Quasigroups and Related Systems **31** (2023), 241 – 259

https://doi.org/10.56415/qrs.v31.19

Commutative and 2-divisible subvarieties of rings of Bol-Moufang type

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Abstract. In the two subvarieties of commutative not necessarily associative rings, and 2-divisible not necessarily associative rings, we show that the sixty Bol-Moufang identities determine exactly five subvarieties in each case, and we completely determine all inclusions and necessary counterexamples. The historically important nonassociative rings of octonions and sedenions may be used as counterexamples in our classification. We utilize both associator and linear forms of each Bol-Moufang identity. The results are analogous to the classification of varieties of loops or quasigroups of Bol-Moufang type (Phillips, Vojtěchovský-2005).

1. Introduction

In this article we study subvarieties of nonassociative rings of Bol-Moufang type, in both the commutative and 2-divisible cases, and we obtain two classifications. One of the motivations for this work was the analogy with the study of commutative quasigroups (Section 12 of [14]).

We use the term *nonassociative* to mean *not necessarily associative* when referring to a binary operation. A Bol-Moufang identity is a property of an algebraic object that tames its nonassociativity without necessarily making it associative. An identity is said to be of *Bol-Moufang type* if it is an equality of two 4th-degree monomial expressions of monomial degree $\{2, 1, 1\}$ with identical variable order, for example: $((xy)x)z = (xy)(xz)$ or $(x(yz))y = x(y(zy))$.

Identities of Bol-Moufang type make sense in any set with binary operation; this gives one the freedom to study varieties of Bol-Moufang type in

2010 Mathematics Subject Classification: 17A30, 17D05

Keywords: nonassociative algebra, nonassociative rings, rings of Bol-Moufang type, octonions, sedenions, Moufang rings, commutative Moufang rings, alternative rings, commutative alternative rings, magma rings, tangent algebras to smooth loops

many algebraic contexts [3], for example in the context of loops [13], quasigroups [14], and rings [7]. The automated theorem prover `prover9` and the finite model builder `mace4`, both developed by McCune [9], have played an important role in these types of investigations.

2. Bol-Moufang notation and duality

If Φ is an identity of Bol-Moufang type, then its dual Φ' is another identity of Bol-Moufang type obtained by reflection, i.e. reading the identity backwards. For example, the dual of the identity Ψ: $((xy)x)z = (xy)(xz)$ is Ψ': $(zx)(yx) = z(x(yx))$. Notice that Ψ' is equivalent to $(xy)(zy) = x(y(zy))$.

Identities of Bol-Moufang type can be indexed using the following scheme.

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

For example, the identity $x((yz)x) = (xy)(zx)$ may be called $D23$. Duality interacts nicely with this notation, i.e. $A \leftrightarrow F$, $B \leftrightarrow E$, $C \leftrightarrow C$, $D \leftrightarrow D$, and $1 \leftrightarrow 5$, $2 \leftrightarrow 4$, $3 \leftrightarrow 3$. For example: $(B34)' = E23$, $(F15)' = A15$, and $(D24)' = D24$. There is a total of $6\binom{5}{2} = 60$ Bol-Moufang identities. Four of the sixty are self-dual identities: $C15, C24, D15, D24$.

Note 2.1. (Duality acts on proofs and models) Suppose you have a proof $\Phi_1 \Rightarrow \Phi_2$. Then duality will act on every line of the proof, yielding a proof $\Phi_1' \Rightarrow \Phi_2'$. For example, if one writes a proof $B14 \Rightarrow A13$, dualizing will provide a proof $E25 \Rightarrow F35$. If $(X, \cdot)$ is a set with binary operation that is a model of an identity Φ, then the dual model $(X', *)$ with $X' = X$ equipped with transposed multiplication $x * y = y \cdot x$, will be a model of the dual identity Φ'. Therefore, for example, if one produces a model $(X, \cdot)$ that satisfies $A13$ but not $B14$, then the dual model $(X', *)$ will be a model that satisfies $F35$ but not $E25$.

3. Nonassociative rings, algebras, and tensors

Definiton 3.1. (Nonassociativer ring) A *nonassociative ring* is a set V with two binary operations $+$ and $\cdot$, such that $(V, +)$ forms an Abelian group, and the two operations obey the distributive law:

$$(a+b)(c+d) = ac + ad + bc + bd \quad \forall a, b, c, d \in V.$$

Nonassociative rings emerge when an Abelian group ($\mathbb{Z}$-module) V is endowed with a multiplication $V \times V \to V$ that is $\mathbb{Z}$-bilinear.

One can generalize from $\mathbb{Z}$-coefficients to coefficients from an arbitrary commutative, associative ring k. Indeed, if V is a k-module endowed with a multiplication $V \times V \to V$ that is k-bilinear, then we call V a nonassociative k-algebra. The following definition is an equivalent definition of a nonassociative k-algebra.

Definition 3.2. (Nonassociative k-algebra) Let k be a commutative, associative ring. A *nonassociative k-algebra* is an underlying k-module V equipped with a multiplication $\cdot$ satisfying:

(1) $(V, +, \cdot)$ *forms a nonassociative ring,*

(2) $(\lambda v)w = \lambda(vw) = v(\lambda w)$ *for all* $v, w \in V$ *and all* $\lambda \in k$.

Note 3.3. (Unital rings and k-algebras) We will assume that all rings and k-algebras appearing in this article are unital, i.e. possess a multiplicative identity.

