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Isotopic class of transversals in finite solvable groups

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Abstract. We prove that if H is a subgroup of a finite solvable group G such that the number of isotopic classes of right transversals of H in G is 1, then H is normal in G.

1. Introduction

Let G be a group and H be a subgroup of G. A *normalized right transversal* (NRT) of H in G is a subset S of G obtained by selecting one and only one element from each right coset of H in G and $1 \in S$. Let S be an NRT of H in G. Then the binary operation of G induces a binary operation on S defined by $\{x \circ y\} = S \cap Hxy$, $x,\ y \in S$. With respect to this binary operation, S is a right loop with identity 1, that is, a right-quasigroup with both-sided identity (see [6, Proposition 4.3.3, p.102]). Conversely, every right loop can be embedded as an NRT in a group with some universal property (see [4, Theorem 3.4]).

Identifying S with the set $H \backslash G$ of all right cosets of H in G, we get a transitive permutation representation $\chi_S : G \longrightarrow Sym(S)$ defined by $\{\chi_S(g)(x)\} = Hxg \cap S$, $g \in G$, $x \in S$. The Kernel Kerχ_S of this action is $\mathrm{Core}_G(H)$, the core of H in G.

Two groupoids $(L, \circ)$ and $(L^{'}, \circ')$ are said to be *isotopic* if there exists a triple (f, g, h), where f, g and h are bijective maps from L to $L^{'}$ such that $f(x) \circ' g(y) = h(x \circ y)$, for all x, y in L. If $f = g = h$, then isotopy (f, g, h) is an isomorphism from L to $L^{'}$. A non-empty subset T of a groupoid L is called a *subgroupoid* of L if the binary operation of L induces the binary operation on T.

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Let $\mathcal{T}(G,H)$ denote the set of all NRTs of H in G. We say that $S,T \in \mathcal{T}(G,H)$ are *isotopic* if their induced right loop structures are isotopic. Let $Itp(G,H)$ denote the set of isotopic classes of NRTs of H in G.

It has been proved in [3, Proposition 3.8] that if G is a finite nilpotent group and H is a subgroup of G such that $|Itp(G,H)| = 1$, then H is normal in G. It has been also proved in [3, Proposition 3.9] that if G is a finite solvable group and H is a subgroup of G. If $(|H|,[G:H]) = 1$ and $|Itp(G,H)| = 1$, then H is normal in G. The main result of this paper is:

Main Theorem. Let G be a finite solvable group and H be a subgroup of G such that $|Itp(G,H)| = 1$. Then H is normal in G.

2. Preliminaries

Definition 2.1. (cf. [1]) Let $(L,\circ)$ be a groupoid. Then the subset

$A = \{a \in L : \text{for all } x,y \in L,\ (x \circ a) \circ y = x \circ (a \circ y)\}$ of L is called the *middle-associator* of $(L,\circ)$.

We use the following results frequently in the paper:

Theorem 2.2. (cf.[1]) *Let $(L,\circ)$ be a groupoid. Then the middle-associator of $(L,\circ)$ is a subgroupoid of $(L,\circ)$ and is a semigroup. If $(L,\circ)$ has identity 1, and $(L,\circ')$ is a groupoid with identity isotopic to $(L,\circ)$, then the middle-associator of $(L,\circ)$ is isomorphic to the middle-associator of $(L,\circ')$.*

Proposition 2.3. *Let G be a finite group, H be a corefree subgroup of G and $S \in \mathcal{T}(G,H)$. Let A be the middle-associator of $(S,\circ)$, where $\circ$ is the binary operation on S induced by the binary operation of G. Then $a \in A$ if and only if $ax \in S$ for all $x \in S$, i.e., if and only if $a \circ x = ax$ for all $x \in S$.*

