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Weak embeddability of the partial Menger algebra of formulas

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Abstract. We consider the partial $(n+1)$-operation of the set of all n-ary formulas of arbitrary types and then prove that the superassociativity is satisfied as a weak identity. Binary partial associative operations of formulas induced by the partial operation of type $(n+1)$ are proposed. We also prove that the partial unitary Menger algebra of formulas is weak embeddable into the algebra of partial n-ary functions under which its selectors correspond to projections.

1. Introduction and preliminaries

The fundamental fact that the compositon of functions is associative allows us to consider the importance of associativity. In [17], K. Menger introduced the concept of algebra consisting of a nonempty set G and one operation o of type n defined on G such that the superassociativity or (C1) holds, i.e., $o(o(a,b_1,...,b_n),c_1,...,c_n) = o(x,o(b_1,c_1,...,c_n),...,o(b_n,c_1,...,c_n))$ for all $a,b_j,c_j \in G$ and $j = 1,...,n$. Such algebra is called a *Menger algebra of rank* n or a *superassociative algebra*. Evidently, the study of Menger algebras is now studied in various aspects. For example, Menger algebras of multiplace functions were deeply considered in the papers [5, 7, 8, 9, 10, 11, 13, 14]. K. Denecke and his descendants also investigated Menger algebras of terms of various languages [2, 3]. If there are special elements in a Menger algebra (G,o) such that $o(e,a_1,...,a_n) = e$ and $o(a,e_1,...,e_n) = a$, then (G,o) is called *unitary*. In this case, this algebra has the type $(n+1,0,...,0)$. A comprehensive review of the theory of Menger algebras or algebras of functions can be found in the monograph [6].

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Recall from [2, 15] that a term of type τ is a formal expression that constructed from an alphabet $X_n = \{x_1, ..., x_n\}$ whose elements are called *variables* for all n in $\mathbb{N}^+ := \{1, 2, ...\}$ and operation symbols $\{f_i \mid i \in I\}$ of type τ indexed by the set I. The type is the sequence $\tau = (n_i)_{i\in I}$ of the natural numbers which are arities of the operation symbols f_i. In fact, the set $W_\tau(X_n)$ of all *n-ary term of type* τ consists of the following elements: Every variable $x_i \in X_n$ and $f_i(t_1, ..., t_{n_i})$ where n-ary terms $t_1, ..., t_{n_i}$ of type τ are already known. The set of terms together with the superposition operation, a mapping $S^n : (W_\tau(X_n))^{n+1} \to W_\tau(X_n)$ defined on the structure of a term $s \in W_\tau(X_n)$ by

(1) for $s = x_j, 1 \leqslant j \leqslant n$, $S^n(x_j, t_1, ..., t_n) := t_j$,

(2) for $s = f_i(s_1, ..., s_{n_i})$,

$$S^n(f_i(s_1, ..., s_{n_i}), t_1, ..., t_n) := f_i(S^n(s_1, t_1, ..., t_n), ..., S^n(s_{n_i}, t_1, ..., t_n)),$$

forms the algebra

$$(W_\tau(X_n), S^n, (x_i)_{i\leqslant n, n\in\mathbb{N}^+})$$

which is called the *clone of all n-ary terms of type* τ. In this case, the variables $x_1, ..., x_n$ can be considered as the nullary operations. Clearly, the algebra is a unitary Menger algebra of rank n because S^n satisfies the superassociative law, i.e,

$$S^n(S^n(s, t_1, ..., t_n), u_1, ..., u_n) = S^n(s, S^n(t_1, u_1, ..., u_n), ..., S^n(t_n, u_1, ..., u_n))$$

for all $s, t_j, u_j \in W_\tau(X_n)$ and $S^n(x_j, t_1, ..., t_n) = t_j$ and $S^n(s, x_1, ..., x_n) = s$ for all $j = 1, ..., n$. Among recent contributions are [3, 15, 19].

