Quasigroups and Related Systems **31** (2023), 221 – 232

https://doi.org/10.56415/qrs.v31.17

Endomorphisms of precyclic n-groups

Sonia Dog and Nikolay A. Shchuchkin

Abstract. We characterize the sets of homomorphisms, endomorphisms and automorphisms of n-ary groups with cyclic retracts.

1. Introduction

Polyadic groups, called also n-ary groups or n-groups, are a generalization of groups. Therefore, n-group theory is closely related to group theory. It is known that for every n-group (G, f) there exists a group $(G, *)$ and its automorphism φ such that $f(x_1, \dots, x_n) = x_1 * \varphi(x_2) * \dots * \varphi^{n-1}(x_n) * b$, $\varphi^{n-1}(x) * b = b * x$ and $\varphi(b) = b$ for some element $b \in G$ (see for example [2]). Then we write $(G, f) = \mathrm{der}_{\varphi,b}(G, *)$. If in the n-group operation f we fix all inner elements, we get the operation $\diamond$ that depends only on two outer elements. The algebra $(G, \diamond)$ obtained in this way is a group called the *retract* of (G, f). All retracts of an n-group $(G, f) = \mathrm{der}_{\varphi,a}(G, *)$ are isomorphic to the group $(G, *)$ (see [3]). Therefore, we can assume that $x \diamond y = f(x, a, \dots, a, y)$. We then write $(G, \diamond) = \mathrm{ret}_a(G, f)$. Moreover, for each $a \in G$, the mapping $\varphi(x) = f(\overline{a}, x, a, \dots, a)$ is an automorphism of the group $(G, \diamond)$ and $(G, f) = \mathrm{der}_{\varphi,b}(\mathrm{ret}_a(G, f))$ for $b = f(\overline{a}, \dots, \overline{a})$, where $\overline{a}$ is such that $f(a, \dots, a, \overline{a}) = a$ (see [3]). An n-group with an abelian retract is called *semiabelian*. In [5] it is shown that an n-group is semiabelian if and only if it is medial (entropic). In this case φ^{n-1} is the identity mapping.

An n-group with a cyclic retract is called *precyclic* (in Russian terminology – *semicyclic*). An infinite precyclic n-group is isomorphic to the n-group $(\mathbb{Z}, f_l) = \mathrm{der}_{1,l}(\mathbb{Z}, +)$, $0 \leqslant l \leqslant \frac{n-1}{2}$, or to the n-group $(\mathbb{Z}, f_{(-1)}) = \mathrm{der}_{-1,0}(\mathbb{Z}, +)$ (for odd n only) [6]. The first is type $(\infty, 1, l)$, the second type $(\infty, -1, 0)$. A finite precyclic n-group of order m is isomorphic to the n-group $\mathrm{der}_{1,l}(\mathbb{Z}_m, +)$ with $l | gcd(m, n-1)$ or to the n-group $\mathrm{der}_{k,l}(\mathbb{Z}_m, +)$,

2010 Mathematics Subject Classification: 20N15.

Keywords: Semiabelian n-group, precyclic n-group, endomorphism, automorphism, $(n, 2)$-semiring.

where $k > 1$, $gcd(k,m) = 1$, $k^{n-1} = 1(\operatorname{mod} m)$, $kl = l(\operatorname{mod} m)$ and $l|gcd(m,S_k)$, $S_k = 1 + k + k^2 + \ldots + k^{n-2} = \frac{k^{n-1}-1}{k-1}$. We say (cf. [6]) that the first is type $(m,1,l)$, the second is type (m,k,l).

First we will show that the set of all homomorphisms from a precyclic n-group into a semiabelian n-group forms an n-group. Next we characterize $(n,2)$-semirings of endomorphisms of precyclic n-groups. Some of our results were inspired by theorems proved in [7] and [8]. We give them in a more general, more useful version. We also provide new, simpler and shorter proofs.

For simplicity, the sequence $x_i, x_{i+1}, \ldots, x_j$ will be written as x_i^j; the sequence $x, x, \ldots, x$ (k times) as $\overset{(k)}{x}$. We also assume that $n > 2$.

2. Homomorphisms of precyclic n-groups

Using the mediality it is not difficult to see that the set $\operatorname{Hom}(G,G')$ of all homomorphisms of an n-group (G,f) into a semiabelian n-group (G',f') forms a semiabelian n-group with respect the n-ary operation F defined by

$$F(h_1,h_2,\ldots,h_n)(x) = f'(h_1(x),h_2(x),\ldots,h_n(x)),$$

where the homomorphism skew to h is defined by $\overline{h}(x) = \overline{h(x)}$.

Note that if an n-group (G',f') has no dempotents, the set $\operatorname{Hom}(G,G')$ may be empty. This is the case, for example, with the 5-groups $(\mathbb{Z}_6,f)$ and $(\mathbb{Z}_4,f')$ 1-derived from the additive groups $\mathbb{Z}_6$ an $\mathbb{Z}_4$, respectively. Indeed, for any homomorphism $h : (\mathbb{Z}_6,f) \to (\mathbb{Z}_4,f')$ there will be $h(0) = c$, $h(1) = hf(0,0,0,0,0) = f'(h(0),h(0),h(0),h(0),h(0)) = c+1$, $h(2) = hf(1,0,0,0,0) = h(1) + 4c + 1 = c + 2$. So, $h(k) = c + k(\operatorname{mod} 4)$. But then $h(1) = hf(1,4,0,0,0) = h(1) + h(4) + 3c + 1 = c + 2(\operatorname{mod} 4)$ which is impossible.

