Quasigroups and Related Systems **31** (2023), 135 – 152

https://doi.org/10.56415/qrs.v31.10

Some types of interior filters in quasi-ordered semigroups

Daniel Abraham Romano

Abstract. In this paper, we introduce the notions of interior filters, quasi-interior filters and weak-interior filters in a quasi-ordered semigroup. Additionally, we study the properties of these types of filters of quasi-ordered semigroups and their interrelationships.

1. Introduction

Semigroups and their substructures are important algebraic structures. In 1987, Kehayopulu in paper [2] introduced the concept of filters in a poe-semigroup. Later on, Kehayopulu ([3]) gave the characterization of the filter of a semigroup in term of the prime ideals. In paper [7], Lee and Lee give the characterization of a left (right) filter of po-semigroups. In addition to standard filters, some non-standard filters of ordered semigroups can be found in the literature. The concept of bi-filters of ordered semigroups was introduced by Y. Cao in [1]. Khan and Mahboob in [5] introduced the concepts of left-m-filters, right-n-filters and (m, n)-filters on ordered semigroup.

In this paper, as a further generalization of filters in a semigroup ordered under a quasi-order relation, we introduce the notions of: interior filters, (left, right) quasi-interior filters and (left, right) weak-interior filters. Additionally, we study the properties of these types of filters and their interrelationships.

2010 Mathematics Subject Classification: 06F05, 20M12.
Keywords: semigroup, quasi-ordered semigroup, (left, right) filter, interior filter, (left, right) weak-interior filter, (left, right) quasi-interior filter.

2. Preliminaries: Filters of quasi-ordered semigroup

Let S be a non-empty set. Total function $w : S \times S \longrightarrow S$ is an internal binary operation on S. This means

(1) $(\forall x, y, u, v \in S)\ ((x = u \wedge y = v) \implies w(x, y) = w(u, v))$.

Assume that the mapping w satisfies the condition

(2) $(\forall x, y, z \in S)\ (w(x, w(y, z)) = w(w(x, y), z))$.

The pair (S, w), where S is a set and w an internal binary operation in S, is called a semigroup. In what follows we will write $x \cdot y$, or xy for short, instead of $w(x, y)$. Therefore, (2) has a form

(3) $(\forall x, y, z \in S)\ (x(yz) = (xy)z)$.

Let $(S, \cdot)$ be a semigroup. If A and B are subsets of the semigroup S, as usual, we write $AB =: \{ab : a \in A \wedge b \in B\}$.

A relation $\preceq$ on S is a quasi-order on S if holds

(4) $(\forall x \in S)\ (x \preceq x)$,

(5) $(\forall x, y, z \in S)\ ((x \preceq y \wedge y \preceq z) \implies x \preceq z)$,

(6) $(\forall x, y, u \in S)\ (x \preceq y \implies (xu \preceq yu \wedge ux \preceq uy))$.

A quasi-order $\preceq$ on a semigroup S is an order on S if

(7) $(\forall x, y \in S)\ ((x \preceq y \wedge y \preceq x) \implies x = y)$.

If a semigroup is ordered by a quasi-order relation (by an order relation), then it is said to be a quasi-ordered semigroup (res. ordered semigroup). We will often, in both cases, write 'ordered semigroup' if there is no confusion.

Example 2.1. Let $S = \{a, b, c, d\}$ be a set. The multiplication is given by

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and a quasi-order is given as $\preceq = \{(a, a), (a, b), (b, b), (c, c), (d, d)\}$. Thus, S is a semigroup ordered under $\preceq$.

Example 2.2. The set $M_{2\times 2}$ of all matrices of size 2×2 of the field of real numbers is a semigroup ordered by the relation '$\leqslant$' defined as follows

$$(\forall A, B \in M_{2\times 2})(A \leqslant B \iff (\forall i, j \in \{1, 2\})(A_{ij} \leqslant B_{ij})).$$

A subset A od a semigroup S is a subsemigroup of S if

(11) $(\forall x, y \in S)\ ((x \in A \ \wedge\ y \in A) \Longrightarrow\ xy \in A)$.

The condition (11) can be written in the form $AA \subseteq A$. A non-empty subset J of a semigroup S is a left (right) ideal of S if holds $SJ \subseteq J$ (res. $JS \subseteq J$). A non-empty subset J is an ideal of a semigroup S if it is a left and right ideal of S. Let us note, that a (left, right) ideal of a semigroup S is a subsemigroup in S. A (left, right) ideal of a quasi-ordered semigroup S is a (left, right) ideal of semigroup S ordered under a quasi-order if

(8) $(\forall x, y \in S)\ ((y \in J \ \wedge\ x \preceq y) \Longrightarrow\ x \in J)$.

For a subset A of a quasi-ordered semigroup S we write

$$(A] =: \{x \in S : (\exists a \in A)(x \preceq a)\}.$$

Therefore, a subset J of a quasi-ordered semigroup S is a (left, right) ideal of S if holds

(9) $J \neq \emptyset$,

(11) $JJ \subseteq J$,

(10) $SJ \subseteq J$ (for left), $JS \subseteq J$ (for right), and

(8) $(J] \subseteq J$.