Proof. Let $\circ$ be the binary operation on S induced by the binary operation of G and $f : S \times S \to H$ be the map defined by $xy = f(x,y)(x \circ y)$, $x,y \in S$. Consider the transitive permutation representation defined as in the introduction. Let θ be the right action of H on S defined by $\chi_S\big|_H$, that is, $\{x\theta h\} = \{\chi_S(h)(x)\} = Hxh \cap S$, $h \in H$, $x \in S$. We observe that the kernel of this action is the $\mathrm{Core}_G(H)$. Let $x,y,z \in S$. Then

$(xy)z = x(yz) \implies (f(x,y)(x \circ y))z = x(f(y,z)(y \circ z))$
$\implies f(x,y)((x \circ y))z) = (x\theta f(y,z))(y \circ z)$
$\implies (f(x,y)(x \circ y))z = x(f(y,z)(y \circ z))$

$$\implies f(x,y)f(x\circ y,z)((x\circ y)\circ z) = f(x\theta f(y,z),(y\circ z))(x\theta f(y,z)\circ(y\circ z))$$
$$\implies (x\circ y)\circ z = x\theta f(y,z)\circ(y\circ z).$$

Now,
$$\begin{aligned} a\in A &\iff (x\circ a)\circ y = x\circ(a\circ y) \quad \text{for all } x,y\in S\\ &\iff x\theta f(a,y)\circ(a\circ y) = x\circ(a\circ y)\\ &\iff x\theta f(a,y) = x \quad \text{for all } x,y\in S\\ &\iff f(a,y)\in Core_G(H)=\{1\} \quad \text{for all } y\in S\\ &\iff f(a,y)=1 \quad \text{for all } y\in S\\ &\iff a\circ y = ay \quad \text{for all } y\in S. \end{aligned}$$
□

Proposition 2.4. *Let G be a finite group and H be a corefree subgroup of G. Let N be a proper subgroup of G containing subgroup H properly. Let $T\in\mathcal{T}(N,H)$ be a subgroup of N and $L\in\mathcal{T}(G,N)$. Let $S=TL\in\mathcal{T}(G,H)$. Then the middle-associator A of S contains T.*

Proof. Let $t\in T$ and $x\in S$. Since $S=TL\in\mathcal{T}(G,H)$, $x=t_1l$ for some $t_1\in T$ and $l\in L$. Since T is a subgroup of N, $tx=(tt_1)l\in S$. Thus by Proposition 2.3, $t\in A$. □

Proposition 2.5. *Let G be a finite group and H be a subgroup of G such that $|Itp(G,H)|=1$. If there exists an NRT of H in G, which is a loop, then H is normal in G.*

Proof. Assume that $S\in\mathcal{T}(G,H)$ is a loop. Then by [3, Corollary 3.2], each NRT of H in G is a loop. Now by [4, Corollary 2.9], H is normal in G. □

3. Proof of Main Theorem

Proposition 3.1. *Let G be a finite group and H be a non-trivial corefree subgroup of G. If G has a non-trivial proper normal subgroup N such that $N\cap H=\{1\}$, then $|Itp(G,H)|>1$.*

Proof. Assume that G has a non-trivial proper normal subgroup N such that $N\cap H=\{1\}$. If possible, assume that $|Itp(G,H)|=1$. Since N is a non-trivial proper normal subgroup of G such that $N\cap H=\{1\}$, $HN\neq G$, for otherwise N is an NRT of H in G and by Proposition 2.5, H is a normal subgroup of G, a contradiction.

Let $L=\{1,l_1,l_2,\ldots,l_{r-1}\}\in\mathcal{T}(G,NH)$ and $S_1=NL$. Since $N\in\mathcal{T}(NH,H)$, $S_1\in\mathcal{T}(G,H)$. Let A_1 be the middle-associator of S_1. Since N is a subgroup of G, by Proposition 2.4, $|A_1|\geqslant|N|$.