Let's start with lemmas that will be needed later. The first lemma is obvious, the second is a modification of Theorem 3 from [4]

Lemma 2.1. *Consider the diagram*

$$\begin{array}{ccc} (G,f) & \xrightarrow{\ \psi\ } & (H,f_1) \\ \downarrow{\scriptstyle \lambda_G} & & \downarrow{\scriptstyle \lambda_H} \\ (G',f') & \xrightarrow{\ \psi'\ } & (H',f_1') \end{array}$$

where ψ and ψ' are isomorphism of the corresponding n-groups. If λ_G, λ_H are homomorphisms of n-groups, and n-groups (G', f'), (H', f') are semiabelian, then $\mathrm{Hom}(G, G')$ *and* $\mathrm{Hom}(H, H')$ *form isomorphic n-groups. This isomorphism acts according to the rule* $\Phi(\alpha) = \psi'\alpha\psi^{-1}$.

The converse is not true. This is the case, for example, when G' has only one element.

Lemma 2.2. *A mapping h from an n-group* $\mathrm{der}_{\varphi,a}(G, *)$ *into a semiabelian n-group* $\mathrm{der}_{\psi,d}(G', \cdot)$ *is an n-group homomorphism if and only if there exists an element $c \in G'$ and a group homomorphism $\beta : (G, *) \to (G', \cdot)$ such that $\beta\varphi = \psi\beta$, $h = R_c\beta$ and $\beta(a) = D(c) \cdot d$, where $R_c(x) = x \cdot c$ for all $x \in G'$ and $D(c) = c \cdot \psi(c) \cdot \psi^2(c) \cdot \ldots \cdot \psi^{n-2}(c)$.*

Proof. Let $(G, f) = \mathrm{der}_{\varphi,a}(G, *)$ and $(G', f') = \mathrm{der}_{\psi,d}(G', \cdot)$ be two n-groups and let (G', f') be semiabelian.

If there exists a group homomorphism $\beta : (G, *) \to (G', \cdot)$ such that $\beta\varphi = \psi\beta$ and $\beta(a) = D(c) \cdot d$ for some fixed $c \in G'$, then for $h(x) = \beta(x) \cdot c$ we have

$$
\begin{aligned}
h(f(x_1^n)) &= \beta(f(x_1^n)) \cdot c = \beta(x_1 * \varphi(x_2) * \ldots * \varphi^{n-1}(x_n) * a) \cdot c \\
&= \beta(x_1) \cdot \beta\varphi(x_2) \cdot \ldots \cdot \beta\varphi^{n-1}(x_n) \cdot \beta(a) \cdot c \\
&= \beta(x_1) \cdot \psi\beta(x_2) \cdot \ldots \cdot \psi^{n-1}\beta(x_n) \cdot D(c) \cdot d \cdot c \\
&= \beta(x_1) \cdot \psi\beta(x_2) \cdot \ldots \cdot \psi^{n-1}\beta(x_n) \cdot c \cdot \psi(c) \cdot \ldots \cdot \psi^{n-2}(c) \cdot d \cdot c \\
&= (\beta(x_1) \cdot c) \cdot \psi(\beta(x_2) \cdot c) \cdot \ldots \cdot \cdot \psi^{n-1}(\beta(x_n) \cdot c) \cdot d \\
&= h(x_1) \cdot \psi h(x_2) \cdot \ldots \cdot \psi^{n-1} h(x_n) \cdot d \\
&= f'(h(x_1), h(x_2), \ldots, h(x_n)).
\end{aligned}
$$

Hence $h : G \to G'$ is an n-group homomorphism.

Conversely, let $h : (G, f) \to (G', f')$ be an n-group homomorphism and $(G, \circ) = \mathrm{ret}_a(G, f)$, $(G', \diamond) = \mathrm{ret}_b(G', f')$. Then $\beta : (G, \circ) \to (G', \diamond)$ defined by $\beta'(x) = f'(h(x), \overset{(n-2)}{h(a)}, \overline{a})$ is a homomorphism. Since $\overline{a}$ and $\overline{b}$ are neutral elements of these groups, $\beta'(\overline{a}) = \overline{b}$.

Let $\overline{a} = h(g)$ for some $g \in G$. Then

$$\beta'(x) = f'(h(x), \overset{(n-2)}{h(a)}, \overline{a}) = f'(h(x), \overset{(n-2)}{h(a)}, h(g)) = h(f(x, \overset{(n-2)}{a}, g)) = h(x \circ g).$$

Thus $h(\overline{a}) = h(g^{-1} \circ g) = h(g^{-1})$.

Now, denoting $h(g^{-1})$ by c', we obtain

$$h(x) = h(x \circ g^{-1} \circ g) = \beta'(x \circ g^{-1}) = \beta'(x) \diamond \beta'(g^{-1}) = \beta'(x) \diamond c'.$$

All retracts of an n-group $\mathrm{der}_{\varphi,b}(G,\star)$ are isomorphic to $(G,\star)$ (cf. [3]), so $(G,\circ)$ and $(G,*)$, also $(G',\diamond)$ and $(G',\cdot)$, are isomorphic. Thus, a group homomorphism β' corresponds to some homomorphism $\beta : (G,*) \to (G,\cdot)$. Hence $h(x) = \beta(x) \cdot c$, i.e. $h = R_c\beta$ for some $c \in G'$.