It should be noted here that, in this case, (11) is a consequence of (10). Indeed, if $JS \subseteq J$ is valid, then we have $JJ \subseteq JS \subseteq J$. So, $SJ \subseteq J \Longrightarrow$ (11). Also, we have $SJ \subseteq J \Longrightarrow JJ \subseteq J$.

A subset T of a semigroup S is called *prime* if $AB \subseteq T \Longrightarrow (A \subseteq T \ \vee\ B \subseteq T)$ for subsets A, B of S.

A subsemigroup F of a semigroup S ordered under a quasi-order $\preceq$ is a *left filter* of S if holds

(12) $(\forall x, y \in S)\ (xy \in F \Longrightarrow\ y \in F)$ and

(13) $(\forall x, y \in S)\ ((x \in F \ \wedge\ x \preceq y), \Longrightarrow\ y \in F)$;

a *right filter* of S if holds (13) and

(14) $(\forall x, y \in S)\ (xy \in F \Longrightarrow\ x \in F)$.

A subsemigroup F that is a left and right filter is called a *filter*.

As already mentioned, F is said to be a prime subset in S if

(15) $(\forall x, y \in S)\ (xy \in F \Longrightarrow\ (x \in F \ \vee\ y \in F))$.

If F is a filter of a quasi-ordered semigroup S, then it satisfies the condition (15) too.

Lemma 2.3 ([3, 7])**.** *Let F be a nonempty subset of an ordered semigroup S. Then F is a* (*left, right*) *filter of S if and only if $F = S$ or $S \backslash F$ is a prime* (*left, right*) *ideal of S.*

We say that the substructure F of S satisfying the conditions (12) and (13), ((14) and (13), res.) is a *generalized left* (*right*) *filter*. If a generalized (left, right) filter F satisfies the condition (15), then we say that F is a *prime generalized* (*left, right*) *filter* of S. It should be noted here that a (left, right) filter F of a quasi-ordered semigroups S is a generalized (left, right) filter of S and F is a prime subset of S.

According to [1] a subset F of a semigroup S is a *bi-filter* if F satisfies (13) and

(16) $(\forall x, y \in S)\ (xyx \in F \implies x \in F)$.

A subsemigroup F of an ordered semigroup S is called a *left-m-filter* (resp. *right-n-filter*) if F satisfies (13) and (17) (resp. (13) and (18), where

(17) $(\forall a, b \in S)\ (ab \in F \implies a^m \in F)$

(18) $(\forall a, b \in S)\ (ab \in F \implies b^n \in S)$.

A subsemigroup F is an *(m, n)-filter* if it is a left-m-filter and a right-n-filter.

Some examples of left-m-filter (right-n-filter) can be found in [5].

Remark 2.4 ([5], Remark 1)**.** In particular for $m = 1$ (resp. $n = 1$), F is a left filter (resp. right filter). Clearly each left filter (resp. right filter, filter) of an ordered semigroup S is a left-m-filter (resp. a right-n-filter, an (m, n)-filter) for each positive integers m and n. Indeed, for any left filter F of S and $a, b \in S$ such that $ab \in F$, as F is a left filter, $a \in F$. Therefore $a^m \in F$. Thus, the concept of a left-m-filter (resp. right-n-filter, (m, n)-filter) is a generalization of the concept of a left filter (resp. right filter, filter). Conversely a left-m-filter (resp. right-n-filter, an (m, n)-filter) need not always be a left filter (resp. right filter, filter).

3. Interior filters

Lajos [6] defined the concept of an interior ideal in a semigroup. Interior ideal in a semigroup was studied by Szasz [10, 11]: A non-empty subset J of a semigroup S is an *interior ideal* in S if it is a subsemigroup of S and holds $SJS \subseteq J$. This means that J satisfies (9), (11) and

(19) $(\forall x, u, v \in S)\ (x \in J \implies uxv \in J)$.

Definition 3.1. Let $((S, =, \neq), \cdot, \preceq)$ be a semigroup ordered under a quasi-order relation $\preceq$. A subset F of S is an *interior filer* of S if holds (13), (15) and

(20) $(\forall u, v, x \in S)\,(uxv \in F \implies x \in F)$.

Remark 3.2. It should be noted here that in determining of the interior filter F in a quasi-ordered semigroup S the requirement that F should be subsemigroup in S is omitted, that is, the condition (11) is omitted. Instead, the consistency requirement (15) is incorporated into the determination of the interior filter.

First, let us show that the concept of interior filters in quasi-ordered semigroups is well defined.

Theorem 3.3. *Let $F\,(\neq S)$ be an interior filter in a quasi-ordered semigroup S. Then the set F^c is an interior ideal of S. If F is a subsemigroup of S, then F^c is a prime interior ideal of S.*

Proof. The subset F^c is non-empty due to the condition $F \neq S$.

Let $x, y \in S$ be arbitrary elements such that $x \in F^c$ and $y \in F^c$. Then $xy \in F$ or $xy \in F^c$. The first option would give $x \in F$ or $y \in F$ by (15) which is contrary to the hypotheses. Therefore, it must be $xy \in F^c$. Thus, the condition (11) is valid.