Assume that $|N| = 2$ and $x(\neq 1) \in N$. Then $x \in Z(G)$, the center of G. Let $h(\neq 1) \in H$ and $S_2 = (S_1 \setminus \{xl_1, xl_2, \ldots, xl_{r-1}\}) \cup \{hxl_1, hxl_2, \ldots, hxl_{r-1}\}$. Let A_2 be the middle-associator of S_2. Since xl_1 is not in S_2 and x, l_1 are in S_2, by Proposition 2.3, x is not in A_2. Further, since $x \in Z(G)$, $xl_i = l_ix$ are not in S_2 and l_i, x are in S_2, by Proposition 2.3, $l_1, l_2, .., l_{r-1}$ are not in A_2. If for any i $(1 \leqslant i \leqslant r-1)$, hxl_i is in A_2, by Proposition 2.3, $hxl_ix^{-1} = hl_ixx^{-1} = hl_i$ is in S_2, a contradiction. Hence hxl_i $(1 \leqslant i \leqslant r-1)$ are not in A_2. This implies that $|A_2| < |N|$. Thus by Theorem 2.2, $|Itp(G, H)| \geqslant 2$, a contradiction. Therefore $|N| > 2$.

Let $x_1(\neq 1) \in N$, $h(\neq 1) \in H$ and

$$S_3 = (S_1 \setminus \{x_1l_1, x_1l_2, \ldots, x_1l_{r-1}\}) \cup \{hx_1l_1, hx_1l_2, \ldots, hx_1l_{r-1}\}.$$

Let A_3 be the middle-associator of S_3. Since x_1l_1 is not in S_3 and x_1, l_1 are in S_3, by Proposition 2.3, x_1 is not in A_3.

Since N is normal in G and $x_1(\neq 1) \in N$, for each i $(1 \leqslant i \leqslant r-1)$, there exists $y_i(\neq 1) \in N$ such that $x_1l_i = l_iy_i$. Further, since l_iy_i are not in S_3 and l_i, y_i are in S_3, by Proposition 2.3, $l_1, l_2, \ldots, l_{r-1}$ are not in A_3. Also, if hx_1l_i are in A_3, by Proposition 2.3, $hx_1l_iy_i^{-1} = hl_i$ are in S_3, a contradiction. Hence hx_1l_i $(1 \leqslant i \leqslant r-1)$ are not in A_3. Let $x(\neq x_1, 1) \in N$. Since N is normal in G, $xl_i = x_1x_1^{-1}xl_i = x_1l_iu$, for some $u \in N$. If xl_i are in A_3, by Proposition 2.3, $xl_iu^{-1} = x_1l_i$ are in S_3, a contradiction. Hence $|A_3| < |N|$. Thus by Theorem 2.2, $|Itp(G, H)| \geqslant 2$, a contradiction. □

Corollary 3.2. *Let G be a finite supersolvable group and H be a subgroup of G such that $|Itp(G, H)| = 1$. Then H is a normal subgroup of G.*

Proof. If possible, assume that H is a non-normal subgroup of G. Let $K = \mathrm{Core}_G(H)$. Since $|Itp(G, H)| = |Itp(G/K, H/K)|$ (see [3, Proposition 3.4]), G/K is supersovable and H/K is corefree subgroup of G/K, we may assume that H is a corefree subgroup of G. Since G is a finite supersolvable group, there exists a minimal normal subgroup N of G such that $|N|$ is a prime number [2, 3B.7(a), p.85]. Thus $N \cap H = \{1\}$. Hence by Proposition 3.1, $|Itp(G, H)| \geqslant 2$, a contradiction. Therefore H is normal in G. □

Corollary 3.3. *Let G be a finite nilpotent group and H be a subgroup of G such that $|Itp(G, H)| = 1$. Then H is a normal subgroup of G.*

Proof. Since every finite nilpotent group is a supersolvable, the corollary follows from the Corollary 3.2. □

Proposition 3.4. *Let G be a finite group G, H be a non-trivial corefree subgroup of G and N be a non-trivial proper normal subgroup of G such that $HN \neq G$. Let K be a subgroup of N such that $K \in \mathcal{T}(HN, H)$. If there exists $L \in \mathcal{T}(G, HN)$ such that $Kl = lK$ for all $l \in L$, then $|Itp(G, H)| > 1$.*

Proof. Let $K = \{1, k_1, k_2, \ldots, k_{s-1}\}$ and $L = \{1, l_1, l_2, \ldots, l_{r-1}\}$ be an NRT of HN in G such that $Kl = lK$ for all $l \in L$. Then $S_1 = KL$ is an NRT of H in G. Let A_1 be the middle-associator of S_1. Since K is a subgroup of N, by Proposition 2.4, $|A_1| \geqslant |K|$.