Since $h : (G,f) \to (G',f')$ is a homomorphism of n-groups,

$$h(f(x_1^n)) = f'(h(x_1), h(x_2), \ldots, h(x_n))$$

implies

$$\beta(f(x_1^n)) \cdot c = f'(\beta(x_1) \cdot c, \beta(x_2) \cdot c, \ldots, \beta(x_n) \cdot c).$$

Consequently,

$$\begin{aligned}&\beta(x_1) \cdot \beta\varphi(x_2) \cdot \beta\varphi^2(x_3) \cdot \ldots \cdot \beta\varphi^{n-1}(x_n) \cdot \beta(a) \cdot c\\ &= (\beta(x_1) \cdot c) \cdot \psi(\beta(x_2) \cdot c) \cdot \psi^2(\beta(x_3) \cdot c) \cdot \ldots \cdot \psi^{n-1}(\beta(x_n) \cdot c) \cdot d.\end{aligned}$$

From this, putting $x_i = \overline{a}$ for all $i = 1, 2, \ldots, n$, we obtain

$$\beta(a) \cdot c = c \cdot \psi(c) \cdot \psi^2(c) \cdot \ldots \cdot \varphi^{n-1}(c) \cdot b = D(c) \cdot \psi^{n-1}(c) \cdot d = D(c) \cdot d \cdot c,$$

which shows that $\beta(a) = D(c) \cdot d$.

Putting in the previous identity $x_2 = x$ and $x_i = \overline{a}$ for other x_i we get

$$\begin{aligned}\beta\varphi(x) \cdot \beta(a) \cdot c &= c \cdot \psi\beta(x) \cdot \psi(c) \cdot \psi^2(c) \cdot \ldots \cdot \varphi^{n-1}(c) \cdot d\\ &= \psi\beta(x) \cdot D(c) \cdot d \cdot c = \psi\beta(x) \cdot \beta(a) \cdot c.\end{aligned}$$

Thus $\beta\varphi = \psi\beta$, which completes the proof. □

As a consequence of the above lemma we obtain

Corollary 2.3. *A mapping h from an n-group $\mathrm{der}_{\varphi,a}(G,*)$ into a semiabelian n-group $\mathrm{der}_{\psi,d}(G',\cdot)$ is an n-group homomorphism if and only if there exists an element $c \in G'$ such that $\beta = h \cdot c^{-1}$ is a group homomorphism from $(G,*)$ into $(G',\cdot)$, $\beta\varphi = \psi\beta$ and $\beta(a) = D(c) \cdot d$.*

Let $(G,f) = \mathrm{der}_{\varphi,a}(G,*)$ and $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$. If (G',f') is a semiabelian n-group, then each homomorphism $h_i \in \mathrm{Hom}(G,G')$ has the form $h_i = R_{c_i}\beta_i$, where β_i and c_i are as in the above lemma. Consequently,

$$
\begin{aligned}
F(h_1^n)(x) &= f'(h_1(x), h_2(x), \dots, h_{n-1}(x), h_n(x)) \\
&= f'(\beta_1(x)\cdot c_1, \beta_2(x)\cdot c_2, \dots, \beta_{n-1}(x)\cdot c_{n-1}, \beta_n(x)\cdot c_n) \\
&= (\beta_1(x)\cdot c_1)\cdot \psi(\beta_2(x)\cdot c_2)\cdot \ldots \cdot \psi^{n-2}(\beta_{n-1}(x)\cdot c_{n-1})\cdot(\beta_n(x_n)\cdot c_n)\cdot d \\
&= \beta_1(x)\cdot\psi\beta_2(x)\cdot\ldots\cdot\psi^{n-2}\beta_{n-1}(x)\cdot\beta_n(x)\cdot c_1\cdot\psi(c_2)\cdot\ldots\cdot\psi^{n-2}(c_{n-1})\cdot c_n\cdot d \\
&= \beta_1(x)\cdot\psi\beta_2(x)\cdot\ldots\cdot\psi^{n-2}\beta_{n-1}(x)\cdot\beta_n(x)\cdot f'(c_1^n) = \beta(x)\cdot f'(c_1^n),
\end{aligned}
$$

where $\beta = \beta_1\cdot\psi\beta_2\cdot\ldots\cdot\psi^{n-2}\beta_{n-1}\cdot\beta_n = \beta_1\cdot\beta_2\varphi\cdot\ldots\cdot\beta_{n-1}\varphi^{n-2}\cdot\beta_n$ is a homomorphism from $(G,*)$ to $(G',\cdot)$. Thus,

$$
F(h_1^n) = R_u\beta, \quad \text{where} \quad u = f'(c_1^n), \quad \beta = \beta_1\cdot\beta_2\varphi\cdot\ldots\cdot\beta_{n-1}\varphi^{n-2}\cdot\beta_n. \tag{1}
$$

Let (G',f') be a semiabelian n-group. Then $(G',\cdot) = \mathrm{ret}_a(G',f')$ is an abelian group (for any $a\in G'$) and $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$ for $d = f'(\overset{(n)}{\overline{a}})$ and $\psi(x) = f'(\overline{a}, x, \overset{(n-2)}{a})$. Moreover, $D(x) = x\cdot\psi(x)\cdot\psi^2(x)\cdot\cdots\cdot\psi^{n-2}(x)$ is an endomorphism of $(G',\cdot)$ such that $\psi(d\cdot D(x)) = d\cdot D(x) = f'(\overset{(n-1)}{x}, \overline{a})$ for every $x\in G'$.

We will use these facts to describe the set of homomorphisms of precyclic n-groups. We'll start with precyclic n-groups of type $(\infty,1,l)$.