Let $x, u, v \in S$ be arbitrary elements such that $x \in F^c$. Then $uxv \in F$ or $uxv \in F^c$. The first option would give $x \in F$ by (20) which is contrary to the hypothesis $x \in F^c$. Therefore, it must be $uxv \in F^c$. So, the set F^c satisfies the requirement (19).

Let $x, y \in S$ be such that $y \in F^c$ and $x \preceq y$. Assuming it is $x \in F$. It would be $y \in F$ by (13) which is impossible by hypothesis. Thus $x \in F^c$. Hence, the set F^c satisfies the requirement (8). □

Theorem 3.4. *Every generalized filter of a quasi-ordered semigroup S is an interior filter of S.*

Proof. Let F be a filter of S. This means that F satisfies the conditions (11), (12), (13), (14) and (15). Let us prove (20). Let $u, v, x \in S$ be such that $uxv \in F$. Then $ux \in F$ because F is a left filter in S. Thus $x \in F$ since F is a right filter in S. So, the set F is an interior filter in S. □

An interior filter of a quasi-ordered semigroup S does not have to be a filter of S in the general case since it does not have to satisfy the condition

(11). The reverse of the previous theorem is realized under one special condition.

Theorem 3.5. *Suppose that a quasi-ordered semigroup S satisfies one additional condition:*

(A) *For every $a \in S$ there exists an element $x_a \in S$ such that $a \preceq ax_a a$.*
Then the generalized filters and the interior filters in S coincide.

Proof. Let F be an interior filter of a quasi-ordered semigroup S that satisfies additional condition (A). Let $a, b \in S$ be such that $ab \in F$. Then there exist elements $x_a, y_b \in S$ such that $a \preceq ax_a a$ and $b \preceq by_b b$. Thus $ab \preceq (ax_a a)(bx_b b)$ by (6) and $(ax_a a)(bx_b b) \in F$ by (13). From here it follows

$$(ax_a)a(by_b b) = (ax_a a)(by_b b) \in F \implies a \in F \text{ and}$$

$$(ax_a a)b(y_b b) = (ax_a a)(by_b b) \in F \implies b \in F$$

because F is an interior filter of S. This proves that F is a generalized filter of S. $\square$

Theorem 3.6. *Suppose that a quasi-ordered semigroup S satisfies one additional condition:*

(B) *For every $a \in S$ there exist element $x_a, y_a \in S$ such that $a \preceq x_a a^2 y_a$.*
Then the generalized filters and the interior filters in S coincide.

Proof. Let F be an interior filter of a quasi-ordered semigroup S that satisfies additional condition (B). Let $a, b \in S$ be such that $ab \in F$. Then there exist elements $x_a, y_a, x_b, y_b \in S$ such that $a \preceq x_a a^2 y_a$ and $b \preceq x_b b^2 y_b$. Thus $ab \preceq (x_a a^2 y_a)(x_b b^2 y_b)$ by (6) and $(x_a a^2 y_a)(x_b b^2 y_b) \in F$ by (13). From here it follows

$$x_a a^2(y_a x_b b^2 y_b) = (x_a a^2 y_a)(x_b b^2 y_b) \in F \implies a^2 \in F \text{ and}$$

$$(x_a a^2 y_a x_b)b^2 y_b = (x_a a^2 y_a)(x_b b^2 y_b) \in F \implies b^2 \in F$$

because F is an interior filter of S. Therefore, $a \in F \wedge b \in F$ by (11). This proves that F is a generalized filter of S. $\square$

Remark 3.7. The class of ordered semigroups that satisfies the condition (A) is recognized as a class of regular quasi-ordered semigroups, while the class of ordered semigroups that satisfy the condition (B) is recognized as a class of quasi-ordered intra-regular semigroups. Analogous claims for interior ideals in ordered semigroups are shown in [4] by Kehayopulu.

However, instead of request (A) or request (B), we can make a request

(C) $(\forall x, y \in S)\ (x \preceq xy)$.

And in that case, every interior filter in S is also a filter in S. Indeed:

Theorem 3.8. *Suppose that a quasi-ordered semigroup S satisfies one additional condition (C). Then any interior filter of S is a generalized filter of S.*

Proof. Let a quasi-ordered semigroup S satisfy condition (C) and let F be an interior filter in S. This means that F satisfies conditions (13), (15) and (20). Let us prove (11).

Let $a, b \in S$ be such that $ab \in F$. From $b \preceq ba$, follows $ab \preceq aba$ by (6), and, hence $ab \in F$ implies $aba \in F$. Thus $b \in F$ by (20). On the other hand, $ab \preceq aab$ follows from $a \preceq aa$, and $aab \in F$ by (13) follows from $ab \in F$. Now, from $aab \in$ it follows $a \in F$ according to (20). Thus, we have $a \in F$ (and $b \in F$ by an analogous procedure) thus proving (11). □

Analogous evidence of the coincidence of interior filters and generalized filters in a quasi-ordered semigroup could be demonstrated under the assumption that the semigroup satisfies the condition $(\forall a, b \in S)(a \preceq ba)$.