Let $h(\neq 1) \in H$ and $S_2 = (S_1 \setminus \{k_i l_j\}) \cup \{hk_i l_j\}$, where $1 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant r-1$. Then $S_2 \in \mathcal{T}(G, H)$. Let A_2 be the middle-associator of S_2. Since $k_1, k_2, \ldots, k_{s-1}$, $l_1, l_2, \ldots, l_{r-1}$ are in S_2 and $k_i l_j$ $(1 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant r-1)$ are not in S_2, by Proposition 2.3, $k_i \notin A_2 (1 \leqslant i \leqslant s-1)$. Since $Kl_j = l_j K$ for all $l_j \in L (1 \leqslant j \leqslant r-1)$, $k_1 l_j = l_j k_{i'}$ for some i' $(1 \leqslant i' \leqslant s-1)$. Since $l_j, k_{i'} \in S_2$ and $l_j k_{i'} = k_1 l_j \notin S_2$, by Proposition 2.3, $l_j \notin A_2$ for $1 \leqslant j \leqslant r-1$. Also, $hk_i l_j = hl_j k_t$ for some t $(1 \leqslant t \leqslant s-1)$. If $hk_i l_j \in A_2$, by Proposition 2.3, $hk_i l_j k_t^{-1} = hl_j k_t k_t^{-1} = hl_j \in S_2$, a contradiction. Hence $hk_i l_j \notin A_2$ $(1 \leqslant i \leqslant s-1, 1 \leqslant j \leqslant r-1)$. Therefore $A_2 = \{1\}$. So $|A_2| < |K|$. Thus by Theorem 2.2, $|Itp(G, H)| > 1$. □

Proposition 3.5. *Let G be a finite group G, H be a non-trivial corefree subgroup of G and N be a non-trivial proper normal subgroup of G such that $HN \neq G$. Let K be a subgroup of N such that $K \in \mathcal{T}(HN, H)$. If there exists an NRT L of HN in G such that $kl = lk'$ for some $k(\neq 1), k'(\neq 1) \in K$, $l(\neq 1) \in L$ and $Kl' \neq l'K$ for some $l'(\neq 1) \in L$, then $|Itp(G, H)| > 1$.*

Proof. Let L be an NRT of HN in G such that $kl = lk'$ for some $k(\neq 1)$, $k'(\neq 1) \in K$, $l(\neq 1) \in L$ and $Kl' \neq l'K$ for some $l'(\neq 1) \in L$. Let A_1 be the middle-associator of $S_1 = KL \in \mathcal{T}(G, H)$. Let $K = \{1, k_1, k_2, \ldots, k_{s-1}\}$ and $L = \{1, l_1, l_2, \ldots, l_{r-1}\}$. By Proposition 2.4, $|A_1| \geqslant |K|$. Since K is a subgroup of N and N is a normal subgroup of G, for any $k_t \in K$, $l_{t'} \in L$ $(1 \leqslant t \leqslant s-1$, $1 \leqslant t' \leqslant r-1)$ either $k_t l_{t'} = l_{t'} k_{s'}$ for some s' $(1 \leqslant s' \leqslant s-1)$ or $k_t l_{t'} = l_{t'} x$ for some $x \in N \setminus K$. Since there exists $k_i \in K$, $l_j \in L$ $(1 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant r-1)$ such that $k_i l_j = l_j k_{i'}$ for some i' $(1 \leqslant i' \leqslant s-1)$ and $Kl_{j'} \neq l_{j'} K$ for some j' $(1 \leqslant j' \leqslant r-1)$, we can get a nonempty subset $C = \{\ k_i l_j |\ 1 \leqslant i \leqslant s-1,\ 1 \leqslant j \leqslant r-1$ and $k_i l_j = l_j k_{i'}$ for some $k_{i'} \in K (1 \leqslant i' \leqslant s-1)\}$ of S_1. Let $h(\neq 1) \in H$ and $D = hC = \{hc \mid c \in C\}$. Let $S_2 = (S_1 \setminus C) \cup D$ and A_2 denote the middle-associator of S_2. Since $C \neq \emptyset$, then we get a nonempty subset E

