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The Ramsey number $R_4(3)$ is not solvable by group partition means

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Abstract. The *Ramsey number* $R_n(3)$ is the smallest positive integer such that colouring the edges of a complete graph on $R_n(3)$ vertices in n colours forces the appearance of a monochromatic triangle. A lower bound on $R_n(3)$ is obtainable by partitioning the non-identity elements of a finite group into disjoint union of n symmetric product-free sets. Exact values of $R_n(3)$ are known for $n \leqslant 3$. The best known lower bound that $R_4(3) \geqslant 51$ was given by Chung. In 2006, Kramer gave a proof of over 100 pages that $R_4(3) \leqslant 62$. He then conjectured that $R_4(3) = 62$. We say that the Ramsey number $R_n(3)$ is *solvable by group partition means* if there is a finite group G such that $|G|+1 = R_n(3)$ and $G \setminus \{1\}$ can be partitioned as a union of n symmetric product-free sets. For $n \leqslant 3$, the Ramsey number $R_n(3)$ is solvable by group partition means. Some authors believe that $R_4(3)$ not be solvable by a group partition approach. We prove this here. We also show that any finite group G whose size is divisible by 3 cannot enjoy $G \setminus \{1\}$ written as a disjoint union of its symmetric product-free sets. We conclude with a conjecture that $R_5(3) \geqslant 257$.

1. Introduction

Let G be a finite group, and S a non-empty subset of G. Then S is said to be *product-free* if $S \cap SS = \varnothing$. A *maximal product-free set* in G is a maximal by cardinality product-free set in G. Let $\lambda(G)$ denote the cardinality of a maximal product-free set in G. Suppose T is any product-free set in a finite group G. For $x_1 \in T$, define $x_1T := \{x_1x_2 | x_2 \in T\}$. As $|x_1T| = |T|$ and $T \cup x_1T \subseteq G$, we have that $2|T| \leqslant |G|$; so $|T| \leqslant \frac{|G|}{2}$. This shows that $\lambda(G) \leqslant \frac{|G|}{2}$; i.e., the size of a product-free set in a finite group G is at most half the size of G.

The value of $\lambda(G)$ is well-known when G is a finite abelian group, following the works of Diananda and Yap [9], as well as Green and Ruzsa [14]. On

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the other hand, the problem of determination of structures and sizes of maximal product-free sets in non-abelian groups is still open, although there has been great progress by many authors, including Kedlaya [17, 18] and Gowers [13]. An interested reader may also see [22, 23, 24, 12, 7, 6, 5, 1, 2, 4, 3] for works on maximal by inclusion product-free sets.

The *Ramsey number* $R_n(3)$ is the smallest positive integer such that colouring the edges of a complete graph on $R_n(3)$ vertices in n colours forces the appearance of a monochromatic triangle. Exact values of $R_n(3)$ are known for $n \leqslant 3$; for instance see [16]. The best known lower bound that $R_4(3) \geqslant 51$ was given by Chung [8] in 1973. Kramer [20], in 2006, after giving a proof of over 100 pages that $R_4(3) \leqslant 62$, conjectured that $R_4(3) = 62$. See also [10, 19].

A symmetric product-free set is a product-free set S such that $S = S^{-1}$. For a finite group G, it is known that if G^* (where $G^* = G \setminus \{1\}$) can be partitioned into disjoint union of m symmetric product-free sets (SPFS for short), then $R_m(3) \geqslant |G| + 1$. Examples by various authors show that the group partition approach gives a sharp lower bound that coincides with the exact value of $R_m(3)$ for $m \leqslant 3$. The main result of this paper is essentially folklore. Here, we show that the group partition approach cannot be used to improve the known lower bound of $R_4(3)$ to r for $52 \leqslant r \leqslant 62$; in particular, we demonstrate that $R_4(3)$ is not solvable by a group partition means. For the rest of this section, we give the following result.