If V is a k-algebra whose underlying k-module is a free module of finite rank, in particular if k is a field and V is a finite-dimensional vector space over k with basis $\{v_1, v_2, \dots v_n\}$, then the n^3 constants (Γ^t_{rs}) given by:

$$v_r v_s = \Gamma^1_{rs} v_1 + \Gamma^2_{rs} v_2 + \cdots + \Gamma^n_{rs} v_n$$

completely determine the multiplication structure on V that make it a k-algebra. The multiplication on V can be interpreted as a k-linear map $V \to \mathrm{Hom}_k(V, V)$, $v \mapsto L_v \in \mathrm{Hom}_k(V, V)$, where L_v is the operator of left-multiplcation by v, i.e. $L_v(w) = vw$. Using the natural isomorphisms of k-modules:

$$\mathrm{Hom}_k(V, \mathrm{Hom}_k(V, V)) \cong V^* \otimes_k \mathrm{Hom}_k(V, V) \cong V^* \otimes_k V^* \otimes_k V,$$

we may interpret the multiplication on V as a rank-3 tensor

$$\Gamma \in V^* \otimes_k V^* \otimes_k V,$$

and the n^3 constants (Γ^t_{rs}) are simply the coefficients of Γ relative to the basis $\{\varepsilon_r \otimes \varepsilon_s \otimes v_t\}$ of $V^* \otimes_k V^* \otimes_k V$. Here $\{\varepsilon_i\}$ is the dual basis to $\{v_i\}$.

Example 3.4. Consider the $\mathbb{R}$-algebra $V = \mathrm{span}_{\mathbb{R}}(1, a, b)$ given by:

$\cdot$	1	a	b
1	1	a	b
a	a	$a+b$	$1+a$
b	b	b	-1

For example: $a \cdot a = a + b = \Gamma^1_{22}1 + \Gamma^2_{22}a + \Gamma^3_{22}b$ and therefore $\Gamma^1_{22} = 0$, $\Gamma^1_{22} = \Gamma^3_{22} = 1$.

In particular, this $\mathbb{R}$-algebra is not commutative since $ab \neq ba$, and not associative since $(ab)a \neq a(ba)$.

$$\begin{aligned} &(ab)a = (1+a)a = a + aa = a + (a+b) = 2a + b. \\ &a(ba) = ab = 1 + a. \end{aligned}$$

Example 3.5. (The algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, $\mathbb{S}$) The algebras of complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ are two and four-dimensional $\mathbb{R}$-algebras respectively: $\mathbb{C} = \mathrm{span}_{\mathbb{R}}(1, i)$, $\mathbb{H} = \mathrm{span}_{\mathbb{R}}(1, i, j, k)$. The algebra $\mathbb{C}$ is commutative and associative, while $\mathbb{H}$ is not commutative, but associative.

$\mathbb{C}$	1	i
1	1	i
i	i	-1

$\mathbb{H}$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

The algebras of octonions $\mathbb{O}$ [6] and sedenions $\mathbb{S}$ [17], [18] are eight and sixteen-dimensional $\mathbb{R}$-algebras respectively. The algebra of octonions is not associative, but alternative, i.e. associators transform via the alternating representation: $\sigma \cdot (x, y, z) = (-1)^\sigma (x, y, z)$, where $(x, y, z) = (xy)z - x(yz)$ is the associator of three elements, σ is a permutation in S_3, and $(-1)^\sigma$ is the sign of the permutation. The octonions are an algebra of Bol-Moufang type, as alternativity is equivalent (in characteristic zero) to the four equivalent Moufang laws $B15, D23, D34, E15$ [7]. For example, we have that $((xy)x)z = x(y(xz))$ for all $x, y, z \in \mathbb{O}$.

On the other hand, the algebra of sedenions $\mathbb{S}$ is not associative and not alternative, but their nonassociativity is tame enough to remain Bol-Moufang type. The sedenions are in fact flexible: $(xy)x = x(yx)$ for all $x, y \in \mathbb{S}$, which is equivalent to the three laws $B45, D24, E12$.

Note 3.6. (Group algebras $k[G]$ and loop algebras $k[Q]$) Let G be a finite group and k a commutative, associative ring. Then the group algebra $k[G]$

(consisting of all k-linear combinations of elements of G) is the associative algebra whose multiplication table is the Cayley table of the group G. If one considers a finite nonassociative group Q, often known as a loop, then the loop ring $k[Q]$ also fits into this scheme (see, for example [2], [4], [5]).

Note 3.7. (Matrix algebras and Lie algebras) Consider the $n \times n$ matrix algebra $M_n(k)$ over a field k with respect to usual matrix multiplication. It has a natural basis $\{e_{ij} \,|\, 1 \leq i \leq j \leq n\}$ with multiplication given by $e_{ij} \cdot e_{jk} = e_{ik}$ and $e_{ij} \cdot e_{kl} = 0$ if $j \neq k$. This algebra is associative and the multiplicative identity is the element $e_{11} + e_{22} + ... + e_{nn}$. If instead $M_n(k) = \mathfrak{gl}_n(k)$ is equipped with the commutator (Lie bracket) multiplication $[X, Y] = XY - YX$, then $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$, and the algebra $\mathfrak{gl}_n(k)$ becomes not associative. It is a Lie algebra since it is skew-symmetric $[X, Y] = -[Y, X]$ and satisfies the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. The Lie algebra $\mathfrak{gl}_n(k)$ has no multiplicative identity, however.

4. Bol-Moufang identities in nonassociative rings

4.1. Linear Bol-Moufang identities

Suppose V is a nonassociative ring satisfying an identity Φ of Bol-Moufang type. The distributive law forces the ring V to satisfy an additional identity, namely the linear version $\mathcal{L}\Phi$ of the Bol-Moufang identity. Since the identity $\mathcal{L}\Phi$ is linear in each variable (unlike the original identity Φ which is quadratic in the repeated variable), if $\mathcal{L}\Phi$ holds on a generating set for V, $\mathcal{L}\Phi$ will hold on all of V.