First, for $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$ and an arbitrary natural l we define the set

$$
G'_{(l,d)} = \{(z,c)\,|\,\psi(z) = z,\ z^l = d\cdot D(c)\} \subseteq G'\times G'.
$$

Using the mediality of (G',f') and the above facts, we can see that $G'_{(l,d)}$ with the operation

$$
g'((z_1,c_1),(z_2,c_2),\dots,(z_n,c_n)) = (z_1\cdot z_2\cdot\ldots\cdot z_n, f'(c_1^n)) \tag{2}
$$

is a semiabelian n-group.

Theorem 2.4. *If the set of all homomorphisms from a precyclic n-group (G,f) of type $(\infty,1,l)$ into a semiabelian n-group $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$ is nonempty, then it forms an n-group isomorphic to the n-group $(G'_{(l,d)}, g')$.*

Proof. Any precyclic n-group of type $(\infty,1,l)$ is isomorphic to the n-group $(\mathbb{Z},f_l) = \mathrm{der}_{1,l}(\mathbb{Z},+)$. Let h be a homomorphism from $(\mathbb{Z},f_l)$ into a semiabelian n-group $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$. Then, according to Lemma 2.2, $h = R_c\beta$ for some homomorphism β from $(\mathbb{Z},+)$ to $(G',\cdot)$, $\beta(x) = \psi(\beta(x))$ and $\beta(l) = d\cdot D(c)$ for some $c\in G'$. Any homomorphism β of a cyclic group is determined by the value of β on the generator of this group. So, if $\beta(1) = z$, then $z = \beta(1) = \psi\beta(1) = \psi(z)$ and $z^l = \beta(l) = d \cdot D(c)$. Hence, any homomorphism $h : (\mathbb{Z}, f_l) \to (G', f')$ determines one pair $(z, c) \in G'_{(l,d)}$.

On the other side, for each pair $(z, c) \in G'_{(l,d)}$ there is only one homomorphism $\beta : (\mathbb{Z}, +) \to (G', \cdot)$ such that $\beta(1) = z$. Hence $\beta(k) = z^k$. Thus $\psi\beta(k) = \psi(z^k) = \psi(z)^k = z^k = \beta(k)$ for every $k \in \mathbb{Z}$. So, $\psi\beta = \beta$ and $\beta(l) = z^l = d \cdot D(c)$.

This shows that the pair (z, c) uniquely determines the homomorphism $h = R_c\beta$ with $\beta(1) = z$. So, there is one-to-one correspondence between elements of the set $\mathrm{Hom}(G, G')$ and elements of the set $G'_{(l,d)}$. Denote this correspondence by τ, i.e. $\tau(h_i) = (z_i, c_i)$ for $h_i = R_{c_i}\beta_i$ and $z_i = \beta_1(1)$. Then $\beta_i(k) = \beta_i(k1) = \beta_i(1)^k = z_i^k$.

Since $\beta(1) = (\beta_1 \cdot \beta_2 \cdot \ldots \cdot \beta_n)(1) = z_1 \cdot z_2 \cdot \ldots \cdot z_n$,

$$\begin{aligned}\tau(F(h_1^n)) &= (z_1 \cdot z_2 \cdot \ldots \cdot z_n, f'(c_1^n)) = g((z_1, c_1), (z_2, c_2), \ldots, (z_n, c_n)) \\ &= g(\tau(h_1), \tau(h_2), \ldots, \tau(h_n)).\end{aligned}$$

Hence τ is an isomorphism. □

Since $z^k = \beta(k) = e$ for $k \in \mathrm{Ker}\,\beta$, the first coordinate of each pair $(z, c) \in G'_{(l,d)}$ has finite order in the group $(G', \cdot)$.

All precyclic n-groups of type $(\infty, -1, 0)$ are idempotent and exist only for odd n. All such n-groups can be identified with the n-group $(\mathbb{Z}, f_{(-1)}) = \mathrm{der}_{-1,0}(\mathbb{Z}, +)$. The homomorphic image of the idempotent n-group is also the n-idempotent group. This means that the homomorphism from $(\mathbb{Z}, f_{(-1)})$ into the n-group (G', f') exists only if (G', f') has at least one idemotent. By Lemma 2.2, any such homomorphism has the form $h = R_c\beta$, where $\beta(0) = D(c) \cdot d$ and $\psi\beta(x) = \beta(x)^{-1}$ for $x \in \mathbb{Z}$. So, $D(c) = d^{-1}$ and $\psi(z) = z^{-1}$ for $z \in \beta(\mathbb{Z})$. Moreover, $h(0) = R_c\beta(0) = c$. Consequently, $c = h(0) = hf_{(-1)}(\overset{(n)}{0}) = f'(\overset{(n)}{h(0)}) = f'(\overset{(n)}{c})$. Thus as a consequence of Lemma 2.2 we obtain

Lemma 2.5. *A mapping h from an n-group $\mathrm{der}_{-1,0}(\mathbb{Z}, +)$ into a semiabelian n-group $\mathrm{der}_{\psi,d}(G', \cdot)$ is an n-group homomorphism if and only if there exists an idempotent $c \in G'$ and a group homomorphism $\beta : (\mathbb{Z}, +) \to (G', \cdot)$ such that $h = R_c\beta$, $D(c) = d^{-1}$ and $\beta(x)^{-1} = \psi\beta(x)$ for $x \in \mathbb{Z}$.*

The proofs of the following theorems is very similar to the proof of Theorem 2.4. So we skip them.