Theorem 1.1. (Idea from [16, Theorem 1.1] and [24, pp. 247–248]) *If G is a finite group such that G^* can be partitioned into disjoint union of m symmetric product-free sets (where $m \geqslant 2$), then $R_m(3) \geqslant |G| + 1$.*

Proof. Suppose $G^* = S_1 \sqcup \cdots \sqcup S_m$ is a disjoint union of m symmetric product-free sets. We assign to the set S_i colour C_i for each $i \in \{1, \ldots, m\}$. Let $K_{|G|}$ be the complete graph on $|G|$ vertices: $v_1, v_2, \ldots, v_{|G|}$. [Note that the vertices of $K_{|G|}$ are the elements of G.] We m-colour $K_{|G|}$ as follows: colour the edge v_iv_j (from v_i to v_j) with colour C_k if $v_iv_j^{-1} \in S_k$. Since S_k is symmetric (i.e., $S_k = S_k^{-1}$), this induces a well-defined edge-colouring of the graph. Let v_a, v_b and v_c be any three vertices of $K_{|G|}$ and consider the triangle on these vertices. Suppose two of its edges say v_av_b and v_bv_c are coloured C_k. This means that $v_av_b^{-1}, v_bv_c^{-1} \in S_k$. Since S_k is product-free, we have that $(v_av_b^{-1})(v_bv_c^{-1}) = v_av_c^{-1} \notin S_k$. So v_av_c must be coloured C_l for $l \neq k$, and no monochromatic triangle is formed. Therefore $R_m(3) > |G|$. □

2. Main results

2.1 A group theoretic motivation

In 1955, Greenwood and Gleason [15] proved that

$$R_{n+1}(3) \leqslant (n+1)(R_n(3)-1)+2$$

for $n \geqslant 1$. This result of Greenwood and Gleason tells us that $R_2(3) \leqslant 6$ and $R_3(3) \leqslant 17$. Note that if $R_m(3) \leqslant k$, then Theorem 1.1 implies that for any group G with $|G| \geqslant k$, it is impossible to partition G^* into m symmetric product-free sets. Hence, if G^* is symmetric and product-free, then $|G| \leqslant 2$ (and clearly the only example is C_2), if G^* has a partition into two symmetric product-free sets, then $|G| \leqslant 5$, and if G^* has a partition into three symmetric product-free sets, then $|G| \leqslant 16$. It is then quick to check by hand that the only examples of groups G for which G^* has a partition into two symmetric product-free sets are $C_4, C_2 \times C_2$ and C_5.

We used GAP [11] to observe that there are only four groups G of order 16 such that G^* has a partition into three symmetric product-free sets. The groups are C_2^4, $C_4 \times C_4$, $(C_4 \times C_2) \rtimes C_2$ and $C_2 \times D_8$, with GAP IDs as $[16, 14]$, $[16, 2]$, $[16, 3]$ and $[16, 11]$ respectively. Each of them when combined with the result of Greenwood and Gleason tells us that $R_3(3) = 17$. The results for the two abelian cases (C_2^4 and $C_4 \times C_4$) are known in the literature; for instance, see [24].

G	An example of a partition of G^* into disjoint union of 3 symmetric product-free sets
$C_2^4 = \langle x_1, x_2, x_3, x_4 \mid x_i x_j = x_j x_i,\ x_i^2 = 1 \text{ for } 1 \leqslant i, j \leqslant 4 \rangle$	$\{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\} \cup$ $\{x_1x_2, x_1x_3, x_2x_4, x_1x_2x_3, x_1x_2x_4\} \cup$ $\{x_1x_4, x_2x_3, x_3x_4, x_1x_3x_4, x_2x_3x_4\}$
$C_4 \times C_4 = \langle x, y \mid x^4 = 1 = y^4,\ xy = yx \rangle$	$\{x, x^3, y, y^3, x^2y^2\} \cup$ $\{x^2, xy, x^3y^3, x^2y, x^2y^3\} \cup$ $\{xy^3, x^3y, y^2, xy^2, x^3y^2\}$
$(C_4 \times C_2) \rtimes C_2 = \langle x, y \mid x^4 = 1 = y^2,\ (xyx)^2 = 1 = (yx^{-1})^4,\ (yxyx^{-1})^2 = 1 \rangle$	$\{y, x, x^3, (xy)^2, x^3yx\} \cup$ $\{yx, x^2, x^2y, x^3y, xyx\} \cup$ $\{x^2yx, xy, yxy, x(xy)^2, x^2(xy)^2\}$
$C_2 \times D_8 = \langle x, y, z \mid x^2 = 1,\ y^2 = 1, z^2 = 1,\ (zx)^2 = 1,\ (zy)^2 = 1,\ (yx)^4 = 1 \rangle$	$\{x, y, xz, (xy)^2, xyxz\} \cup$ $\{xy, z, yx, xyx, yz\} \cup$ $\{xyz, yxy, yxz, yxyz, (xy)^2z\}$

We now end this section with some GAP [11] programs that can be used to get the table above and investigate more groups.