Consider the Moufang identity $B15$,

$$B15: \qquad ((xy)x)z = x(y(xz)).$$

Suppose that this identity holds for all elements x, y, z in a ring V. If we make the substitutions $x = a + c$, $y = b$, $z = d$, distribute, and then cancel the appropriate terms related to $B15$, we obtain a linear version of the identity

$$\mathcal{L}B15: \qquad ((ab)c)d + ((cb)a)d = a(b(cd)) + c(b(ad)).$$

In general, $\mathcal{L}\Phi$ has the same bracket structure as Φ, but $\mathcal{L}\Phi$ involves transpositions of two of the variables in the addends on each side. The two transposed variables always occupy the positions of the *repeated* variable in the original Bol-Moufang identity Φ.

In the example above, the identity $B15$: $(xy)x)z = x(y(xz))$ has the repeated variable in positions 1 and 3, and the linear identity $\mathcal{L}\Phi$ involves transpositions of the first and third variables a and c.

4.2. Bol-Molfang identities via associators

If V is a nonassociative ring, the associator of three elements $x, y, z \in V$ is defined $(x, y, z) = (xy)z - x(yz)$. In a nonassociative ring, any Bol-Moufang identity may be written as an associator identity via the following scheme.

$$\begin{array}{r|l} 12 & o(o,o,o) = 0 \\ 13 & (o,o,oo) = 0 \\ 14 & o(o,o,o) + (o,oo,o) = 0 \\ 15 & (o,o,oo) + (oo,o,o) = 0 \\ 23 & o(o,o,o) - (o,o,oo) = 0 \\ 24 & (o,oo,o) = 0 \end{array}$$

For example, in a nonassociative ring the identity $B14 : x(y(xz)) = (x(yx))z$ may be written as follows:

$$x(y, x, z) + (x, yx, z) = 0.$$

We will sometimes denote the associator form of the identity using an $\mathcal{A}$-prefix, for example $\mathcal{A}B14$. As another example, the identity $C24$: $x((yy)z) = (x(yy))z$ has associator form $\mathcal{A}C24 : (x, yy, z) = 0$. Dual identities can be easily obtained by reading backwards, for example $\mathcal{A}E14$: $x(y, z, y) + (x, yz, y) = 0$ dualizes to $\mathcal{A}B25 : 0 = (y, zy, x) + (y, z, y)x$.

Additionally, linear identities may be written in associator form, where the associator expression simply alternates sign under the transposition of the variables that occur in the repeated positions in the original identity. For example, the Bol-Moufang identity $B15 : ((xy)x)z = x(y(xz))$ has linear form $\mathcal{L}B15 : ((ab)c)d + ((cb)a)d = a(b(cd)) + c(b(ad))$, which may be written equivalently with associators, by requiring that the expression $(o, o, oo) + (oo, o, o)$ alternate sign under the transposition of the first and third variables. The equivalence of a linear identity and its associator form is not immediately obvious, and requires some algebraic manipulation.

$$\mathcal{A}\mathcal{L}B15 : (a, b, cd) + (ab, c, d) = -\left[(c, b, ad) + (cb, a, d)\right].$$

5. Two classifications

The heart of this paper is the two classifications (Theorems 5.1.1 and 5.2.1). We determine the Hasse diagrams of equivalences and incidences between the sixty identities of Bol-Moufang type in two separate cases: first in the class of commutative nonassociative rings, and second in the class of 2-divisible nonassociative rings.

The definition of *commutative* is well known: $xy = yx$ for all x, y in the ring V. We will use the following definition of (additively) 2-*divisible*: $x + x = 0$ implies that $x = 0$, for all x in the ring V.

In the general case of nonassociative rings of Bol-Moufang type (i.e. with no additional conditions), the Hasse diagram of all equivalences and implications is as follows [7].

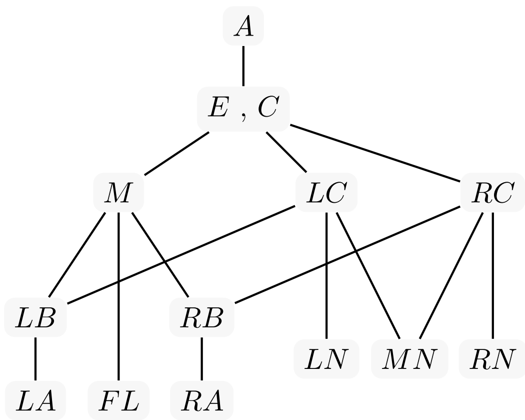

Figure 1: The varieties of rings of Bol-Moufang type.

There are thirteen varieties: the left alternative variety ($LA = A13$, $A45$, $C12$) and its dual right alternative variety ($RA = C45$, $F12$, $F35$), the flexible variety ($FL = B45$, $D24$, $E12$), the left nuclear-square variety ($LN = A35$) and its dual right nuclear-square variety ($RN = F13$), the middle nuclear-square variety ($MN = C24$), the left Bol variety ($LB = B14$) and its dual right Bol variety ($RB = E25$), the Moufang variety ($M = B15$, $D23$, $D34$, $E15$), the left-C variety ($LC = A14$, $A15$, $A34$, $C14$) and its dual right-C variety ($RC = C25$, $F23$, $F15$, $F25$), the Extra, C variety ($E = B23$, $D15$, $E34$, $C = C15$), and finally the associative variety (A) consisting of the remaining thirty Bol-Moufang identities.