Theorem 2.6. *If the set of all homomorphisms from a precyclic n-group of type $(\infty,-1,0)$ into a semiabelian n-group $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$ is nonempty, then it forms an n-group isomorphic to the n-group (G''_d, g''), where*

$$G''_d = \{(z,c)\,|\,\psi(z) = z^{-1},\ D(c) = d^{-1}\} \subseteq G' \times G' \quad \text{and}$$

$$g''((z_1,c_1),(z_2,c_2),\ldots,(z_n,c_n)) = (z_1\cdot z_2^{-1}\cdot z_3\cdot z_4^{-1}\cdot\ldots\cdot z_{n-1}^{-1}\cdot z_n, f'(c_1^n)).$$

Theorem 2.7. *If the set of all homomorphisms from a precyclic n-group of type (m,k,l) with $k \geqslant 1$, into a semiabelian n-group $(G',f') = \mathrm{der}_{\psi,d}(G',\cdot)$ is nonempty, then it forms an n-group isomorphic to the n-group $(G'_{(l,d)}, g')$.*

Example 2.8. Let us consider three 5-groups: $(G_1,f_1) = \mathrm{der}_{5,3}(\mathbb{Z}_6,+)$, $(G_2,f_1) = \mathrm{der}_{1,1}(\mathbb{Z},+)$ and $(G',f') = \mathrm{der}_{1,1}(\mathbb{Z}_4,+)$. Then, as already mentioned, the set $\mathrm{Hom}(G_1,G')$ is empty. The set $\mathrm{Hom}(G_1,G')$ contains four homomorphisms. They are defined by $h_c(x) = r + c(\mathrm{mod}\,4)$, where $x = 4t+r$, $0 \leqslant r < 4$ and $c = 0,1,2,3$. $\mathrm{Hom}(G',G')$ also contains four homomorphisms, namely $h_c(x) = x + c(\mathrm{mod}\,4)$, $c = 0,1,2,3$.

3. Endomoprhisms of precyclic n-groups

Recall that an $(n,2)$-*nearring* $(G,f,\cdot)$ is an n-group (G,f) with an associative multiplication such that

$$a\cdot f(x_1^n) = f(\{a\cdot x_i\}_1^n) \quad \text{and} \quad f(x_1^n)\cdot a = f(\{x_i\cdot a\}_1^n)$$

for all $a, x_1^n \in G$. An $(n,2)$-nearring $(G,f,\cdot)$ with a semiabelian n-group (G,f) is called an $(n,2)$-*semiring*; with an abelian n-group – an $(n,2)$-*ring*.

In [5] it is noted that the set $\mathrm{End}(G,f)$ of all endomorphisms of a semiabelian n-group (G,f) forms an $(n,2)$-semiring with respect to the n-ary operation F defined as for homomorphisms and an ordinary superposition of endomorphisms. The set of all endomorphisms of an abelian n-group forms an $(n,2)$-ring with unity.

Based on the results of the previous section, we can characterize $(n,2)$-semirings of endomorphisms of precyclic n-groups. For this we will use the following lemma which is a consequence of Lemma 2.2.

Lemma 3.1. *A mapping $h : \mathbb{Z} \to \mathbb{Z}$ is an endomorphism of an n-group $(\mathbb{Z}, f_l)$ of type $(\infty, 1, l)$ if and only if here exists an element $c \in \mathbb{Z}$ and an endomorphism β of $(\mathbb{Z}, +)$ such that $h = R_c\beta$ and $\beta(l) = (n-1)c + l$.*

Let $h = R_c\beta$ be an endomorphism of $(\mathbb{Z}, f_l)$. Then $h(0) = c$. Hence, if $\beta(1) = m$, then $\beta(l) = \beta(l1) = l\beta(1) = lm$. So, $lm = (n-1)c + l$, i.e. for fixed m, n and l there is only one c satisfying this equation. This means that each endomorphism of $(\mathbb{Z}, f_l)$ depends only on m and has the form $h_m(x) = xm + c_m$, where $c_m = h_m(0)$ and $ml = l(\mathrm{mod}\,(n-1))$. So, $\tau(h_m) = m$ is a bijection from the set $\mathrm{End}(\mathbb{Z}, f_l)$ onto the set

$$\mathbb{Z}_{(l,n)} = \{m \,|\, ml = l(\mathrm{mod}\,(n-1))\} \subseteq \mathbb{Z}.$$

This is an $(n, 2)$-semiring with respect to the operation

$$g'(m_1, m_2, \ldots, m_n) = m_1 + m_2 + \ldots + m_n$$

and an ordinary multiplication of numbers.

Since $lm = (n-1)c + l$ means that $c = \frac{l(m-1)}{n-1}$, we have

$$\begin{aligned}
&F(h_{m_1}, h_{m_2}, \ldots, h_{m_n})(z) = f_l(h_{m_1}(z), h_{m_2}(z), \ldots, h_{m_n}(z))\\
&= (zm_1 + c_{m_1}) + (zm_2 + c_{m_2}) + \ldots + (zm_n + c_{m_n}) + l\\
&= z(m_1 + m_2 + \ldots + m_n) + f_l(c_{m_1}, c_{m_2}, \ldots, c_{m_n})\\
&= z(m_1 + m_2 + \ldots + m_n) + \tfrac{l(m_1+m_2+\ldots+m_n-n)}{n-1} + l\\
&= z(m_1 + m_2 + \ldots + m_n) + \tfrac{l(m_1+m_2+\ldots+m_n-1)}{n-1}\\
&= z(m_1 + m_2 + \ldots + m_n) + c_{m_1+m_2+\ldots+m_n} = h_{m_1+m_2+\ldots+m_n}(z).
\end{aligned}$$

Hence $\tau(F(h_{m_1}, h_{m_2}, \ldots, h_{m_n})) = g'(\tau(h_{m_1}), \tau(h_{m_2}), \ldots, \tau(h_{m_n}))$.