of L such that $k_il_j = l_jk_{i'}$ for some $i, i^{'}(1 \leqslant i,\ i^{'} \leqslant s-1)$ and for $l_j \in E$ $(1 \leqslant j \leqslant r-1)$. Since $k_il_j = l_jk_{i'}$ are not in S_2, by Proposition 2.3, k_i and $l_j \in E$ are not in A_2.

Let $l_j \in L \setminus E$. Then for any $k_i \in K$ $k_il_j = l_jx_i$ for some $x_i \in N \setminus K$. If $l_j \in A_2$, $l_j^{-1} \in A_2$ and by Proposition 2.3, $l_j^{-1}k_il_j = l_j^{-1}l_jx_i = x_i \in S_2$, a contradiction. Hence $l_j \in L \setminus E$ are not in A_2. Therefore $l_j \notin A_2$ for all $1 \leqslant j \leqslant r-1$.

Assume that $k_il_j \notin C$ for some i and j $(1 \leqslant i \leqslant s-1,\ 1 \leqslant j \leqslant r-1)$. Then $k_il_j = l_ju_i$ for some $u_i \in N \setminus K$. If $k_il_j \in A_2$, $(k_il_j)^{-1} \in A_2$ and by Proposition 2.3, $(k_il_j)^{-1}l_j = (l_ju_i)^{-1}l_j = u_i^{-1}l_j^{-1}l_j = u_i^{-1} \in S_2$, a contradiction. Hence if $k_il_j \notin C$, it is not in A_2.

Let $y \in D$. Then $y = hc = hk_il_j = hl_jk_{i'}$ for some i' $(1 \leqslant i' \leqslant s-1)$. If $y \in A_2$, by Proposition 2.3, $yk_{i'}^{-1} = hk_il_jk_{i'}^{-1} = hl_jk_{i'}k_{i'}^{-1} = hl_j \in S_2$, a contradiction. Thus if $y \in D$, it is not in A_2. This implies that $|A_2| < |K|$. Therefore by Theorem 2.2, $|Itp(G, H)| > 1$. □

Proposition 3.6. *Let G be a finite group G, H be a corefree subgroup of G and N be a non-trivial proper normal subgroup of G such that $HN \neq G$. Let K be a subgroup of N such that $K \in \mathcal{T}(HN, H)$. If $L \in \mathcal{T}(G, HN)$ such that for any $k(\neq 1) \in K$ and $l(\neq 1) \in L$ $kl = lx$ for some $x \in N \setminus K$ and if A_1 denotes the middle-associator of $S_1 = KL \in \mathcal{T}(G, H)$, then $|A_1| = |K|$.*

Proof. Let $L \in \mathcal{T}(G, HN)$ such that for any $k(\neq 1) \in K$ and $l(\neq 1) \in L$, $kl = lx$ for some $x \in N \setminus K$ and A_1 be the middle-associator of $S_1 = KL \in \mathcal{T}(G, H)$. Then by Proposition 2.4, $|A_1| \geqslant |K|$. Let $k(\neq 1) \in K$ and $l(\neq 1) \in L$. Since $kl = lx$ for some $x \in N \setminus K$, l is not in A_1, for otherwise $l^{-1} \in A_1$ and by Proposition 2.3, $l^{-1}kl = l^{-1}lx = x \in S_1$, a contradiction. Further, if $kl \in A_1$, $(kl)^{-1} \in A_1$ and by Proposition 2.3, $(kl)^{-1}l = (lx_1)^{-1}l = x_1^{-1}l^{-1}l = x_1^{-1} \in S_1$ for some $x_1 \in N \setminus K$, a contradiction. Hence $kl \notin A_1$. Therefore $|A_1| = |K|$. □