Requirement (C) may be weakened:

Theorem 3.9. *Suppose a quasi-ordered semigroup S satisfies the condition*

(D) $(\forall x \in S)\ (x \preceq x^2)$.

Then any interior filter of S is a generalized filter of S.

Proof. Let F be an interior filter of S. It is only necessary to prove that the condition (11) is valid.

Let $a, b \in S$ be such that $ab \in F$. Since $a \preceq a^2$ and $b \preceq b^2$, we have $ab \preceq a^2b$ and $ab \preceq ab^2$ by (6). Thus $a^2b \in F$ and $ab^2 \in F$ by (13). Hence $a \in F$ and $b \in F$ by (20). This means that F is a generalized filter of S. □

The family $\mathfrak{Intf}(S)$ of all interior filters of a semigroup S ordered under a quasi-order is not empty because $\emptyset, S \in \mathfrak{Intf}(S)$.

Theorem 3.10. *The family $\mathfrak{Intf}(S)$ of all interior filters of a semigroup S ordered under a quasi-order is a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of interior filters of a quasi-ordered semigroup $(S,\cdot,\preceq)$.

(a) Let $x,y\in S$ such that $xy\in\bigcup_{i\in I}F_i$. Then there exists an index $k\in I$ such that $xy\in F_k$. Thus $x\in F_k\subseteq\bigcup_{i\in I}F_i$ and $y\in F_k\subseteq\bigcup_{i\in I}F_i$ by (15). This means that the set $\bigcup_{i\ inI}F_i$ satisfies the condition (15).

Let $x,u,v\in S$ be such $uxv\in\bigcup_{i\in I}F_i$. Then there exists an index $k\in I$ such that $uxv\in F_k$. Thus $x\in F_k\subseteq\bigcup_{i\in I}F_i$ by (20) because F_k is an interior filter of S. So, the set $\bigcup_{i\in I}F_i$ satisfies the condition (20).

Let $x,y\in S$ be arbitrary elements such that $x\in\bigcup_{i\in I}F_i$ and $x\preceq y$. Then there exists an index $k\in I$ such that $x\in F_k$. Thus $y\in F_k\subseteq\bigcup_{i\in I}F_i$ by (13).

This shows that the set $\bigcup_{i\in I}F_i$ is an interior filter of S.

(b) Let X be the family of all interior filters of S contained in $\bigcap_{i\in I}F_i$. Then $\cup X$ is the maximal interior filter of S contained in $\bigcap_{i\in I}F_i$ according to part (a) of this evidence.

(c) If we put $\sqcup_{i\in I}F_i=\bigcup_i F_i$ and $\sqcap_{i\in I}F_i=\cup X$ then $(\mathfrak{Intf}(S),\sqcup,\sqcap)$ is a complete lattice. □

Corollary 3.11. *For any inhabited subset X of S there is the maximal interior filter contained in X.*

Proof. The proof of this Corollary follows directly from part (b) in the proof of the previous theorem. □

Corollary 3.12. *For any element x of S there is the maximal ordered interior filter F_x of S such that $x\notin F_x$.*

Proof. Proof of this Corollary is obtained if in the previous Corollary we take $X=\{u\in S:u\neq x\}$. □

Remark 3.13. It should be noted here that the previous theorem would not be valid if the interior filter F in a quasi-ordered semigroup S were required to be a subsemigroup in S.

Example 3.14. Let $S=\{0,1,2,3,4\}$ and operation '$\cdot$' defined on S as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	3	2
3	0	3	3	3	3
4	0	4	2	3	4

Then S forms a semigroup. The quasi-order relation on S is given by

$$\preceq = \{(0,0),\ (0,1),(0,2),(0,3),(0,4),(1,1),(2,1),(2,2),(3,1),\\ (3,2),(3,3),(3,4),(4,1),\ (4,2),\ (4,4)\}.$$

Then S is a quasi-ordered semigroup. By direct verification one can establish that the sets $\{1,2,3,4\}$, $\{1,2,4\}$ and $\{1\}$ are interior filters S.

Example 3.15. The interval $S = \langle 0,1\rangle \subseteq \mathbb{R}$ is a semigroup ordered under the order $\leqslant$. Filters in this semigroup have a form of $\langle a,1\rangle$ for any element $a \in \langle 0,1\rangle$. So, the filter $\langle a,1\rangle$ is an interior filter in S by Theorem 3.4.

4. Weak-interior filters

According to [8] a non-empty subset J of a semigroup S is said to be a *left weak-interior ideal* of S if J is a subsemigroup of S and holds $SJJ \subseteq J$. In other words, J is a left weak-interior ideal of a semigroup S if it satisfies (9), (11) and

(21) $(\forall x,u,v \in S)((u \in J \wedge v \in J) \implies xuv \in J)$.

A non-empty subset J of a semigroup S is said to be a *right weak-interior ideal* of S if it satisfies (9), (11) and

(22) $(\forall x,u,v \in S)((u \in J \wedge v \in J) \implies uvx \in J)$.