We will state our results in a way that describes how this general Hasse diagram coalesces when the additional assumptions of commutativity and 2-divisibility are imposed.

5.1. The commutative subvariety

Theorem 5.1.1. *The class of commutative rings of Bol-Moufang type consists of five varieties: the flexible variety FL, the middle nuclear-square variety MN, the left and right nuclear-square variety LN, RN, the Moufang variety M, and finally the associative variety A. The Hasse diagram of incidences is given.*

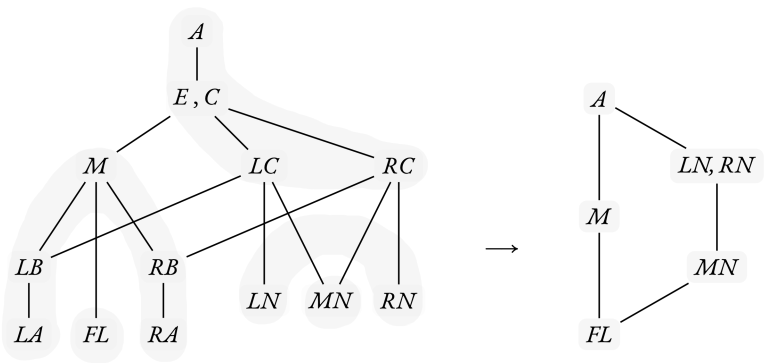

Figure 2: The five varieties of commutative rings of BM type.

Note 5.1.2. All commutative rings are automatically flexible. This follows easily from: $(xy)x = x(xy) = x(yx)$. Also note that if a commutative ring satisfies an identity Φ, then it automatically satisfies the dual identity Φ'. For example, if a commutative ring is left Bol ($LB = B14$), then it is automatically right Bol ($RB = E25$) and vice versa.

Proposition 5.1.3. (E, C, LC, $RC \Leftrightarrow A$) *In the variety of commutative rings, each of the identities E, C, LC, and RC are equivalent to associativity.*

Proof. First we will show that LC, RC imply C. If a commutative ring is LC, it must also be RC and vice versa. If a commutative ring is LC and RC, it must be C since:

$$\begin{aligned}
(xx)(yz) &= (x(xy))z && \text{(start from the } LC \text{ law } A34) \\
(yy)(xz) &= (y(yx))z && \text{(swap roles } x \leftrightarrow y) \\
(yy)(zx) &= ((xy)y)z && \text{(commutativity)} \\
(y(yz))x &= ((xy)y)z && \text{(use } A34 \text{ to rewrite LHS)} \\
x(y(yz)) &= ((xy)y)z && \text{(commutativity once more),}
\end{aligned}$$

which proves the C law $C15$. Thus the LC, RC, C, E laws are equivalent for commutative rings, since in general the E, C laws are equivalent and imply the LC, RC laws [7].

We complete the proof by showing that if the ring is commutative, the two equivalent Extra laws $B23$: $x((yx)z) = (xy)(xz)$ and $E34$: $(x(yz))y = (xy)(zy)$ together imply associativity. Swap the roles played by x and y in $B23$ to obtain: $y((xy)z) = (yx)(yz)$. Then apply commutativity to $E34$ to obtain: $(x(yz))y = (yx)(yz)$. Thus we have $y((xy)z) = (x(yz))y$ and once again by commutativity, $((xy)z)y - (x(yz))y = 0$, i.e. $(x, y, z)y = 0$. This identity is quadratic in the variable y, so we consider its linear version:

$$(a, b, c)d + (a, d, c)b = 0.$$

By setting $d = 1$, we obtain $(a, b, c) + 0 \cdot b = 0$, which implies $(a, b, c) = 0$ for all a, b, c in the ring. This completes the proof. □

Proposition 5.1.4. ($LN, RN \Rightarrow MN$) *In the class of commutative rings, the left and right nuclear-square identities imply the middle nuclear-square identity.*

Proof. This is a straightforward use of commutativity, left nuclear-squares LN: $((xx)y)z = (xx)(yz)$, and right-nuclear squares RN: $(xy)(zz) = x(y(zz))$

$$\begin{aligned}
(x(yy))z &= ((yy)x)z = (yy)(xz) \\
&= (xz)(yy) = x(z(yy)) \\
&= x((yy)z).
\end{aligned}$$

This demonstrates the middle nuclear-square law MN. □

Proposition 5.1.5. (LA, RA, LB, $RB \Leftrightarrow M$) *In the class of commutative rings, the left and right alternative as well as the left and right Bol identities are equivalent to the Moufang identities.*

Proof. First note that for arbitrary rings, any pair of LA, FL, RA implies the third. For example, if a ring satisfies LA and RA it will satisfy $\mathcal{L}LA$ and $\mathcal{L}RA$. If we use LA: $(x, x, y) = 0$ and $\mathcal{L}RA$: $(a, b, c) = -(a, c, b)$, we can derive FL as follows: $0 = (x, x, y) = -(x, y, x)$, which obviously implies $0 = (x, y, x)$, which is FL.

Secondly, we will show that if an arbitrary ring is all three LA, FL, RA, then it satisfies the four Moufang laws ($B15$, $D23$, $D34$, $E15$). Our strategy will be to show: $(LA, FL, RA) \Rightarrow B15 \Rightarrow D23 \Rightarrow (LA, FL, RA)$. By dualizing, and noting that (LA, FL, RA) is self-dual, we will obtain: $(LA, FL, RA) \Rightarrow E15 \Rightarrow D34 \Rightarrow (LA, FL, RA)$.