Program A. This checks whether a set T is product-free

```
PFTest:=function(T) local x,y; for x in T do for y in T do
if x*y in T then return 1; fi; od; od; return 0; end;
```

Program B. This gives a partition of G^* into k product-free sets if such partition exists

```
PGk:=function(G,k) local LL, AA, g, P, p, PPk, PPkA, PPP;
LL:=List(G);; AA:=[];; for g in LL do if Order(g)>1 then Add(AA,g);
fi; od; AA:=Set(AA);; PPk:=PartitionsSet(AA,k);; PPkA:=[];;
for P in PPk do for p in P do if PFTest(p)=1 then Add(PPkA,P); fi;
od; od; PPkA:=Set(PPkA);; PPP:=Difference(PPk,PPkA);;
if Size(PPP)>0 then return PPP[1]; else return []; fi; end;
```

Program C. All groups G of order n such that G^* has a partition into k product-free sets

```
GGnk:=function(n,k) local M, MM, G, GG;
MM:=[];; GG:=AllSmallGroups(n);
for G in GG do M:=PGk(G,k); if Size(M)>0 then Add(MM,[IdGroup(G),M]);
fi; od; return MM; end;
```

2.2 $R_4(3)$ is not solvable by a group partition means

Recall that $51 \leqslant R_4(3) \leqslant 62$. We say a finite group G is *m-partitioned* if the non-identity elements of G can be partitioned into disjoint union of m symmetric product-free sets. A natural question is whether Chung's lower bound for $R_4(3)$ can be improved to r for $52 \leqslant r \leqslant 62$. We shall use an algorithmic approach to show that the group partition approach cannot be used to improve Chung's lower bound to r for $52 \leqslant r \leqslant 62$. We begin with Lemma 2.1 below.

Lemma 2.1. *If G is a finite group such that G^* has a partition into m symmetric product-free sets (where $m \geqslant 2$), then $|G|$ is not divisible by* 3.

Proof. Let G be a finite group such that $G^* = \bigcup_{i=1}^{m} S_i$, where $m \geqslant 2$ and each S_i is a symmetric product-free set in G. Suppose for contradiction that $|G|$ is divisible by 3. Then G has an element of order 3; say x. Without loss of generality, let $x \in S_1$. As S_1 is symmetric, $x^{-1} \in S_1$. But $x^{-1} = x^2$, a contradiction; as S_1 is product-free. Therefore $|G|$ is not divisible by 3. $\square$

We used GAP [11] to observe that there are 56 groups whose sizes are from 51 up to 61; in particular, there are 1, 5, 1, 15, 2, 13, 2, 2, 1, 13 and 1 group(s) of orders 51, 52, 53, 54, 55, 56, 57, 58, 59, 60 and 61 respectively. In the light of Lemma 2.1, we discard 31 groups from the list, and only work with 25 groups; those whose order is one of $52, 53, 55, 56, 58, 59$ and 61.

Lemma 2.1 tells us that the group partition approach into symmetric product-free sets cannot be used to check whether $R_4(3)$ is 52. The next result (Theorem 2.2) shows that the group partition approach into SPFS cannot be used to prove the conjecture of Kramer that $R_4(3) = 62$.

Theorem 2.2. *The group of order* 61 *cannot be* 4*-partitioned.*

Proof. Suppose we 4-colour the edges of K_{61}. Choose any vertex v_0 of K_{61}. Suppose we edge join v_0 with each of the vertices $v_1, v_2, \ldots, v_m$ respectively. Consider the complete graph K_m on those m vertices. If we colour any edge in K_m with the first colour, then we force the appearance of a triangle in the first colour. So we only colour edges of K_m with any of the remaining three colours. As $R_3(3) = 17$, in order not to have a monochromatic triangle in K_m, we have that $m \leqslant 16$. This argument shows that the largest size of any symmetric product-free set involved in any 4-partition of C_{61} is 16.