Also $\tau(h_{m_1} \circ h_{m_2}) = \tau(h_{m_1}) \cdot \tau(h_{m_2})$.

So, τ is an isomorphism between $(\mathrm{End}(\mathbb{Z}, f_l), F, \circ)$ and $(\mathbb{Z}_{(l,n)}, g', \cdot)$.

Theorem 3.2. *The set of endomorphisms of a precyclic n-group of type $(\infty, 1, l)$ forms an $(n, 2)$-semiring isomorphic to $(\mathbb{Z}_{(l,n)}, g', \cdot)$.*

Endomorphisms of precyclic n-groups of type $(\infty, -1, 0)$ are characterized by

Lemma 3.3. *A mapping $h : \mathbb{Z} \to \mathbb{Z}$ is an endomorphism of a precyclic n-group of type $(\infty, -1, 0)$ if and only if $h(x) = mx + c$ for some $m, c \in \mathbb{Z}$.*

Using the same method as in the proof of Theorem 3.2 we obtain

Theorem 3.4. *The set of all endomorphisms of a precyclic n-group $(\mathbb{Z}, f_{(-1)})$ of type $(\infty, -1, 0)$ forms an $(n,2)$-semiring isomorphic to the $(n,2)$-semiring $(\mathbb{Z}\times\mathbb{Z}, g, *)$, where*

$$g((m_1,c_1),(m_2,c_2),\ldots,(m_n,c_n)) = (f_{(-1)}(m_1^n), f_{(-1)}(c_1^n)) \quad \textit{and}$$

$$(m_1,c_1)*(m_2,c_2) = (m_1m_2, m_1c_2+c_1).$$

For endomorphisms of precyclic n-groups of type (m,k,l) we have

Theorem 3.5. *A mapping $h\colon \mathbb{Z}_m \to \mathbb{Z}_m$ is an endomorphism of an n-group $(\mathbb{Z}_m, f_{(k,l)})$ if and only if $h(x) = tx + c(\operatorname{mod} m)$ and $tl = S_kc + l(\operatorname{mod} m)$ for some $t, c \in \mathbb{Z}_m$. Such endomorphisms forms an $(n,2)$-semiring isomorphic to the $(n,2)$-semiring $(\mathbb{Z}_m^{(k,b)}, g, *)$, where*

$$\mathbb{Z}_m^{(k,l)} = \{(t,c)\,|\, t,c\in\mathbb{Z}_m,\ tl = S_kc + l(\operatorname{mod} m)\},$$

$$g((t_1,c_1),(t_2,c_2),\ldots,(t_n,c_n)) = (f_{(k,0)}(t_1^n), f_{(k,l)}(c_1^n)) \quad \textit{and}$$

$$(t_1,c_1)*(t_2,c_2) = (t_1t_1, t_1c_2+c_1).$$

Proof. Each endomorphism of $(\mathbb{Z}_m, +)$ has the form $\beta(x) = tx(\operatorname{mod} m)$. Hence, by Lemma 2.2, $h\colon \mathbb{Z}_m \to \mathbb{Z}_m$ is an endomorphism of an n-group $(\mathbb{Z}_m, f_{(k,l)})$ if and only if $h(x) = \beta(x) + c = tx + c(\operatorname{mod} m)$ for some $c \in \mathbb{Z}_m$ such that $\beta(l) = D(c) + l$. But $D(c) = c + kc + k^2c + \ldots + k^{n-2}c = S_kc$. So, $\beta(l) = tl = S_kc + l(\operatorname{mod} m)$. In $\mathbb{Z}_m$ there is only one c satisfying this equation. Indeed, if $\beta(l) = D(c) + l$, then $\beta(x) + d = h(x) = \beta(x) + c$, whence $c = d(\operatorname{mod} m)$. Thus, $\tau(h) = (t,c)$ is a bijection between the set of all endomorphism of $(\mathbb{Z}_m, f_{(k,l)})$ and $\mathbb{Z}_m^{(k,l)}$.

Moreover, for all $h_1,\ldots,h_n \in \operatorname{End}(\mathbb{Z}_m, f_{(k,l)})$ we have

$$F(h_1^n)(x) = f_{(k,l)}(h_1(x), h_2(x),\ldots,h_n(x))$$
$$= (t_1x+c_1)+k(t_1+c_1)+k^2(t_2x+c_2)+\ldots+k^{n-2}(t_{n-1}x+c_{n-1})+(t_nx+c_n)+l$$
$$= (t_1 + kt_2 + \ldots + k^{n-2}t_{n-1} + t_n)x + (c_1 + kc_2 + \ldots + k^{n-2}c_{n-1} + c_n + l)$$
$$= f_{(k,0)}(x) + f_{(k,l)}(c_1^n) = h_{f_{(k,0)}} + f_{(k,l)}.$$

Hence

$$\tau(F(h_1^n)) = (f_{(k,0)}, f_{(k,b)}) = g((t_1,c_1),(t_2,c_2),\ldots,(t_n,c_n))$$
$$= g(\tau(h_1),\tau(h_2),\ldots,\tau(h_n)),$$

which shows that τ is an isomorphism. □

Observe that in the above proof for fixed k and l the element c is uniquely determined by t, so an endomorphism $h = R_c\beta$ of $(\mathbb{Z}_m, f_{(k,l)})$ is uniquely determined by the value of $t = \beta(1)$. Thus, the the set $\mathbb{Z}_m^{(k,l)}$ can be identified with the set $\mathbb{P}_m^{(k,l)} = \{t \in \mathbb{Z}_m \,|\, tb = S_k c + l (\operatorname{mod} m)\}$. Consequently, the $(n,2)$-semiring $(\mathbb{Z}_m^{(k,l)}, g, *)$ can be identified with the $(n,2)$-semiring $(\mathbb{P}_m^{(k.l)}, f_{(k,0)}, \cdot)$, where $\cdot$ is an ordinary multiplication modulo m.