Suppose a ring satisfies the three laws: $(xx)y = x(xy), (xy)x = x(yx)$, $(xy)y = x(yy)$. These will imply the linear laws $\mathcal{L}LA$, $\mathcal{L}FL$, $\mathcal{L}RA$, and therefore associators alternate: $\sigma \cdot (x, y, z) = \mathrm{sgn}(\sigma) \cdot (x, y, z)$, $\sigma \in S_3$. Following the technique of Schafer [16] we obtain:

$$\begin{aligned}
((xy)x)z - x(y(xz)) &= ((xy)x)z - (xy)(xz) + (xy)(xz) - x(y(xz)) \\
&= (xy, x, z) + (x, y, xz) \\
&= -(x, xy, z) - (x, xz, y) \\
&= -(x(xy))z + x((xy)z) - (x(xz))y + x((xz)y) \\
&= -(x^2y)z - (x^2z)y + x((xy)z + (xz)y) \\
&= -x^2(yz) - x^2(zy) + x(x(yz) + x(zy)) \quad \text{(using } \mathcal{L}RA) \\
&= -x^2(yz + zy) + x^2(yz + zy) \\
&= 0.
\end{aligned}$$

This proves $B15$. Now suppose $B15$: $((xy)x)z = x(y(xz))$. If we set $z = 1$ we obtain the flexible law $(xy)x = x(yx)$, and if we set $y = 1$ we obtain the left-alternative law $(xx)z = x(xz)$. These together imply alternativity $\sigma \cdot (x, y, z) = \mathrm{sgn}(\sigma) \cdot (x, y, z)$, $\sigma \in S_3$, which in particular implies $(y, z, x) = (z, x, y)$. Now we calculate:

$$\begin{aligned}
(xy)(zx) - x((yz)x) &= (x, y, zx) + x(y(zx)) - x((yz)x) \\
&= -(x, zx, y) - x((yz)x) + x(y(zx)) \\
&= -(x(zx))y + x((zx)y) - x((yz)x) + x(y(zx)) \\
&= -((xz)x)y + x((zx)y - (y, z, x)) \\
&= -x(z(xy)) + x((zx)y - (y, z, x)) \\
&= -x(z(xy) - (zx)y + (y, z, x)) \\
&= -x(-(z, x, y) + (y, z, x)) \\
&= -x \cdot 0 = 0.
\end{aligned}$$

This proves $D23$. Now suppose $D23$: $(xy)(zx) = x((yz)x)$. If we set $z = 1$ we obtain the flexible law $(xy)x = x(yx)$. Additionally we know that $D23$ implies the linear version $\mathcal{L}D23$, which (upon setting the four variables equal to 1, one at a time) implies alternativity $\sigma\cdot(x,y,z) = \mathrm{sgn}(\sigma)\cdot(x,y,z)$, $\sigma \in S_3$. The combination of the flexible law $(x,y,x) = 0$ with the linear $\mathcal{L}LA$ law $(x,y,z) = -(y,x,z)$ together imply $0 = (x,y,x) = -(y,x,x)$, which implies $(y,x,x) = 0$, i.e. the right-alternative law (RA). Similarly, we can derive the left-alternative law (LA). Therefore $D23 \Rightarrow (LA, FL, RA)$.

We can now complete the original goal of this proof. Suppose a commutative ring is any one of LA, RA, LB, RB. Since it is a commutative ring, it must be both LA and RA. By the work above, it must therefore be Moufang. Conversely, if the ring is Moufang it must be each of LA, RA, LB, RB. ☐

Note 5.1.6. (Moufang rings and alternative rings) Embedded within the proof of (5.1.5) is a proof that, in general, Moufang rings are precisely those rings satisfying the three laws LA, FL, and RA. If we take the linear versions $\mathcal{L}LA$, $\mathcal{L}FL$, $\mathcal{L}RA$, we may conclude that *Moufang rings are alternative rings* (in the traditional sense that associators transform via the alternating representation of S_3, i.e. $\sigma \cdot (x,y,z) = \mathrm{sgn}(\sigma)\cdot(x,y,z)$ where σ is a permutation in S_3 and $\mathrm{sgn}(\sigma)$ is its sign). If the ring is 2-divisible (i.e. for all x, $2x = 0$ implies $x = 0$), then Moufang rings are in fact equivalent to alternative rings, since the linear laws $\mathcal{L}LA$, $\mathcal{L}FL$, $\mathcal{L}RA$ are equivalent to the laws LA, FL, RA. Only in characteristic two can one produce an example of an alternative ring that is not Moufang (see example 5.1.2 in [7]).

Example 5.1.7. (Commutative flexible ring that is not Moufang, MN) Consider the 8-element commutative ring $R = \mathbb{F}_2[1,\alpha,\beta]$ where:

$\cdot$	α	β
α	β	0
β	0	1

Since this ring is commutative, it must be flexible. However, it is not left alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$ and therefore not Moufang; it is not middle nuclear-square since $(\alpha(\alpha\alpha))\beta \neq \alpha((\alpha\alpha)\beta)$. This example is of minimal order.