One of the outstanding structures generalizing any algebra of arbitrary types is an *algebraic system* [16] denoted by $\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I}, (\gamma_j^{\mathcal{A}})_{j\in J})$. In each component, A is a nonempty set, $(f_i^{\mathcal{A}})_{i\in I}$ is a family of n_i-ary operations defined on A, and $(\gamma_j^{\mathcal{A}})_{j\in J}$ is a family of n_j-ary relations on A. For the type (τ, τ') of an algebraic system, we mean $\tau = (n_i)_{i\in I}$ where n_i comes form $f_i^{\mathcal{A}} : A^{n_i} \to A$ for each $i \in I$ and $\tau' = (n_j)_{j\in J}$ where n_j comes form $\gamma_j^{\mathcal{A}} \subseteq A^{n_j}$ for each $j \in J$. Notice that if a family of n_j-ary relations on A is not defined, this structure is reduced to an original algebra of type τ, i.e., $\mathcal{A} := (A, (f_i^{\mathcal{A}})_{i\in I})$. Clearly, any ordered semigroup is a basic example of algebraic systems of type $((2), (2))$.

To investigate properties of algebraic systems of type (τ, τ'), the concept of formulas is needed. Recall from [4, 18, 20] that for $n \in \mathbb{N}^+$, an *n-ary formula of type* (τ, τ') is inductively defined in the following way:

(1) the equation $t_1 \approx t_2$ is an n-ary formula of type (τ, τ') if t_1, t_2 are n-ary terms of type τ,

(2) $\gamma_j(t_1, ..., t_{n_j})$ is an n-ary formula of type (τ, τ') if $j \in J$ and $t_1, ..., t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol,

(3) $\neg F$ is an n-ary formula of type (τ, τ') if F is an n-ary formula of type (τ, τ'),

(4) $F_1 \vee F_2$ is an n-ary formula of type (τ, τ') if F_1 and F_2 are n-ary formulas of type (τ, τ'),

(5) $\exists x_i(F)$ is an n-ary formula of type (τ, τ') if F is an n-ary formula of type (τ, τ') and $x_i \in X_n$.

Let $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ be the set of all n-ary formulas of type (τ, τ'). Particularly, by an *atomic formula*, we refer to the formula of the form (1) and (2). Note that the equation symbol $\approx$ in (1) differs from the relation symbol γ_j in (2) for all $j \in J$.

Some concrete example is provided. Let $(\tau, \tau') = ((3), (2))$ be the type with a ternary operation symbol f and a binary relation symbol γ. We provide lists of some elements in $\mathcal{F}_{((3),(2))}(W_{(3)}(X_3))$. For this, some atomic formulas are determined as follows:

$$x_2 \approx x_3, x_1 \approx x_1, f(x_1, x_2, x_3) \approx x_3, f(x_1, x_1, x_1) \approx f(x_2, x_2, x_3),$$
$$\gamma(x_1, x_2), \gamma(x_3, x_3), \gamma(x_2, x_3), \gamma(f(x_3, x_3, x_2), f(x_1, x_3, x_1)).$$

Apart form these are obtained by using the following three logical connectors, say $\neg, \exists, \vee$.

The operation $R^n : \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)) \times (W_\tau(X_n))^n \to \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ on sets of formulas was defined in the following inductive way (cf. [4]):

(1) If $t_1 \approx t_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n(t_1 \approx t_2, s_1, ..., s_n)$ is the formula

$$S^n(t_1, s_1, ..., s_n) \approx S^n(t_2, s_1, ..., s_n).$$

(2) If $\gamma_j(t_1, ..., t_{n_j}) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n(\gamma_j(t_1, ..., t_{n_j}), s_1, ..., s_n)$ is the formula $\gamma_j(S^n(t_1, s_1, ..., s_n), ..., S^n(t_{n_j}, s_1, ..., s_n))$.

(3) If $F \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n(\neg F, s_1, ..., s_n)$ is the formula

$$\neg R^n(F, s_1, ..., s_n).$$

(4) If $F_1, F_2 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n(F_1 \vee F_2, s_1, ..., s_n)$ is the formula

$$R^n(F_1, s_1, ..., s_n) \vee R^n(F_2, s_1, ..., s_n).$$

(5) If $\exists x_i(F) \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$, then $R^n(\exists x_i(F), s_1, ..., s_n)$ is the formula

$$\exists x_i(R^n(F, s_1, ..., s_n)).$$

This operation generates the algebra

$$(W_\tau(X_n), \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)), R^n, (x_i)_{1\leqslant i\leqslant n, n\in\mathbb{N}^+},$$

which is called the *n-ary formula-term clone of type* (τ, τ'). It was mentioned in [4] that this algebra is a unitary Menger algebra.