Proposition 3.7. *Let G be a finite group G, H be a non-trivial corefree subgroup of G and N be a non-trivial proper normal subgroup of G such that $HN \neq G$. Let K be a subgroup of N such that $K \in \mathcal{T}(HN, H)$. If there exists $L \in \mathcal{T}(G, HN)$ such that for any $k(\neq 1) \in K$ and $l(\neq 1) \in L$ $kl = lx$ for some $x \in N \setminus K$ and $l_1^{-1}(\neq l_1) \in L$ for some $l_1 \in L$, then $|Itp(G, H)| > 1$.*

Proof. Let $L \in \mathcal{T}(G, HN)$ such that for any $k(\neq 1) \in K$ and $l(\neq 1) \in L$ $kl = lx$ for some $x \in N \setminus K$ and $l_1^{-1}(\neq l_1) \in L$ for some $l_1 \in L$. Let A_1

be the middle-associator of $S_1 = KL \in \mathcal{T}(G,H)$. Then by Proposition 3.6, $|A_1| = |K|$.

Let $l_1^{-1}(\neq l_1) \in L$. Let $k(\neq 1) \in K$, $h(\neq 1) \in H$, $S_2 = (S_1 \setminus \{kl_1\}) \cup \{hkl_1\}$ and A_2 be the middle-associator of S_2. Thus by Proposition 2.3, $k \notin A_2$ and as argued in the proof of Proposition 3.6, $k'l \notin A_2$ for any $k'(\neq k) \in K$ and $l(\neq 1) \in L$. Let $l(\neq 1, l_1)$ and $kl = lx_1$ for $x_1 \in N \setminus K$. If $kl \in A_2$, $(kl)^{-1} \in A_2$ and by Proposition 2.3, $(kl)^{-1}l = (lx_1)^{-1}l = x_1^{-1}l^{-1}l = x_1^{-1} \in S_2$, a contradiction. Hence $kl \notin A_2$. Also, $l \notin A_2$, for otherwise $l^{-1} \in A_2$ and by Proposition 2.3, $l^{-1}kl = l^{-1}lx_1 = x_1 \in S_2$, a contradiction. If $l_1 \in A_2$, by Proposition 2.3, $l_1kl_1^{-1} = l_1l_1^{-1}x_2 = x_2 \in S_2$ for some $x_2 \in N \setminus K$, a contradiction. Hence $l_1 \notin A_2$. If $hkl_1 \in A_2$, by Proposition 2.3, $hkl_1l_1^{-1} = hk \in S_2$, a contradiction. Thus $hkl_1 \notin A_2$. Therefore $|A_2| < |K|$ and by Theorem 2.2, $|Itp(G,H)| > 1$. □

Proof of Main Theorem: Let G be a finite solvable group and H be a subgroup of G such that $|Itp(G,H)| = 1$. Now, if possible assume that H is not normal in G. Let $K = \mathrm{Core}_G(H)$, the core of H in G. Since $|Itp(G/K, H/K)| = |Itp(G,H)|$ [3, Proposition 3.4, p.413], we may assume that H is a corefree subgroup of G. Since G is a finite solvable group, there exists a non-trivial proper minimal normal subgroup N of G. Then N is an elementary abelian p-group [5, 5.4.3, p.148] for some prime p. If $N \cap H = \{1\}$, by Proposition 3.1, $|Itp(G,H)| \geqslant 2$, a contradiction. Hence $N \cap H \neq \{1\}$. Also, $N \cap H \neq N$, for otherewise $N \subset H$, a contradiction. Since $N \cap H$ is a subgroup of elementary abelian p-group N, there exists an NRT M of $N \cap H$ in N which is a subgroup of N. Hence $M \in \mathcal{T}(NH,H)$, for $\mathcal{T}(N, N \cap H) \subseteq \mathcal{T}(NH,H))$. If $NH = G$, by Proposition 2.5, H is a normal subgroup of G, a contradiction. Hence $NH \neq G$.