4. Automoprhisms of precyclic n-groups

A binary composition (superposition) of automorphisms of a fixed n-group is an automorphism of this n-group. Thus for a given n-group (G,f) the set $\operatorname{Aut}(G,f)$ of all its automorphisms is a group contained in the semigroup $\operatorname{End}(G,f)$. Hence, as a consequence of the above results, we obtain

Proposition 4.1. *A mapping $h : G \to G$ is an automorphism of a semiabelian n-group $(G,f) = \operatorname{der}_{\varphi,a}(G,\cdot)$ if and only if there exists $c \in G$ and an automorphism β of $(G,\cdot)$ such that $\beta\varphi = \varphi\beta$, $h = R_c\beta$ and $\beta(a) = D(c)\cdot a$.*

Theorem 3.10 in [1] implies the following characterization:

Proposition 4.2. *A mapping $h : G \to G$ is an automorphism of a semiabelian n-group $(G,f) = \operatorname{der}_{\varphi,a}(G,\cdot)$ if and only if $h = R_c\beta$, where β is an automorphism of $(G,\cdot)$, $\beta(a) = a$ and $\varphi(c) = c = c^n$.*

Corollary 4.3. *Let $(G,f) = \operatorname{der}_{\varphi,a}(G,\cdot)$ be a precyclic n-group, $c \in G$ and $\beta \in \operatorname{Aut}(G,\cdot)$. Then $h = R_c\beta \in \operatorname{Aut}(G,f)$ if and only if $R_c \in \operatorname{Aut}(G,f)$ and $\beta \in \operatorname{Aut}(G,f)$.*

Proof. If $h = R_c\beta$ is an automorphism of $(G,f) = \operatorname{der}_{\varphi,a}(G,\cdot)$, then, by the above Propositions, $\beta\varphi = \varphi\beta$ and $\beta(a) = a$. Hence, as it is not difficult to see, β is an automorphism of (G,f). Consequently, also $R_c = h\beta^{-1}$ is an automorphism of (G,f). The converse statement is obvious. □

The above fact also follows from the results proven in [1].

Theorem 4.4. *If $(G,f) = \operatorname{der}_{\varphi,a}(G,\cdot)$ is a precyclic n-group, then*

$$\operatorname{Aut}(G,f) \cong \mathcal{R}_\varphi(G,f) \ltimes \operatorname{Aut}_a(G,\cdot),$$

where

$$\mathcal{R}_\varphi(G,f) = \{R_c \,|\, \varphi(c) = c = c^n\} \quad \text{and}$$
$$\operatorname{Aut}_a(G,\cdot) = \{\beta \in \operatorname{Aut}(G,\cdot) \,|\, \beta(a) = a\}.$$

Proof. $\mathcal{R}_\varphi(G,f)$ and $\mathrm{Aut}_a(G,\cdot)$ are subgroups of $\mathrm{Aut}(G,f)$ and $\mathrm{Aut}(G,\cdot)$, respectively. $(G,\cdot)$ is abelian, so $\mathcal{R}_\varphi(G,f)$ is a normal subgroup. Moreover, if $\psi\in\mathcal{R}_\varphi(G,f)\cap\mathrm{Aut}_a(G,\cdot)$, then $\varphi=R_c=\beta$. Thus, $R_c(a)=\beta(a)=a$, which gives $c=e$. Therefore, $\mathcal{R}_\varphi(G,f)\cap\mathrm{Aut}_a(G,\cdot)=\{\varepsilon\}$. Consequently, $\mathrm{Aut}(G,f)\cong\mathcal{R}_\varphi(G,f)\ltimes\mathrm{Aut}_a(G,\cdot)$. □

Theorem 4.5. *If a precyclic n-group $(G,f)=\mathrm{der}_{\varphi,a}(G,\cdot)$ has at least one idempotent, then*

$$\mathrm{Aut}(G,f)\cong\mathcal{R}_{E(G,f)}\ltimes\mathrm{Aut}(G,\cdot),$$

where $\mathcal{R}_{E(G,f)}$ is a group of right translations of $(G,\cdot)$ determined by idempotent elements.

Proof. Let $(G,f)=\mathrm{der}_{\varphi,a}(G,\cdot)$ be a precyclic n-group containing at least one idempotent. We will show first that (G,f) is isomorphic to $(G,g)=\mathrm{der}_\varphi(G,\cdot)$.