Example 5.1.8. (Commutative MN ring that is not Moufang, LN, RN) Consider the 8-element commutative ring $R = \mathbb{F}_2[1,\alpha,\beta]$ where:

$\cdot$	α	β
α	β	0
β	0	β

The ring R is in fact middle nuclear-square, as one checks that β is middle nuclear (i.e. $(x, \beta, y) = 0$ for all $x, y \in R$), and so all squares $\alpha^2 = \beta$, $\beta^2 = \beta$, $(1+\alpha)^2 = 1+\beta$, $(1+\beta)^2 = 1+\beta$, $(\alpha+\beta)^2 = 0$, $(1+\alpha+\beta)^2 = 1$ are middle nuclear as well. However, this ring is not left nuclear-square LN since $((\alpha\alpha)\alpha)\alpha \neq (\alpha\alpha)(\alpha\alpha)$, and thus not RN since it is commutative; and it is not left-alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$, and thus it cannot be Moufang. This example is of minimal order.

Example 5.1.9. (Commutative LN, RN ring that is not Moufang) Consider the 8-element commutative ring $R = \mathbb{F}_2[1, \alpha, \beta]$ where:

$\cdot$	α	β
α	0	1
β	1	0

The ring R is in fact left/right nuclear-square, as one checks that all squares $\alpha^2 = 0$, $\beta^2 = 0$, $(1+\alpha)^2 = 1$, $(1+\beta)^2 = 1$, $(\alpha+\beta)^2 = 0$, $(1+\alpha+\beta)^2 = 1$ are nuclear. However, this ring is not Moufang since it is not left alternative $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$. This example is of minimal order.

Example 5.1.10. (Commutative Moufang ring that is not MN) Consider the commutative ring of order 3^7 given by $R = \mathbb{F}_3[1, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta]$ where:

$\cdot$	α	β	γ	δ	ε	ζ
α	0	δ	0	0	ζ	0
β	δ	0	ε	0	0	0
γ	0	ε	0	$-\zeta$	0	0
δ	0	0	$-\zeta$	0	0	0
ε	ζ	0	0	0	0	0
ζ	0	0	0	0	0	0

This is a folklore example attributed to I. Kaplansky of an alternative ring (and therefore Moufang by 5.1.5 above) that is not associative. It is the ring-theory analogue of a commutative Moufang loop that is not associative [11]. There is computational evidence that this example may be of minimal order. It is in fact not MN since $(\beta K)\gamma \neq \beta(K\gamma)$ where $K = (1+\alpha)^2 = 1 + 2\alpha$. It is interesting to note that this example exists in characteristic 3 (rather than characteristic 2 like the previous examples). It is also interesting to note that this ring does not admit a magma-basis (unlike the previous examples which do). In fact, we have verified that any commutative Moufang ring admitting a magma-basis is necessarily associative. The argument is as follows: a commutative Moufang ring V satisfies the linear law $\mathcal{L}D23$, and so if V admits a magma-basis $\beta = \{v_1, v_2, ..., v_n\}$, the identity $\mathcal{L}D23$ must also hold on the basis elements, i.e. $a((bc)d) + d((bc)a) = (ab)(cd) + (db)(ca)$ for all a, b, c, d in the basis β. This would require the equality of unordered pairs $\{a((bc)d), d((bc)a)\} = \{(ab)(cd), (db)(ca)\}$ for all a, b, c, d in the basis β. In particular, for all elements of the basis β, we have $a((bc)d) = (ab)(cd)$ or $d((bc)a) = (ab)(cd)$, and upon setting $d = 1$ gives: $a(bc) = (ab)c$ or $(bc)a = (ab)c$. Commutativity then implies that the magma-basis is associative, and therefore the entire ring V must be associative.

The above propositions and examples completely classify the Hasse diagram of commutative rings of Bol-Moufang type.

5.2. The 2-Divisible Subvariety

Theorem 5.2.1. *The class of additively* 2*-divisible rings of Bol-Moufang type consists of five varieties: the left alternative variety LA, its dual right alternative variety RA, the flexible variety FL, the usual Moufang variety M, and finally the associative variety A. The Hasse diagram of incidences is given.*

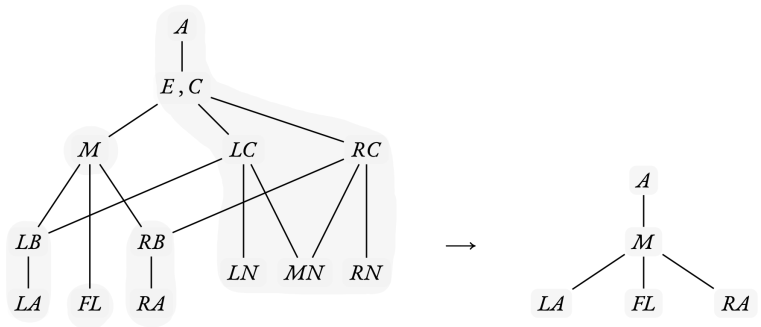

Figure 3: The five varieties of 2-divisible rings of BM type.

Proposition 5.2.2. (E, C, LC, RC, LN, MN, $RN \Leftrightarrow A$) *In the class of*

2-divisible rings, all identities E, C, LC, RC, LN, MN, RN are equivalent to associativity.

Proof. The argument is essentially the same for all identities except for $B23$, $D15$, $E34$ (the E or "Extra" identities). For the non-Extra identities one simply considers the linear identity, sets one of the variables equal to the multiplicative identity, then uses the 2-divisibility condition. For example the C identity $C15$; first we consider the linear identity $\mathcal{L}C15$: $a(b(cd)) + a(c(bd)) = ((ab)c)d + ((ac)b)d$, and set $b = 1$ to obtain $a(cd) + a(cd) = (ac)d + (ac)d$, i.e. $2a(cd) = 2(ac)d$, and thus $a(cd) = (ac)d$ by 2-divisibility. Since this holds for all a, c, d in the ring, it must be associative.