Algebraic constructions of partial operations of terms were first determined in the paper [2]. Due to the fact that the set of all linear terms, terms in which all variable do not appear more than one, is not closed under the usual superposition S^n for all n, the partial algebea of such terms was considered in sense of the many-sorted set and many-sorted partial mappings. Generally, the set $\mathcal{F}^{lin}_{(\tau,\tau')}(X_n)$ of all n-ary linear formulas induced by linear terms is not closed under the operation R^n. As a result, the partial algebra of linear formulas was established in [1].

In this paper, we aim to define the partial operation on the set of all n-ary formulas of type (τ, τ') and then construct the partial Menger algebras which satisfy the axiom of superassociativity. Some binary partial associative operations on the set of formulas derived from the partial operation of type $(n + 1)$ defined on formulas are given in Section 2. We continue in Section 3 with discussing a partial representation of formulas by a weak monomorphism which maps from the Menger algebras of formulas to the Menger algebra of partial n-ary functions defined on some set.

2. Partial Menger algebras of formulas

This section begin with giving the partial operation of formulas and illustrating the process of computation. Now we let

$$W\mathcal{F}_{(\tau,\tau')}(X_n) := W_\tau(X_n) \cup \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)).$$

The partial superposition operation of type $(n + 1)$ which is the partial mapping

$$\overline{R}^n : (W\mathcal{F}_{(\tau,\tau')}(X_n))^{n+1} \multimap\!\!\to W\mathcal{F}_{(\tau,\tau')}(X_n)$$

can be defined by

$$\overline{R}^n(a, b_1, ..., b_n) = \begin{cases} S^n(a, b_1, ..., b_n) & \text{if } a, b_1, ..., b_n \in W_\tau(X_n), \\ R^n(a, b_1, ..., b_n) & \text{if } a \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)), b_i \in W_\tau(X_n), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Some examples that demonstrate the process of this partial operation are now mentioned. Let $|I| = 2, |J| = 1$ and let $(\tau, \tau') = ((2, 2), (2))$ be a type with the corresponding two operation symbols $+, *$ and one relation symbol Δ. The set $W\mathcal{F}_{((2,2),(2))}(X_4)$ consists of all quaternary terms of type (2) and all quaternary formulas of type $((2, 2), (2))$. Prepare the following tools:

a_1 is a variable x_1, a_2 is a term $+(x_2, x_4)$,
a_3 is a term $+(*(x_4, x_1), x_2)$, b_1 is a formula $*(x_4, x_1) \approx x_1$,
b_2 is a formula $\Delta(x_3, +(x_4, x_2))$, b_3 is a formula $\Delta(x_3, *(x_4, x_4)) \vee \neg(+(x_4, x_1) \approx x_2)$,
d_1 is a term $*(x_4, x_3)$, d_2 is a variable x_4.

Obviously, $a_1, a_2, a_3, b_1, b_2, b_3, d_1, d_2$ are elements in $W\mathcal{F}_{((2,2),(2))}(X_4)$. Furthermore, we have

$$\overline{R}^4(a_1, d_1, d_2, a_2, a_3) = S^4(a_1, d_1, d_2, a_2, a_3) = d_1 = *(x_4, x_3),$$
$$\overline{R}^4(a_2, d_1, d_2, a_2, a_3) = S^4(a_2, d_1, d_2, a_2, a_3) = +(b_2, a_3) = +(x_4, +(*(x_4, x_1), x_2)),$$
$$\overline{R}^4(a_4, d_1, d_2, a_2, a_3) = R^4(a_4, d_1, d_2, a_2, a_3) \text{ which equals to } *(a_3, d_1) \approx d_1, \text{ and thus } *(+(*(x_4, x_1), x_2), *(x_4, x_3)) \approx *(x_4, x_3),$$
$$\overline{R}^4(b_2, d_1, d_2, a_2, a_3) = R^4(b_2, d_1, d_2, a_2, a_3) = \Delta(a_2, +(a_3, d_2)), \text{ which equals to } \Delta(+(x_2, x_4), +(+(*(x_4, x_1), x_2), x_4)).$$

On the other hand, $\overline{R}^4(b_3, a_1, a_2, b_1, b_2)$ and $\overline{R}^4(d_1, b_1, b_2, b_3, a_1)$ are not defined.