A non-empty subset J of a semigroup S is said to be a *weak-interior ideal* of S if J is a subsemigroup of S and J is left and right weak-interior ideal of S.

If S is a semigroup ordered under a quasi-order, the determination of the notion of (left, right) weak-interior filters in S must be adapted to the specific order requirement of this semigroup with the condition (13) (see [9]). The appropriate counterparts of these types of ideals in quasi-ordered semigroups are introduced by the following definition:

Definition 4.1. Let $S =: ((S, =, \neq), \cdot, \preceq)$ be a quasi-ordered semigroup and let F be a subset of S.

(i) The subset F is a *left weak-interior filter* of S if the conditions (12), (13) are valid and the following holds

(23) $(\forall x, u, v \in S)(xuv \in F \implies (u \in F \vee v \in F))$.

(ii) The subset F is a *right weak-interior filter* of S if the conditions (13), (14) are valid and the following holds

(24) $(\forall x, u, v \in S)(uvx \in F \implies (u \in F \vee v \in F))$.

(iii) The subset F is a *weak-interior filter* of S if it is a left and right weak-interior filter of S.

Remark 4.2. As in the case of determination of interior filters, here, too, in the determination of (left, right) weak-interior filters, it should be noted that the condition (11) is omitted from the definition of this filter class. It should also be noted that a (left, right) weak-interior filter in a quasi-ordered semigroup S is a consistent subset of S.

Example 4.3. Let $\mathbb{Q}$ be a field of rational numbers, $S = \{\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in \mathbb{Q}\}$ be a semigroup of matrices over the filed $\mathbb{Q}$. The operation in S is the standard multiplication of matrices. Then S is an ordered semigroup. Then $F =: \{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \mid d \in \mathbb{Q} \wedge d \neq 0\}$ is a right weak-interior filter of the semigroup S and F is neither a left filter nor a right filter, not a weak-interior filter and not an interior filter of the semigroup S.

Our first proposition shows that the concept of the right weak-interior filter is well defined.

Proposition 4.4. *Let $F\,(\neq\ S)$ be a right weak-interior filter of a quasi-ordered semigroup S. Then the set F^c is a right weak-interior ideal of S.*

Proof. It needs to be proven:

$F^c \neq \emptyset$,

$(\forall x, y \in S)((x \in F^c \wedge y \in F^c) \implies xy \in F^c$,

$(\forall x, u, v \in S)((u \in F^c \wedge v \in F^c) \implies uvx \in F^c)$, and

$(\forall x, y \in S)((y \in F^c \wedge x \preceq y) \implies x \in F^c)$.

The condition $F \neq S$ ensures that the set F^c is non-empty.

Let $x, y \in S$ be such that $x \notin F$ and $y \notin F$. Then $xy \notin F$ or $xy \in F$. The second option would give $x \in F$ by (14) which is in contradiction with the hypotheses $x \in F^c$. Therefore, it must be $xy \notin F$. This means $xy \in F^c$.

Let $x, u, v \in S$ be arbitrary elements such that $u \notin F$ and $v \notin F$. Then $uvx \notin F \vee uvx \in F$. The second option would give $u \in F \vee v \in F$ according to (24) which is contrary to the hypotheses $u \notin F$ and $v \notin F$. So it must be $uvx \notin F$. This means $uvx \in F^c$.

Let $x, y \in S$ be arbitrary elements such that $x \preceq y$ abd $y \in F^c$. Then $x \notin F \vee x \in F$. The second option would be to give $(x \preceq y \wedge y \in F) \implies y \in F$ by (13). It contradict to the hypotheses. Therefore, it must be $x \notin F$. This means $x \in F^c$. Thus proving the validity of the condition (8). □

The concept of right weak-interior filters is a generalization of the notion of filters in quasi-ordered semigroups, as shown by the following theorem.

Theorem 4.5. *Any right filter of a semigroup S ordered under quasi-order is a right weak-interior filter of S.*

Proof. Let $S =: ((S, \cdot), \preceq)$ be a quasi-ordered semigroup and let F be a right filter of S. This means that F satisfies the conditions (11), (13) and (14). It needs to be proven (24). Let $x, u, v \in S$ be such that $uvx \in F$. Then $uv \in F$ because F is a right filter of S. Thus $u \in F \vee v \in F$ by (11). This proves that F is a right weak-interior filter of S. □

The reverse, of course, may not be true since the determination of a right weak interior filter cannot satisfy the condition (11). The reverse of the previous theorem can be demonstrated in one special case:

Theorem 4.6. *Let S be a quasi-ordered semigroup which satisfies the condition*

(D) $(\forall x \in S)\ (x \preceq x^2)$.

Then the right generalized filters and the right weak-interior filters in S coincide.

Proof. Suppose that S is a quasi-ordered semigroup which satisfies the condition (D) and F is a right weak-interior filter of S. Let $x, y \in S$ be arbitrary elements such that $xy \in F$. How from (D) it follows $xy \preceq x^2y$, we have that $xy \in F$ implies $x^2y \in$ according to (13). Hence $x^2 \in F$ by (24) and $x \in F$ by (14). This means that F is a right generalized filter od S. □

Similar to the above, the concept of right weak-interior filter in a quasi-ordered semigroup S is a generalization of the concept of interior filters in S as shown by the following theorem.

