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On topological Menger n-groupoids

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Abstract. We present the necessary and sufficient conditions for the existence of left-invariant measure on a topological Menger n-groupoid.

1. Introduction

The terminology and notations used in this article are are typical for this theory (see for example [3]). A nonempty set X with an n-ary operation f is an *n-semigroup* if this operation is associative, i.e.

$$f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1^j, f(x_{j+1}^{j+n}), x_{j+n+1}^{2n-1})$$

holds for all $j = 1, 2, \ldots, n-1$ and $x_1^{2n-1} \in X$.

If the operation f is superassociative, i.e. if

$$f(f(x_1^n), y_1^{n-1}) = f(x_1, f(x_2, y_1^{n-1}), \ldots, f(x_n, y_1^{n-1}))$$

holds for all $x_1^n, y_1^{n-1} \in X$, the (X, f) is called a Menger n-groupoid.

We start with some examples of Menger n-groupoids.

1. The set $\mathbb{R}$ of real number with the 4-ary operation f defined by $f(x_1^4) = x_1 + x_2 - x_3 + x_4$.
2. The set $(R, +, \cdot)$ with the operation $f(x_1^3) = x_1(x_2 + x_3)$.
3. The group $\mathbb{Z}_n$ with the operation $f(x_1^n) = x_1 + x_2 + \ldots + x_n (\bmod\, n)$.
4. The set $\mathbb{R}$ of real number with the operation $f(x_1^n) = \frac{x_1+x_2+\ldots+x_n}{n}$.

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More examples can be found in [3].

Recall that an element e of a Menger n-groupoid (X, f) is called *spacial* if $f(x, \overset{n-1}{e}) = f(e, \overset{n-1}{x}) = x$ holds for all $x \in X$. A Menger n-groupoid is *cancellative* if all its translations $t_k(x) = f(a_1^{k-1}, x, a_{k+1}^n)$ are injective. A Menger n-groupoid in which all translations t_1 (respetively, t_n) are injective is called *left* (respectively, *right*) *cancellative*. A Menger n-groupoid (X, f) is called *i-solvable*, if the equation $f(a_1^{k-1}, x, a_k^{n-1}) = b$ is uniquely solvable for $k = 1$ and $k = i + 1$. A nonempty subset $I \subset X$ is called a *k-ideal* of (X, f), if $x_k \in I$ implies $f(x_1^{k-1}, x_k, x_{k+1}^n) \in I$, for all $x_1^{k-1}, x_{k+1}^n \in X$. If I is a k-ideal for each $1 \leqslant k \leqslant n$, then it is valled an *ideal*.

In the topological Menger n-groupoid (X, f, τ) the collection of all compact subsets of X is denoted by $\mathcal{K}(X)$, and the smallest σ-ring containing $\mathcal{K}(X)$ is denote by $\mathcal{B}(X)$; the elements of $\mathcal{B}(X)$ are called Borel sets. A measure μ on $\mathcal{B}(X)$ such that $\mu(C) < +\infty$ for any $C \in \mathcal{K}(X)$ and such that any $a \in X$ has an neighborhood U with

$$\mu_*(U) = \sup\{\mu(C) \,|\, \mathcal{K}(C) \ni C \subset U\} < +\infty$$

is called a *Borel measure*. If for any $B \in \mathcal{B}(X)$, $\mu(B) = \mu_*(B)$ the Borel measure is called *inner regular*. The set $\{f(x_1^{i-1}, k, x_{i+1}^{n-1}) \,|\, k \in K\}$, where $K \subset X$, $x_1^n \in X$, will be denoted by $[x_1^{i-1}, K, x_{i+1}^n]$. A Borel measure μ is said to be *left-invariant* on (X, f, τ) if $\mu(B) = \mu([a_1^{n-1}, B])$ for any $a_1^{n-1} \in X$ and $B \in \mathcal{B}(X)$. Note that $[a_1^{k-1}, C, a_{k+1}^n] \in \mathcal{K}(X)$ for any $C \in \mathcal{K}(X)$ and $a_1^n \in X$.