Now we show how to deal with the Extra identities. Consider for example $B23$ and its linear version $\mathcal{L}B23$: $a((bc)d) + c((ba)d) = (ab)(cd) + (cb)(ad)$. Setting $a = 1$ yields $(bc)d + c(bd) = b(cd) + (cb)d$, i.e. the transposition $(12) \in S_3$ acts by $+1$ on associators $(b, c, d) = (c, b, d)$. Whereas, setting $d = 1$ yields $a(bc) + c(ba) = (ab)c + (cb)a$, i.e. the transposition $(13) \in S_3$ acts by -1 on associators $(a, b, c) = -(c, b, a)$. Therefore the cycle $(123) = (13)(12) \in S_3$ will act by -1 on associators $(a, b, c) = -(c, a, b)$ and its inverse $(321) = (123)^2$ will act by $+1$ on associators $(a, b, c) = (b, c, a)$. Together, these facts will force all associators to be zero, as one calculates $(a, b, c) = -(c, a, b) = -(a, b, c)$, which implies that $2(a, b, c) = 0$ and $(a, b, c) = 0$ by 2-divisibility. The arguments for $D15$ and $E34$ are similar. □

Proposition 5.2.3. ($LA \Leftrightarrow LB$:$RA \Leftrightarrow RB$) *In the class of 2-divisible rings, the left/right alternative identity is equivalent to the left/right Bol identity.*

Proof. As LB implies LA, we need to show that LA implies LB. Suppose that a ring satisfies LA: $(x, x, y) = 0$ and therefore it satisfies $\mathcal{L}LA$: $(a, b, c) = -(b, a, c)$. We need to show LB: $x(y(xz)) = (x(yx))z$.

Let α denote the expression $\alpha := (x^2, y, z) + (y, x^2, z)$, notice that $\mathcal{L}LA$ implies that $\alpha = 0$. Using left-alternativity we may rewrite α as follows:

$$\begin{aligned}\alpha &= (x^2, y, z) + (y, x^2, z)\\ &= ((xx)y)z - (xx)(yz) + (y(xx))z - y((xx)z)\\ &= (x(xy))z - x(x(yz)) + (y(xx))z - y(x(xz))\\ &= (x, xy, z) + x(x, y, z) + (y(xx))z - y(x(xz))\\ &= -(xy, x, z) - x(y, x, z) + (y(xx))z - y(x(xz)).\end{aligned}$$

Let β denote the expression $\beta := -(x, yx, z) - (yx, x, z) + (x, y, x)z + (y, x, x)z$, and once again by left-alternativity, $\beta = 0$ and we may rewrite β

as follows:

$$\begin{aligned}\beta &= -(x,yx,z)-(yx,x,z)+(x,y,x)z+(y,x,x)z\\ &= -(x(yx))z+x((yx)z)-((yx)x)z+(yx)(xz)\\ &\quad +((xy)x)z-(x(yx))z+((yx)x)z-(y(xx))z\\ &= -(x(yx))z+x((yx)z)+(yx)(xz)\\ &\quad +((xy)x)z-(x(yx))z-(y(xx))z \quad \text{(cancel the } ((yx)x)z).\end{aligned}$$

Adding α and β yields:

$$\begin{aligned}0=\alpha+\beta &= -(xy,x,z)-x(y,x,z)+(y(xx))z-y(x(xz))-(x(yx))z\\ &\quad +x((yx)z)+(yx)(xz)+((xy)x)z-(x(yx))z-(y(xx))z\\ &= -(xy,x,z)-x(y,x,z)-y(x(xz))-(x(yx))z\\ &\quad +x((yx)z)+(yx)(xz)+((xy)x)z-(x(yx))z\\ &= -(xy,x,z)-x(y,x,z)+(y,x,xz)-(x(yx))z\\ &\quad +x((yx)z)+((xy)x)z-(x(yx))z\\ &= -(xy,x,z)-x(y,x,z)-(x,y,xz)-(x(yx))z\\ &\quad +x((yx)z)+((xy)x)z-(x(yx))z\\ &= 2x(y(xz))-2(x(yx))z.\end{aligned}$$

Finally, using 2-divisibility, we have LB: $x(y(xz))-(x(yx))z=0$. Using duality, we have $RA \Leftrightarrow RB$. $\square$

Example 5.2.4. (2-divisible LA ring that is not FL, RA) Consider the 81-element ring $R=\mathbb{F}_3[1,\alpha,\beta,\gamma]$ where:

$\cdot$	α	β	γ
α	α	β	0
β	γ	γ	0
γ	0	0	0

This ring R is in fact left-alternative but not flexible since $(\alpha\beta)\alpha \neq \alpha(\beta\alpha)$, and not right-alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$. This example is of minimal order.

Example 5.2.5. (2-divisible FL ring that is not LA, RA) Consider the 27-element ring $R=\mathbb{F}_3[1,\alpha,\beta]$ where:

$\cdot$	α	β
α	β	0
β	0	1

This ring is flexible, but it is not left-alternative since $(\alpha\alpha)\beta \neq \alpha(\alpha\beta)$, and it is not right-alternative since $(\alpha\beta)\beta \neq \alpha(\beta\beta)$. This example is of minimal order.

Another famous example is the infinite $\mathbb{R}$-algebra of the sedenions $\mathbb{S}$ [17], [18] which are flexible but not left or right alternative.

