebraic Graph Theory*, Graduate Texts in Mathematics **207**, Springer-Verlag, New York, 2001.

[5] **F. Harary**, *Graph Theory*, Addison-Wesley, Reading, MA, 1994.

[6] **C. Jordan**, *Sur les assemblages de lignes*, (French), J. Reine Angew. Math. **70** (1869) 185 – 190.

[7] **P. Klavík, R. Nedela and P. Zeman**, *Jordan-like characterization of automorphism groups of planar graphs*, J. Comb. Theory Ser. B **157** (2022) 1 – 39.

[8] **P. Klavík and P. Zeman**, *Automorphism groups of geometrically represented graphs*, 32nd Intern. Symp. Theor. Aspects Computer Sci., 540 – 553, LIPIcs. Leibniz Int. Proc. Inform., 30, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015.

[9] **J.J. Rotman**, *An Introduction to the Theory of Groups*, Fourth edition. Graduate Texts in Mathematics, 148. Springer-Verlag, New York, 1995.

[10] **The GAP Team**, *Gap – Groups, Algorithms, and Programming*, version 4.4. http://www.gap-system.org, 2006.

[11] **Wolfram Research Inc.**, *Mathematica*, Version 10.0, Wolfram Research, Inc., Champaign, Illinois, 2014.

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Faculty of Mathematical Sciences, Department of Pure Mathematics,
University of Kashan, Kashan 87317-53153 , I. R. Iran