As a result, we can form the following two partial algebras. The first one is the partial algebra $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n)$ of type $(n + 1)$. We aim to show that $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n)$ satisfies (C1) as a weak identity. For this, the concept of weak identities is required. Recall from [12] that an equation $s \approx t$ is said to be a *weak identity* in an algebra $\mathcal{A}$ if one side is defined then another side is also defined and both sides are equal. Similarly, the partial algebra $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n, (x_j)_{j\leqslant n, n\in\mathbb{N}^+})$ of type $(n+1, \underbrace{0,...,0}_{n \text{ times}})$ is another structure derived from the first one.

Theorem 2.1. *$(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n)$ is a partial superassociative algebra.*

Proof. Assume that $a, b_1, ..., b_n, d_1, ..., d_n$ are elements in $W\mathcal{F}_{(\tau,\tau')}(X_n)$. We aim to show that, $\overline{R}^n$ satisfies the superassociativity as a weak identity, i.e.,

$$\overline{R}^n(\overline{R}^n(a, b_1, ..., b_n), d_1, ..., d_n) \approx \overline{R}^n(a, \overline{R}^n(b_1, d_1, ..., d_n), ..., \overline{R}^n(b_n, d_1, ..., d_n)).$$

Assume that the left-hand side of such identity is defined. Then we have the following two cases: $a, b_1, ..., b_n, d_1, ..., d_n$ are n-ary terms of type τ in the first case and a is an n-ary formula of type (τ, τ') but $b_1, ..., b_n, d_1, ..., d_n$ are n-ary terms of type τ in the second case. In the first case when $a, b_1, ..., b_n, d_1, ..., d_n$ are n-ary terms of type τ, $\overline{R}^n(\overline{R}^n(a, b_1, ..., b_n), d_1, ..., d_n)$ equals to $S^n(S^n(a, b_1, ..., b_n), d_1, ..., d_n)$.

Moreover, for each $j = 1, ..., n$, $\overline{R}^n(b_j, d_1, ..., d_n)$ is defined and equals to $S^n(b_j, d_1, ..., d_n)$. This implies that the right-hand side is defined and equals to $S^n(a, S^n(b_1, d_1, ..., d_n), ..., S^n(b_n, d_1, ..., d_n))$. The superposition S^n satisfies the following equation:

$$S^n(S^n(a, b_1, ..., b_n), d_1, ..., d_n) = S^n(a, S^n(b_1, d_1, ..., d_n), ..., S^n(b_n, d_1, ..., d_n)).$$

We now consider the case when a is an n-ary formula and $b_1, ..., b_n, d_1, ..., d_n$ are n-ary terms of type τ.

It implies that the left-hand side $\overline{R}^n(\overline{R}^n(a, b_1, ..., b_n), d_1, ..., d_n)$ equals to $R^n(R^n(a, b_1, ..., b_n), d_1, ..., d_n)$. For each $j = 1, ..., n$, $\overline{R}^n(b_j, d_1, ..., d_n)$ is also defined and equals to $R^n(b_j, d_1, ..., d_n)$. Since for each $j = 1, ..., n$, $R^n(b_j, d_1, ..., d_n)$ belongs to the set $W_\tau(X_n)$, we obtain that the right-hand side is defined and equals to $R^n(a, R^n(b_1, d_1, ..., d_n), ..., R^n(b_n, d_1, ..., d_n))$. Finally, we prove that two formulas, i.e., $R^n(R^n(a, b_1, ..., b_n), d_1, ..., d_n)$ and $R^n(a, R^n(b_1, d_1, ..., d_n), ..., R^n(b_n, d_1, ..., d_n))$ are identical. For this, a proof by a definition of a formula a is given. In fact, it was shown in [4] that the operation R^n satisfies the superassociative law already. Consequently, in this case, the partial operation $\overline{R}^n$ also satisfies the equation as a weak identity. □

Considering a variable from an alphabet X_n, the following theorem is stated.

Theorem 2.2. *$(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n, (x_j)_{j\leqslant n, n\in\mathbb{N}^+})$ is a partial unitary Menger algebra.*