Let c be an idempotent of (G,f). Then

$$c=f(c,c,\dots,c)=c\cdot\varphi(c)\cdot\varphi^2(c)\cdot\ldots\cdot\varphi^{n-2}(c)\cdot c\cdot a. \tag{3}$$

Thus,

$$a\cdot c^{-1}=c^{-1}\cdot\varphi(c^{-1})\cdot\varphi^2(c^{-1})\cdot\ldots\cdot\varphi^{n-2}(c^{-1})\cdot c^{-1}. \tag{4}$$

Hence

$$\begin{aligned}&R_{c^{-1}}f(x_1^n)=x_1\cdot\varphi(x_2)\cdot\varphi^2(x_3)\cdot\ldots\cdot\varphi^{n-2}(x_{n-1})\cdot x_n\cdot a\cdot c^{-1}\\ &\overset{(4)}{=}x_1\cdot c^{-1}\cdot\varphi(x_2)\cdot\varphi(c^{-1})\cdot\varphi^2(x_3)\cdot\varphi^2(c^{-1})\cdot\ldots\cdot\varphi^{n-2}(x_{n-1})\cdot\varphi^{n-2}(c^{-1})\cdot x_n\cdot c^{-1}\\ &=x_1\cdot c^{-1}\cdot\varphi(x_2\cdot c^{-1})\cdot\varphi^2(x_3\cdot c^{-1})\cdot\ldots\cdot\varphi^{n-2}(x_{n-1}\cdot c^{-1})\cdot x_n\cdot c^{-1}\\ &=R_{c^{-1}}(x_1)\cdot\varphi R_{c^{-1}}(x_2)\cdot\varphi^2 R_{c^{-1}}\cdot\ldots\cdot\varphi^{n-2}R_{c^{-1}}(x_{n-1})\cdot R_{c^{-1}}(x_n)\\ &=g(R_{c^{-1}}(x_1),R_{c^{-1}}(x_2),\dots,R_{c^{-1}}(x_n)).\end{aligned}$$

Therefore $R_{c^{-1}}:(G,f)\to(G,g)$ is a homomorphism. Since it is a bijection, $(G,f)\cong(G,g)$. Then also $\mathrm{Aut}(G,f)\cong\mathrm{Aut}(G,g)$ and $\mathcal{R}_{E(G,f)}\cong\mathcal{R}_{E(G,g)}$. So it is sufficient to prove our theorem for (G,g).

The neutral element of $(G,\cdot)$ is an idempotent of (G,g). Thus the set $\mathcal{R}_{E(G,g)}$ is nonempty and $R_bR_c=R_{c\cdot b}$ for all $R_c,R_b\in\mathcal{R}_{E(G,g)}$ because, by (3), $c{\cdot}b$ is an idempotent. Thus $\mathcal{R}_{E(G,g)}$ is a subgroup of $\mathrm{Aut}(G,g)$ such that $(R_b\beta)^{-1}\circ R_c\circ R_b\beta=R_{\beta^{-1}(c)}$ for $R_b\beta\in\mathrm{Aut}(G,g)$ and $R_c\in\mathcal{R}_{E_{(G,g)}}$. Since, $\beta^{-1}(c)=\beta^{-1}g(c,c,\dots,c)=g(\beta^{-1}(c),\beta^{-1}(c),\dots,\beta^{-1}(c))$, by Corollary

4.3, $\beta^{-1}(c)$ is an idempotent of (G, g). Consequently, $R_{\beta^{-1}(c)} \in \mathcal{R}_{E(G,g)}$, which shows that $\mathcal{R}_{E(G,g)}$ is a normal subgroup of $\mathrm{Aut}(G, g)$. Moreover, $\mathcal{R}_{E(G,g)} \cap \mathrm{Aut}(G, \cdot) = \{\varepsilon\}$. So, $\mathrm{Aut}(G, g) \cong \mathcal{R}_{E(G,g)} \ltimes \mathrm{Aut}(G, \cdot)$. □

Corollary 4.6. *If a precyclic n-group $(G, f) = \mathrm{der}_{\varphi,a}(G, \cdot)$ has only one idempotent, then*

$$\mathrm{Aut}(G, f) \cong \mathrm{Aut}(G, \cdot)$$

References

[1] **W.A. Dudek,** *Automorphisms of n-ary groups*, Results Math. **77** (2022), paper no. 46.

[2] **W.A. Dudek, K. Głazek,** *Around the Hosszú-Gluskin Theorem for n-ary groups*, Discrete Math. **308** (2008), 4861 – 4876.

[3] **W.A. Dudek, J. Michalski,** *On a generalization of Hosszú theorem*, Demonstratio Math. **15** (1982), 783 – 805.

[4] **W.A. Dudek, J. Michalski,** *On retracts of polyadic groups*, Demonstratio Math. **17** (1984), 281 – 301.

[5] **K. Głazek, B. Gleichgewicht,** *Abelian n-groups*, Coll. Math. Soc. J. Bolyai, 29. Universal Algebra, Esztergom (Hungary) 1977, 321 – 329.

[6] **N.A. Shchuchkin,** *Semicyclic n-ary groups,* (Russian), Izv. F. Skaryna Univ., Gomel, **3** (2009), 186 – 194.

[7] **N.A. Shchuchkin,** *Homomorphisms from infinite semilcyclic n-groups to a semiabelian n-group*, (Russian), Chebyshevskiĭ Sb. **22** (2021), 340 – 352.

[8] **N.A. Shchuchkin,** *Endomorphisms of semicyclic n-groups*, (Russian), Chebyshevskiĭ Sb. **22** (2021), 353 – 369.

Received August 24, 2023

S. Dog
22 Pervomayskaya str, 39600 Kremenchuk, Ukraine
Temporary address: WSB Merito University, Wroclaw, Poland
Email: soniadog2@gmail.com

N.A. Shchuchkin
Volgograd State Pedagogical University, Lenina prosp., 27, 400131 Volgograd, Russia
Email: nikolaj_shchuchkin@mail.ru