The only possibilities of such partition is using SPFS of sizes 16, 16, 16 and 12 or SPFS of sizes 16, 16, 14 and 14. Hence, we only need to work with SPFS of sizes 12, 14 and 16 in our programs for such partition. Using Program E below, we see that there are 27060, 13680 and 3975 symmetric product-free sets of sizes 12, 14 and 16 respectively in C_{61}. We then use Program F below to check for either four SPFS of sizes 16, 16, 16 and 12 whose size of their union is 60 or those of sizes 16, 16, 14 and 14 whose size of their union is 60, and found none. Therefore C_{61} cannot be 4-partitioned. □

Remark 2.3. The same reasoning used for the group of order 61 in the proof of Theorem 2.2 above shows that the maximum size of any of the symmetric product-free sets in a 4-partition of any of the groups we consider here is 16. We shall use this repeatedly in our computations.

Algorithm D. This gives all SPFS of respective sizes (up to 16) in a finite group G

1. For $x \in G$, if $o(x) > 2$, then select only one element from the pair $\{x, x^{-1}\}$. Let A be a collection of all the selected elements. (In this case, $|A| = \frac{|G|-1-|InvG|}{2}$, where $InvG$ is the set of all involutions in G.)

2. Form all subsets of A whose sizes are from 1 up to 8. Test for product-freeness of each subset of A of respective sizes, and make sets T_i consisting of product-free sets of size i for each $i \in \{1, \ldots, 8\}$.

3. Create a non-empty set U_i for each $i \in \{1, \ldots, 8\}$. For each set M in each T_i, if the union of M and M^{-1} is product-free, then add the union to U_i. Repeat

this for each $i \in \{1, \dots, 8\}$. Let spf be the collection of all the U_i's; i.e., $spf := [U_1, U_2, \dots, U_8]$, where each U_i consists of all symmetric product-free sets of size $2i$; not containing an involution.

4. Let $InvG$ be the set of all involutions in G. Take subsets of sizes 1 up to 16 of $InvG$. Test for product-freeness. Let $Ispf$ be the set of all such product-free sets of respective sizes. Let $sprf$ be an empty set. Check whether the union of any set in spf and $Ispf$ is product-free. Add all such union which are product-free of size less than 17 to $sprf$. Also, add all members of spf and $Ispf$ to $sprf$. Then $sprf$ is the set of all SPFS of respective sizes up to 16 in G when $|G|$ is even.

Remark 2.4.
1. We apply only steps 1, 2 and 3 if $|G|$ is odd, and all the steps 1, 2, 3 and 4 if $|G|$ is even.

2. The motivation for treating the sets of involutions separately is to reduce computational time; since we know that $\binom{|G|-1}{16} > \binom{\frac{|G|}{2}+3}{16}$, where $\frac{|G|}{2} + 3$ is the maximum number of involutions in the groups involved.

3. We used Algorithm above (instead of program) because the actual program spreads up to 3 pages of the manuscript. An interested reader can request a copy of the GAP program used. We call the function in Algorithm D, SPFS. It takes only one input which is a finite group of our choice.

Program E. This gives the number of SPFS of various sizes (up to 16) in G

```
SizeSPFS:=function(G) local S,A,i,a; S:=SPFS(G); A:=[];
for i in S do a:=Size(i); if a>0 then Add(A,[Size(i[1]),a]);
fi; od; return A; end;
```

An example of Program E above is given below.

```
gap> SizeSPFS(CyclicGroup(61));
[ [ 2, 30 ], [ 4, 405 ], [ 6, 3000 ], [ 8, 12285 ], [ 10, 26166 ],
[ 12, 27060 ], [ 14, 13680 ], [ 16, 3975 ]]
```

Program F. It decides if G^* can be partitioned into SPFS of sizes a, b, c and d

```
IsPartG:=function(G,a,b,c,d)
local S,Sa,Sb,Sc,Sd,i,j,k,l;
S:=SPFS(G); Sa:=S[a]; Sb:=S[b]; Sc:=S[c]; Sd:=S[d];
for i in Sa do for j in Sb do for k in Sc do for l in Sd do
if Size(Set(Union(i,j,k,l)))=Size(G)-1 then Print([i,j,k,l]); fi;
od; od; od; od; end;
```

The next in the sequel is to have an understanding of the number of iterations we will perform to check all the groups of orders among $52, 53, 55, 56, 58$ and 59.