Let $S_1 = ML \in \mathcal{T}(G,H)$, where $L = \{1, l_1, l_2, \ldots, l_{r-1}\} \in \mathcal{T}(G,NH)$ and A_1 be the middle-associator of S_1. Since M is a subgroup of N, by Proposition 2.4, $M \subseteq A_1$ and so $|A_1| \geqslant |M|$.

If $ml = lm'$ for some $m' \in M$, where $m \in (M \setminus \{1\})$ and $l \in (L \setminus \{1\})$, by Propositions 3.4 and 3.5, $|Itp(G,H)| > 1$, a contradiction. Hence $ml = lx$ for some $x \in N \setminus M$, for all $m(\neq 1) \in M$ and for all $l(\neq 1) \in L$. Thus by Proposition 3.6, $|A_1| = |M|$.

Next, assume that $H \nsubseteq N$ and $h \in H \setminus N$. Hence for all $m(\neq 1) \in M$, $hm \notin N$. Let $m'(\neq 1) \in M$ and $S_2 = (S_1 \setminus \{m'\}) \cup \{hm'\} \in \mathcal{T}(G,H)$. Let A_2 be the middle-associator of S_2. Then as argued in the proof of Proposition 3.6, l_i, ml_i are not in A_2 for any $m(\neq 1) \in M$, $1 \leqslant i \leqslant r-1$. If $hm' \in A_2$, by Proposition 2.3, $hm'l_i \in S_2$, a contradiction. Hence

$hm' \notin A_2$ and $|A_2| < |M|$. Therefore by Theorem 2.2, $|Itp(G,H)| > 1$, a contradiction. Thus $H \subset N$. If there exists $l \in G$ such that $Nl \neq Nl^{-1}$, we can find an NRT L_2 contaning l, l^{-1} of N in G. Hence in this case, by Proposition 3.7, $|Itp(G,H)| \geqslant 2$, a contradiction. Thus for all $g \in G$ $Ng = Ng^{-1}$, that is, $g^2 \in N$ for all $g \in G$. If $|N| = 2^s$ for some $s \in \mathbb{N}$, G is a 2-group and so by Corollary 3.3, H is a normal subgroup of G, a contradiction. Therefore N is not 2-group.

Since N is an elementary abelian p-group, $p > 2$, we have $|M| > 2$ and there is no element of M of order 2. Let $S_3 = (S_1 \setminus \{m_1 l_1\}) \cup \{hm_1 l_1\} \in \mathcal{T}(G,H)$, where $h(\neq 1) \in H$ and $m_1(\neq 1) \in M$. Let A_3 be the middle-associator of S_3. Then as argued in Proposition 3.6, $l_i(\neq l_1), ml_i, m_1$ (for any $m(\neq 1) \in M$, $1 \leqslant i \leqslant r-1$) are not in A_3. Since $|M| > 2$, there exists $m_2(\neq 1, m_1) \in M$. Further, since $m_2 l_1 = l_1 x_1$ for some $x_1 \in N \setminus M$, therefore if $l_1 \in A_3$, by Proposition 2.3, $l_1^{-1} l_1 x_1 = x_1 \in S_3$, a contradiction. Hence $l_1 \notin A_3$. Since N is an abelian, $hm_1 l_1 = m_1 h l_1$. Hence $m_1^{-1}(\neq m_1) \notin A_3$, for otherwise by Proposition 2.3, $m_1^{-1} m_1 h l_1 = h l_1 \in S_3$, a contradiction. Therefore $|A_3| < |M|$ and so by Theorem 2.2, $|Itp(G,H)| \geqslant 2$, a contradiction. Thus H is normal in G. □

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