Theorem 4.7. *Any interior filter of a quasi-ordered semigroup S is a right weak-interior filter of S.*

Proof. Let F be an interior filter of a quasi-ordered semigroup S. Then (15), (13) and (20) are valid formulas. Let us prove (24). Let $x, u, v \in S$ be such that $uvx \in F$. Then $v \in F$ by (20). Thus $u \in F \vee v \in F$. So, F is a right weak-interior filter of S. □

Theorem 4.8. *The family $\mathfrak{W}_\mathfrak{r}\mathfrak{intf}(S)$ of all right weak-interior filters of a quasi-ordered semigroup S forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be family of right weak-interior filters of a quasi-ordered semigroup S.

(a) Let $x, y \in S$ be such that $xy \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $xy \in K_k$. Thus $x \in F_k \subseteq \bigcup_{i\in I} F_i$ by (14). This means that the set $\bigcup_{i\in I} F_i$ satisfies the condition (14).

Let $u, v, x \in S$ be such that $uvx \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $uvx \in F_k$. Thus $u \in F_k \subseteq \bigcup_{i\in I} F_i$ or $v \in K_k \subseteq \bigcup_{i\in I} F_i$. Thus, the set $\bigcup_{i\in I} F_i$ satisfies the condition (24).

Let $x, y \in S$ be such that $x \in \bigcup_{i\in I} F_i$ and $x \preceq y$. Then there exists an index $k \in I$ such that $x \in F_k$. Thus $y \in F_k \subseteq \bigcup_{i\in I} F_i$ by (13).

Therefore, the set $\bigcup_{i\in I} F_i$ is a right weak-interior filter of S.

(b) Let X be the family of all right weak-interior filters of S contained in $\bigcap_{i\in I} F_i$. Then $\cup X$ is the maximal right weak-interior filter of S contained in $\bigcap_{i\in I} F_i$ according to part (a) of this proof.

(c) If we put $\sqcup_{i\in I} f_i = \bigcup_{i\in I} F_i$ and $\sqcap_{i\in I} F_i = \cup X$, then $(\mathfrak{W}_\mathfrak{r}\mathfrak{intf}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 4.9. *For any subset X of a semigroup S there is the maximal right weak-interior filter of S contained in X.*

Corollary 4.10. *For any element $a \in S$ there is the maximal right weak-interior filter F_a of S such that $a \notin F_a$.*

Claims for (left) weak-interior filters of a quasi-ordered semigroup can be designed without major difficulties analogously to previous claims. Accepting this belief allows us to create the following theorem:

Theorem 4.11. *Suppose that a quasi-ordered semigroup S satisfies condition (D). Then any weak-interior filter of S is an interior filter of S.*

Proof. Let F be a weak-interior filter of S. Then F is a generalized filter of S by Theorem 4.6. Thus F is an interior filter of S by Theorem 3.4. □

5. Quasi interior filters

A non-empty subset J of a semigroup S is said to be a *left quasi-interior ideal* of S if J is a subsemigroup of S and holds $SJSJ \subseteq J$, i.e. it satisfies (9), (11) and

(25) $(\forall x, y, u, v \in S)\ ((u \in J \wedge v \in J) \implies xuyv \in J)$.

If we add (8) to (9), (11) and (25) that the subset J of a quasi-ordered semigroup S must meet in order for it to be a left quasi-interior ideal of S, we get the determination of the left quasi interior ideal of a quasi-ordered semigroup S (see [9]).

A non-empty subset J of S is said to be a *right quasi-interior ideal* of S if it satisfies (9), (11) and

(26) $(\forall x, y, u, v \in S)((u \in J \wedge v \in J) \implies uxvy \in J)$.

If we (8) to (9), (11) and (26) that the subset J of a quasi-ordered semigroup S must meet in order for it to be a left quasi-interior ideal of S, we get the determination of the ordered right quasi-interior ideal of a quasi-ordered semigroup S (see [9]).

A *quasi-interior ideal* of S if it is both a left quasi-interior ideal and a right quasi-interior ideal of S.

Definition 5.1. Let $S =: ((S, =, \neq), \cdot)$ be a semigroup ordered under a quasi-order $\preceq$. The subset F of S is:

a *left quasi-interior filter* of S if it satisfies (12), (13) and

(27) $(\forall x, y, u, v \in S)(xuyv \in F \implies (u \in F \vee v \in F))$;

a *right quasi-interior filter* of S if it satisfies (13), 14) and

(28) $(\forall x, y, u, v \in S)(uxvy \in F \implies (u \in F \vee v \in F))$;

a *quasi-interior filter* of S and it is a left and right quasi-interior filter of S.

Remark 5.2. As in the case of determination of interior filters and (left, right) weak-interior filters, here, too, in the determination of (left, right) quasi-interior filters, it should be noted that the condition (11) is omitted from the definition of this class of filters.