Example 5.2.6. (2-divisible Moufang ring that is not associative) Consider the 3^8-element ring given by $R = \mathbb{F}_3[1, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta]$ where:

$\cdot$	α	β	γ	δ	ε	ζ	η
α	0	α	ε	0	0	0	ζ
β	δ	1	η	α	ε	ζ	γ
γ	ζ	γ	0	ε	0	0	0
δ	0	δ	ζ	0	0	0	ε
ε	0	ζ	0	0	0	0	0
ζ	0	ε	0	0	0	0	0
η	ε	η	0	ζ	0	0	0

This ring was found by searching for a magma-basis that satisfies the linear-Moufang laws (which are equivalent in the 2-divisible case to the Moufang laws). Since the basis is linear-Moufang, the entire ring R is Moufang (and hence also alternative). This ring is an example of a Moufang/alternative ring that is not associative, for example $(\alpha\beta)\gamma \neq \alpha(\beta\gamma)$. Another famous example is the infinite $\mathbb{R}$-algebra of the octonions $\mathbb{O}$, which are Moufang/alternative, but not associative.

The above propositions and examples completely classify the Hasse diagram of 2-divisible rings of Bol-Moufang type.

6. The significance of the Moufang Variety

One of the main goals of classifications such as this is to emphasize the relative importance of certain subvarieties of objects, and the present paper certainly highlights the importance of Moufang rings. In fact, in the 2-divisible case, Moufang rings are equivalent to the historically important and well-studied class of alternative rings, i.e. where associators transform via the alternating representation of S_3. In this section we include some results, with proofs, that are, perhaps, part of the folklore about commutative alternative rings. These results are analogous to results in the setting of commutative Moufang loops, cf. Theorem 2.1 in [12].

Theorem 6.1. *Let V be a commutative Moufang ring and $x \in V$ any element. The element $3x$ is nuclear, i.e. associates with all elements of V.*

Proof. We make use of the following "half-Akivis identity", valid for all x, y, z in any ring:

$$(x, y, z) + (y, z, x) + (z, x, y) = [xy, z] + [yz, x] + [zx, y],$$

where $(u, v, w) = (uv)w - u(vw)$ and $[u, v] = uv - vu$ are the usual associator and commutator. Since the ring V in question is assumed Moufang (and therefore alternative), we have $(x, y, z) = (y, z, x) = (z, x, y)$. Since V is also assumed to be commutative, the other side of the identity vanishes, yielding:

$$3(x, y, z) = 0.$$

Finally, using linearity of the associator: $0 = 3(x, y, z) = (3x, y, z) = (x, 3y, z) = (x, y, 3z)$, which gives the result. □

Theorem 6.2. *Let V be a commutative Moufang ring and $x \in V$ any element. The element x^3 is nuclear, i.e. x^3 associates with all of V.*

Proof. First we need to show that for any elements $x, y, z \in V$, we have $(y, x^3, z) = 0$. Then, by alternativity, $0 = -(x^3, y, z) = -(y, z, x^3)$ and therefore all associators $(x^3, y, z) = (y, x^3, z) = (y, z, x^3) = 0$ vanish, as required.

Since V is alternative, $x^3 = (xx)x = x(xx)$ is well-defined. We will establish two identities, which hold for all elements of V:

(1) $a^3b = a(a(ab))$

(2) $a(b(cb)) = ((ab)c)b = b((ba)c)$.

The identity (1) follows by applying one of the four equivalent Moufang identities $B15$: $x(y(xz)) = ((xy)x)z$ specialized to the case of $x = a$, $y = a$, and $z = b$. The identity (2) follows by applying another equivalent Moufang identity $E15$: $x(y(zy)) = ((xy)z)y$ specialized to the case of $x = a$, $y = b$, and $z = c$ to obtain the first equality of (2), and the second equality of (2) follows by commutativity. Now let x, y, z be elements of V and compute:

$$\begin{aligned}
y(x^3z) &= y(x(x(xz))) && \text{using (1)}\\
&= y(x((xz)x)) && \text{commutativity}\\
&= ((yx)(xz))x && \text{(2) with } a = y,\ b = x,\ c = xz\\
&= x((xz)(xy)) && \text{commutativity}\\
&= z(x((xy)x)) && \text{(2) with } b = x,\ a = z,\ c = xy\\
&= z(x(x(xy))) && \text{commutativity}\\
&= z(x^3y) && \text{using (1)}\\
&= (yx^3)z. && \text{commutativity}
\end{aligned}$$

As required, this proves that $(y, x^3, z) = (yx^3)z - y(x^3z) = 0$. $\square$

7. Acknowledgements

We thank Murray Bremner, as the motivation to study nonassociative rings of Bol-Moufang type grew out of a separate but related problem posed by Bremner to study tangent algebras to smooth loops of Bol-Moufang type, an intriguing form of nonassociative Lie theory (see for example [1], [8], [10], [15]). One may consider a nonassociative $\mathbb{R}$-algebra $\mathfrak{q}$ of Bol-Moufang type with set of units $U \subseteq \mathfrak{q}$. If U is closed under multiplication it becomes a smooth loop of Bol-Moufang type and $T_1U \cong \mathfrak{q}$ is its tangent algebra with natural commutator $[X, Y] = XY - YX$ and associator $(X, Y, Z) = (XY)Z - X(YZ)$ structures. It remains to determine the appropriate conditions on an $\mathbb{R}$-algebra $\mathfrak{q}$ of Bol-Moufang type so that its set units $U \subseteq \mathfrak{q}$ is closed under multiplication (and therefore a loop). This would be an important preliminary result on the way to classifying tangent algebras to smooth loops of Bol-Moufang type. The associator forms of the Bol-Moufang identities determined in section 4.2 may also play an important role in this classification as well.

We also thank the anonymous referee for his or her careful reading of the paper, and for his or her helpful suggestions.

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Received April 22, 2023

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