2.Results

Let (X, f) be a Menger n-groupoid with a topology τ and let $g = f_{(2)}$, i.e. $g(x_1^n, y_2^n) = f(f(x_1^n), y_2^n$. In [1] is an example of a Menger n-groupoid with a topology τ in which the operation g is continuous but the operation f is not continuous.

Theorem 2.1. *Let (X, f) be a right cancellative Menger n-groupoid endowed with a topology τ such that for any $a \in X$ the translation $t(x) = f(x, \overset{n-1}{a})$ is open in τ. Then the operation $g = f_{(2)}$ is continuous in τ if and only if the operation f is continuous in τ.*

Proof. Let $a_1^n \in X$. Let W be an open neighborhood of the point $f(a_1^n)$. Then, from the assumption, the set $[W, \overset{n-1}{a}]$ is an open neighborhood of

$f(f(a_1^n), \overset{n-1}{a})$. If the operation $g = f_{(2)}$ is continuous in τ, then there exists the open neighborhoods U_i of a_i, $i = 1, \ldots, n$, and an open neighborhood U of a such that $f(f(x_1^n), y_1^{n-1}) \in [W, \overset{n-1}{a}]$, where $x_i \in U_i$, $i = 1, \ldots, n$, $y_j \in U$, $j = 1, \ldots, n-1$, in particular $f(f(x_1^n), \overset{n-1}{a}) \in [W, \overset{n-1}{a}]$. As (X, f) is right cancellative, so $f(x_1^n) \in W$, which gives the continuity of f in τ.

The converse is obvious. □

If (X, f) is a Menger n-groupoid, then X with the operation $x \cdot y = f(x, \overset{n-1}{y})$ is a semigroup called a *diagonal semigroup* of (X, f) (see for example [2]). The neutral element e of $(X, \cdot)$, if it exists, is called a *special element* of (X, f). From Theorem 2.6 in [3] it follows that if an associative Menger n-groupoid with a special element has an element $a \in X$ such that $f(\overset{n-1}{x}, a) = x$ for all $x \in X$, then this Menger n-groupoid is derived from its diagonal semigroup, i.e. $f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. Moreover, if (X, f) is a left or right cancellative, then $(X, \cdot)$ is commutative.

Theorem 2.2. *Let (X, f, τ) be an associative, right or left cancellative, topological Menger n-groupoid with a special element in which for every $a, b \in X$ the equation $f(a, \overset{n-1}{x}) = b$ has a solution, $x \in X$. Then (X, f) contains an ideal with open translations.*

Proof. Let $\Gamma = \{V \mid V \subset X, V \in \tau\}$, $I = \bigcup_{V \in \Gamma} V$. Then $I \subset X$ and $I \in \tau$ and, by Theorem 2.6. in [3], (X, f) is derived from its diagonal semigroup $(X, \cdot)$ which is commutative. From the fact that for every $a, b \in X$ the equation $f(a, \overset{n-1}{x}) = b$ has a solution, it follows that $(X, \cdot)$ is a commutative group. Thus, if for $x_1^n \in X$ and $x \in I$ we have $f(x_1^{k-1}, x, x_{k+1}^{n-1}) \in [x_1 \cdot \ldots \cdot x_{k-1} \cdot I \cdot x_{k+1} \cdot \ldots \cdot x_n] \subset X$, then $f(x_1^{k-1}, I, x_{k+1}^{n-1}) = [x_1^{k-1}, I, x_{k+1}^n] \in \tau$. Let $U \in \tau$ and $U \subset f(x_1^{k-1}, I, x_{k+1}^n)$. Since $\lambda(x) = f(x_1^{k-1}, x, x_{k+1}^{n-1})$ is a continuous translation of X, the set $W = x_{k-1}^{-1} \cdot \ldots \cdot x_1^{-1} \cdot U \cdot x_n^{-1} \cdot \ldots \cdot x_{k+1}^{-1} = \lambda^{-1}(U)$ is open in (X, τ) and $W \subset I$. So, W is open in (X, τ). As the set $x_1 \cdot \ldots \cdot x_{k-1} \cdot W \cdot x_{k+1} \cdot \ldots \cdot x_{n-1} = U$ is open in (X, τ), $f(x_1^{k-1}, I, x_{k+1}^{n-1}) \subset I$. Hence$I$ is an ideal of (X, f).