Example 5.3. Let $\mathbb{Q}$ be a field of rational numbers, $S := \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{Q}\}$ be the semigroup of matrices over the filed $\mathbb{Q}$. The operation in S is the standard multiplication of matrices. Then S is an ordered semigroup. Then $F =: \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Q} \wedge a \neq 0\}$ is a left quasi-interior filter of the semigroup S.

Let us show, for the sake of illustration, that the concept of left quasi-interior filters of a quasi-ordered semigroup is correctly defined.

Proposition 5.4. *Let $F\,(\neq S)$ be a left quasi-interior filter of a quasi-ordered semigroup S. Then the set F^c is a left quasi-interior ideal of S.*

Proof. It needs to be prove:

$F^c \neq \emptyset$,

$(\forall x, y \in S)((x \in F^c \wedge y \in F^c) \implies xy \in F^c)$,

$(\forall x, y, u, v \in S)((u \in F^c \wedge v \in F^c) \implies xuyv \in F^c)$.

The condition $F \neq S$ ensures that the set F^c is not empty.

Let $x, y \in S$ such that $x \in F^c$ and $y \in F^c$. Then $xy \in F$ or $xy \notin F$. The first option would give $y \in F$ by (12). We got a contradiction according to the hypotheses. So, must be $xy \notin F$. This means $xy \in F^c$.

Let $x, y, u, v, t \in S$ be arbitrary elements such that $u \in F^c$ and $v \in F^c$. Then $xuyv \notin F$ or $xuyv \in F$. The second option would be give $u \in F$ or $v \in F$ by (26). This is impossible. So, must be $xuyv \notin F$. This means $xuyv \in F^c$.

Let $x, y \in S$ be such that $x \preceq y$ and $y \in F^c$. Then $x \in F$ or $x \notin F$. The first option would be give $y \in F$ what is impossible. So, must be $x \in F^c$. □

The concept of left quasi-interior filters is a generalization of the notion of left filters in a quasi-ordered semigroup, which shows the following theorem:

Theorem 5.5. *Any left filter of a quasi-ordered semigroup S is a left quasi-interior filter of S.*

Proof. Let F be a left filter of a quasi-ordered semigroup S. This means that the set F satisfies the conditions (11), (12) and (13). Let us prove (27).

Let $x, y, u, v \in S$ be such that $xuyv \in F$. Then $v \in F$ by (12). Thus $u \in F \vee v \in F$. So, F is a left quasi-interior filter of S. □

The opposite statement may not be true since a left quasi-ordered filter does not satisfy the condition (11).

In addition to the above, the concept of left quasi-interior filters is a generalization of the concept of interior filters in quasi-ordered semigroups.

Theorem 5.6. *Any interior filter of a quasi-ordered semigroup S is a left quasi-interior filter of S.*

Proof. Let F be an interior filter of a quasi-order semigroup S. It needs to be proven (27). Let $x, y, u, v \in S$ such that $xu(yv) = xuyv \in F$. Then $u \in F$ by (20). Thus $u \in f \vee v \in F$. So, F is a left quasi-interior filter of S. □

The reverse of the previous theorem can be proved if the quasi-ordered semigroup S satisfies the condition (C). Indeed:

Theorem 5.7. *Suppose that a quasi-ordered semigroup S satisfies the condition (C). Then the interior filters and the left quasi-interior filters in S coincide.*

Proof. Suppose that a quasi-ordered semigroup S satisfies the condition (C) and let F be a left quasi-interior filter of S. Let $u, v, x, y \in S$ be arbitrary elements such that $uxv \in F$. On the other hand, we have $v \preceq vx$ by (C). Hence it follows $uxv \preceq uxvx$ according to (6). Therefore, from $uxv \in F$ and $uxv \preceq uxvx$ it follows $uxvx \in F$ according to (13). Thus $x \in F$ by (27). This proves that F is an interior filter of S. □

The previous theorem allows us to establish a connection between the left filter and the left quasi-interior filter in a quasi-ordered semigroup.

Theorem 5.8. *Let a quasi-ordered semigroup S satisfies the condition (C). Then the left generalized filter of S and the left quasi-interior filter of S coincide.*

Proof. Let F be a left quasi interior filter of S. Then F is an interior filter of S by Theorem 5.7. Thus F is a left generalized filter of S by Theorem 3.8. □

Theorem 5.9. *The family $\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intf}(S)$ of all left quasi-interior filters of a quasi-ordered semigroup S forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of left quasi-interior filters of a quasi-ordered semigroup S.

(a) Let $x, y \in S$ be such that $xy \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $xy \in F_k$. Thus $y \in F_k \subseteq \bigcup_{i\in I} F_i$. by (12). This means that the set $\bigcup_{i\in I} F_i$ satisfies condition (12).

Let $x, y, u, v \in S$ be arbitrary elements such that $xuyv \in \bigcup_{i\in I} F_i$. Then there exists an index $k \in I$ such that $xuyv \in F_k$. Thus $u \in F_k \subseteq \bigcup_{i\in I} F_i$ or $v \in F_k \subseteq \bigcup_{i\in I} F_i$ by (27).