Program G1. This tells us the iterations to perform for each group G of even order n

```
ExpMathEven:=function(n)
local A, i,j,k,l,B;
A:=[2..16];; B:=[];;
for i in A do for j in A do for k in A do for l in A do
if i<=j and j<=k and k<=l and i+j+k+l=n-1 then Add(B,[i,j,k,l]); fi;
od; od; od; od; return B; end;
```

Program G2. This tells us the iterations to perform for each group G of odd order n

```
ExpMathOdd:=function(n)
local A, i,j,k,l,B,C;
A:=[2..16];; C:=[];; B:=[];;
for i in A do if IsEvenInt(i) then Add(C,i); fi; od;
for i in C do for j in C do for k in C do for l in C do
if i<=j and j<=k and k<=l and i+j+k+l=n-1 then Add(B,[i,j,k,l]); fi;
od; od; od; od; return B; end;
```

We now give some examples of Programs G1 and G2. |small

```
gap> [Size(ExpMathOdd(53)), ExpMathOdd(53)];
[ 9, [ [ 4, 16, 16, 16 ], [ 6, 14, 16, 16 ], [ 8, 12, 16, 16 ],
[ 8, 14, 14, 16 ], [ 10, 10, 16, 16 ], [ 10, 12, 14, 16 ],
[ 10, 14, 14, 14 ], [ 12, 12, 12, 16 ], [ 12, 12, 14, 14 ] ] ]
gap> [Size(ExpMathEven(58)), ExpMathEven(58)];
[ 11, [ [ 9, 16, 16, 16 ], [ 10, 15, 16, 16 ], [ 11, 14, 16, 16 ],
[ 11, 15, 15, 16 ], [ 12, 13, 16, 16 ], [ 12, 14, 15, 16 ],
[ 12, 15, 15, 15 ], [ 13, 13, 15, 16 ], [ 13, 14, 14, 16 ],
[ 13, 14, 15, 15 ], [ 14, 14, 14, 15 ] ] ]
```

The example above tells us that there are 9 (respectively 11) ways of choosing $[a, b, c, d]$ to be used in Program F, as well as what the possibilities are when $|G| = 53$ (respectively $|G| = 58$).

We now check the total possibilities across all groups of order n, where $n \in \{52, 53, 55, 56, 58, 59\}$.

```
A:=[52, 53, 55, 56, 58, 59];; B:=[];; for n in A do
if IsEvenInt(n) then Add(B,NrSmallGroups(n)*Size(ExpMathEven(n)));
else Add(B,NrSmallGroups(n)*Size(ExpMathOdd(n))); fi; od;
gap> B;
```

```
[ 195, 9, 12, 234, 22, 3 ]
gap> Sum(B);
475
```

We have checked all the 475 trials, and did not find such partition of any of the groups. By Lemma 2.1 and Theorem 2.2 therefore, no group of order from 51 up to 61 can be 4-partitioned.

2.3 Concluding remarks

In this paper, we have shown that, while $R_1(3)$, $R_2(3)$ and $R_3(3)$ are solvable by group partition means, the folklore that $R_4(3)$ is not solvable by group partition means is indeed true. It will be interesting to know which Ramsey numbers $R_k(3)$ are solvable by group partition means for $k \geqslant 5$. An interested reader may see [21, pp. 42–43] for bounds on $R_k(3)$ for some $k \geqslant 5$. It is known that $162 \leqslant R_5(3) \leqslant 307$, $538 \leqslant R_6(3) \leqslant 1838$ and $1682 \leqslant R_7(3) \leqslant 12861$. We anticipate that $R_5(3)$ is solvable by group partition means. We are motivated by our computer searches to conjecture that $R_5(3) \geqslant 257$, and that the lower bound can be obtained by partitioning the non-identity elements of a non-cyclic group of order 256 into a disjoint union of five SPFS.

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