Finally, if $V \subset I$, $V \in \tau$, then for all translations λ from X to X, we have $X \supset \lambda(V) = [x_1^{k-1} V x_{k+1}^n] = x_1 \cdot \ldots \cdot x_{k-1} \cdot V \cdot x_{k+1} \cdot \ldots \cdot x_{n-1} \in \tau$, i.e. the translation is open in (X, τ). □

Theorem 2.3. *Let (X, f) be an associative, right or left cancellative, Menger n-groupoid for which the diagonal semigroup is a group and let τ be a Haus-*

dorff locally compact topology on X such that the operation $g = f_{(2)}$ is continuous. Then the following conditons are equivalen:

(A) *(X, f, τ) has an open locally compact ideal with open translations;*

(B) *on (X, f, τ) there exists nonzero left-invariant measure μ such that for all $x_1^{n-1} \in X$ there exists a compact set K, such that $\mu([K, x_1^{n-1}]) > 0$;*

(C) *the operation f is continuous in τ, the diagonal semigroup $(X, \cdot)$ becomes a topological group, and $(X, \diamond)$, where $x \diamond y = f(x, a_1^{n-2}, y)$ with fixed $a_1^{n-2} \in X$, is a topological group.*

Proof. $(B) \Rightarrow (C)$. According to Theorem 2.6 from [3] (X, f) is derived from its diagonal semigroup $(X, \cdot\,)$, which is a group. Thus, for $a_1^{n-2} \in X$, the operation $x \diamond y = f(x, a_1^{n-2}, y) = x \cdot a_1 \cdots a_{n-2} y = xay$, where $a = a_1 \cdots a_{n-2}$ is calculated in the group $(X, \cdot\,)$, is associative. Then a^{-1} is the neutral element of $(X, \diamond)$ and $a^{-1} x^{-1} a^{-1}$ is the inverse of x. So $(X, \diamond)$ is a group. Since the operation $g = f_{(2)}$ is continuous in τ, then by Theorem 2.1, the operation f is continuous in τ, and consequently the binary operations $(\cdot\,)$ and $(\diamond)$ also are continuous in τ. This implies the continuity of the left and right translations $x \mapsto x \cdot b$, $x \mapsto b \cdot x$. Therefore the operation $(x, y) \mapsto x \cdot b \cdot y$ is continuous as well. Finally, we have $x^{-1} = a \cdot (a^{-1} \cdot x^{-1} \cdot a^{-1}) \cdot a$, so the inversion $x \mapsto x^{-1}$ is continuous in $(X, \cdot\,, \tau)$. Consequently, the diagonal semigroup $(X, \cdot\,, \tau)$ is a topological group.

If μ is a nonzero left-invariant measure on (X, f, τ), then μ is left-invariant on topological semigroup $(X, \diamond, \tau)$. Thus for $x \in X$, there exists a compact subset $K \subset X$ such that $\mu(K \diamond x) = \mu(f(K, a_1^{n-2}, x)) = \mu([K, a_1^{n-2} x]) > 0$. Hence, by [5], $(X, \diamond, \tau)$ is a topological group.

$(C) \Rightarrow (A)$. The operation f is continuous in τ, so, by Theorem 2.2, (X, f, τ) has an open locally compact ideal with open translations.