Let $x, y \in S$ be such that $y \in \bigcup_{i\in I} F_i$ and $x \preceq y$. Then there exists an index $k \in I$ such that $y \in F_k$. Thus $x \in F_k \subseteq \bigcup_{i\in I} F_i$ by (13).

This proves that the set $\bigcup_{i\in I} K_i$ is a left quasi-interior filter of S.

(b) Let X be the family of all lest quasi-interior filters of S included in $\bigcap_{i\in I} F_i$. Then the set $\cup X$ is the maximal lest quasi-interior filter of S included in $\bigcap_{i\in I} F_i$.

(c) If we put $\sqcup_{i\in I} F_i = \bigcup_{i\in I} F_i$ and $\sqcap_{i\in I} F_i = \cup X$, then $(\mathfrak{Q}_{\mathfrak{l}}\mathfrak{intf}(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 5.10. *For any subset X of a quasi-ordered semigroup S there is the maximal lett quasi-interior filter of S contained in X.*

Corollary 5.11. *For any element $a \in S$ there is the maximal left quasi-interior filter F_a of S such that $a \notin F_a$.*

Remark 5.12. Claims for (right) quasi-interior filters of a quasi-ordered semigroup can be designed without major difficulties analogously to previous claims.

Thus, for example, we transform Theorem 5.7 into the following theorem:

Theorem 5.13. *Suppose that a quasi-ordered semigroup S satisfies one additional condition:*

(E) *For every elements $a, b \in S$ the following holds $a \preceq ba$.*

Then the interior filters and the right quasi-interior filters in S coincide.

The concept of left quasi-interior filters of a quasi-ordered semigroup S is a generalization of the concept of weak-interior filters of S.

Theorem 5.14. *Any left weak-interior filter of a quasi-ordered semigroup S is a left quasi-interior filter of S.*

Proof. Let F be a left weak-interior filter of a quasi-ordered semigroup S. This means that F satisfies the conditions (12), (13) and (23). Let us prove (27).

Let $x, y, u, v \in S$ be such that $xuyv \in F$. Then $u \in F$ or $yv \in F$ by (23). Thus $u \in F$ of $v \in F$ by (12). It is shown that F is a left quasi-interior filter of S. □

Further on, it can be demonstrated that any left quasi-interior filter of a quasi-ordered semigroup S is a left weak-interior filter of S if S satisfies the condition (C).

Theorem 5.15. *Suppose that a quasi-ordered semigroup S satisfies the condition (C). Then any quasi-interior filter of S is a weak-interior filter of S.*

Proof. Suppose that a quasi-ordered semigroup S satisfies the condition (C). Let F be a left quasi-interior filter of S. This means that F satisfies the conditions (12), (13) and (27). Let us prove (23). Let $x, y, v \in S$ be such that $xuv \in F$. On the other hand, from $u \preceq ux$ it follows $xuv \preceq xuxv$ by (6). Therefore, $xuxv \in F$ by (13). Hence $u \in F \vee v \in F$ by (27). □

References

[1] **Y. Cao**, *Chain deconposition of ordered semigroups*, Semigroup Forum, **65** (2002), 83 – 106.

[2] **N. Kehayopulu**, *On weakly commutative poe-semigroups*, Semigroup Forum, **34** (1987), 367 – 370.

[3] **N. Kehayoupulu**, *On left regular ordered semigroups.* Math. Japon., **35** (1990), no. 6, 1057 – 1060.

[4] **N. Kehayopulu**, *Note on interior ideals, ideal elements in ordered semigroups*, Scientiae Math., **2**(1999), no. 3, 407 – 409.

[5] **N. M. Khan and A. Mahboob**, *Left-m-filter, right-n-filter and (m, n)-filter on ordered semigroup*, J. Taibah Un. Sci., **13**(2019), no. 1, 27 – 31.

[6] **S. Lajos**, *$(m; k; n)$-ideals in semigroups.* In: *Notes on Semigroups II*, Karl Marx Univ. Econ., Dept. Math. Budapest, **1** (1976), pp. 12 – 19.

[7] **S.K. Lee and S.S. Lee**, *Left (right) filters on po-semigroups*, Kangweon-Kyungki Math. J., **8** (2000), 43 – 45.

[8] **M.M. Krishna Rao**, *Quasi-interior ideals and weak-interior ideals*, Asia Pac. J. Math., **7** (2020), ID: 21, DOI: 10.28924/APJM/7-21

[9] **D.A. Romano**, *A note on weak-interior and quasi-interior ideals in quasi-ordered semigroups*, Discuss. Math., Gen. Algebra Appl., **43** (2023), https://doi.org/10.7151/dmgaa.1414.

[10] **G. Szasz**, *Interior ideals in semigroups.* In: *Notes on semigroups IV*, Karl Marx Univ. Econ., Dept. Math. Budapest, **5** (1977), pp. 1 – 7.

[11] **G. Szasz**, *Remark on interior ideals of semigroups.* Studia Scient. Math. Hung., **16** (1981), 61 – 63.

Received May 20, 2022

International Mathematical Virtual Institute
Kordunaška street 6, 78000 Banja Luka
Bosnia and Herzegovina
E-mail: daniel.a.romano@hotmail.com