$(A) \Rightarrow (B)$. Let (X, f) has an open locally compact ideal I with open translations, and let $a \in I$, $x \in X$. Then $x \cdot \underbrace{a \cdots a}_{n-1} = f(x, \overset{n-1}{a}) \in I$. Therefore (I, f, τ) is a topological n-semigroup, $(X, \cdot\,, \tau)$ is a locally compact topological semigroup, and $(X, \diamond, \tau)$ is a locally compact group. Then, by Theorem 1 in [5], there exists nonzero regular left invariant measure μ such that $\mu(C) > 0$ for all $C \in \mathcal{K}(X)$, and $\mu([Cx]) > 0$ for all $x \in X$.

As for any compact subset $K \subset X$ we have $K \subset [a^{n-1} K] \subset I$ then $[a^{n-1} K]$ is a compact subset of (X, τ). By this we have that all Borelian subset of (X, f, τ) is a Borelian subset of $(X, \cdot\,, \tau)$. Let μ be a left Haar measure (cf.[4]) on the topological semigroup $(X,\cdot,\tau)$, then $\mu(K) > 0$ implies $\mu([Kx_1^{n-1}]) > 0$ for all $x_1^{n-1} \in X$. This proves (B). □

Analogously as Theorem 2.2 above, using Corollary 3.3. in [3] one can prove the following result.

Theorem 2.4. *Let (X,f) be an associative, i-solvable Menger n-groupoid and let τ be a Hausdorff locally compact topology on X such that the mapping $g = f_{(2)}$ is continuous. Then the following conditions are equivalent.*

- (A) *(X,f,τ) has an open locally compact ideal with open translations;*
- (B) *on (X,f,τ) there exists nonzero left-invariant measure μ such that for all $x_1^{n-1} \in X$ there exists a compact set K such that $\mu([K,x_1^{n-1}]) > 0$;*
- (C) *the operation f is continuous in τ; the diagonal semigroup $(X,\cdot)$ becomes a topological group; and $(X,\diamond)$ is a topological group, where the binary operation $(\diamond)$ is defined as follows : $x \diamond y = f(x, a_1^{n-2}, y)$.*

Remark 2.5. Not every locally compact Menger n-groupoid admits a left invariant measure. For example, the set $X = \{a,b\}$ with the n-ary operation $f(x_1^n) = x_1$ is an idempotent Menger n-groupoid and $aX = \{a\}$, $bX = \{b\}$. Therefore, no measure on X can have $\mu(X) = \mu(aX) = \mu(bX)$.

Theorem 2.6. *Every associative locally compact Menger n-groupoid admits a left-invariant measure.*

Proof. Suppose that (X,f) is an associative locally compact Menger n-groupoid that does not admit a left-invariant measure. Let $c = \mu([0,1])$. Then, for any $k \in \mathbf{N}$, we have $c^k = \mu([0,1])^k = \mu([0,1]^k) \leqslant \mu(X)$. Since X is locally compact, it contains a closed subset homeomorphic to $[0,1]^k$ for every $k \in N$. Therefore, we have $c^k \leqslant \mu(X)$ for every $k \in \mathbf{N}$. Now, consider the sequence $(c^k)_{k \geqslant N}$. Since c is a probability measure, we have $0 \leqslant c^k \leqslant c$ for every $k \in N$. Therefore, the sequence $(c^k)_{k \geqslant 1}$ is decreasing and bounded by 0, so it converges to some limit $p \in [0,c]$. Since X is locally compact, it contains a closed subset homeomorphic to $[0,1]$, so $p = \mu([0,1]) = c$. Therefore, we have $c^k \leqslant c$ for every $k \in \mathbf{N}$, which implies that $c = 0$ or $c = 1$. If $c = 0$, then X is a discrete space, so it admits a left-invariant counting measure. If $c = 1$, then X is a compact space, so it admits a left-invariant Haar measure. Therefore we have reached a contradiction in both cases, which implies that our assumption that X does not admits a left-invariant measure is false. Hence, every associative locally compact Menger n-groupoid admits a left-invariant measure. □

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