Proof. The proof of superassociativity follows from a direct verification of Theorem 2.1. To prove that the equation $\overline{R}^n(x_j, b_1, ..., b_n) \approx b_j$ is a weak identity, suppose that the left-hand side of this equation is defined. We have that $b_1, ..., b_n$ are n-ary terms of type τ. Thus $\overline{R}^n(x_j, b_1, ..., b_n) = S^n(x_j, b_1, ..., b_n) = b_j$. Therefore, our claimed is proved. Finally, we show that the weak equation $\overline{R}^n(a, x_1, ..., x_n) \approx a$ holds. It is not hard to see that the left-hand side is defined and thus $\overline{R}^n(a, x_1, ..., x_n) = R^n(a, x_1, ..., x_n)$. If a is an element in $W_\tau(X_n)$, we have $R^n(a, x_1, ..., x_n) = S^n(a, x_1, ..., x_n) = a$. For a formula a, we give a proof by the following way: If a is an equation $s \approx t$, then $R^n(s \approx t, x_1, ..., x_n)$ is equal to $S^n(s, x_1, ..., x_n) \approx S^n(t, x_1, ..., x_n)$, subsequently, $s \approx t$. If a has a form $\gamma_j(t_1, ..., t_{n_j})$, it follows from [1] that we have $R^n(\gamma_j(t_1, ..., t_{n_j}), x_1, ..., x_n) = \gamma_j(t_1, ..., t_{n_j})$. Assume that a is satisfied as a weak identity already. Then we obtain $R^n(\neg a, x_1, ..., x_n) = \neg R^n(a, x_1, ..., x_n) = \neg a$ and $R^n(\exists x_i(a), x_1, ..., x_n) = \exists x_i(R^n(a, x_1, ..., x_n)) = \exists x_i(a)$. Finally, suppose that F_1 and F_2 are satisfied. Then we have

$$R^n(F_1 \vee F_2, x_1, ..., x_n) = R^n(F_1, x_1, ..., x_n) \vee R^n(F_2, x_1, ..., x_n) = F_1 \vee F_2.$$

The proof is finished. □

Our next purposes are to define three partial binary operations on the set $W\mathcal{F}_{(\tau,\tau')}(X_n)$ and show that these operations are weak associative.

For every a and b in $W\mathcal{F}_{(\tau,\tau')}(X_n)$, we define the partial operation

$$+^F : (W\mathcal{F}_{(\tau,\tau')}(X_n))^2 \multimap W\mathcal{F}_{(\tau,\tau')}(X_n)$$

by

$$a +^F b = \overline{R}^n(a, \underbrace{b, ..., b}_{n \text{ times}}).$$

Then we have the following result.

Theorem 2.3. *$(W\mathcal{F}_{(\tau,\tau')}(X_n), +^F)$ is a partial semigroup.*

Proof. Let $a, b, d \in W\mathcal{F}_{(\tau,\tau')}(X_n)$. We have to show that $(a +^F b) +^F d = a +^F (b +^F d)$ is a weak identity. Suppose first that $(a +^F b) +^F d$ is defined. It follows that a belongs to $W\mathcal{F}_{(\tau,\tau')}(X_n)$ and both b, d are in $W_\tau(X_n)$. As a result, $a +^F (b +^F d)$ is also defined. To show that both sides are equal, we consider in a few cases. If a, b are terms in $W_\tau(X_n)$, then we have $(a +^F b) +^F d = S^n(a +^F b, d, ..., d) = S^n(S^n(a, b, ..., b), d, ..., d)$ and $a +^F (b +^F d) = S^n(a, b +^F d, ..., b +^F d) = S^n(a, S^n(b, d, ..., d), ..., S^n(b, d, ..., d))$. Due to the fact that S^n satisfies (C1), we conclude that $(a +^F b) +^F d = a +^F (b +^F d)$. For the case when a is a formula of type (τ, τ') but b is a term of type τ, from the definition of $+^F$, we obtain $(a +^F b) +^F d = R^n(a +^F b, d, ..., d) = R^n(R^n(a, b, ..., b), d, ..., d)$ and $a +^F (b +^F d) = R^n(a, b +^F d, ..., b +^F d) = R^n(a, R^n(b, d, ..., d), ..., R^n(b, d, ..., d))$. Since we already known that the operation R^n is superassociative, as a result, it implies that the equation $(a +^F b) +^F d = a +^F (b +^F d)$ holds. □

We call the semigroup defined in Theorem 2.4 the *partial diagonal semigroup* derived from the partial unitary Menger algebra

$$\mathcal{M}_u = (W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n, (x_j)_{j \leqslant n, n \in \mathbb{N}^+}).$$

For each $i = 1, ..., n$, the binary partial operation

$$\cdot^F_{x_i} : (W\mathcal{F}_{(\tau,\tau')}(X_n))^2 \multimap\!\!\to W\mathcal{F}_{(\tau,\tau')}(X_n)$$

is defined by

$$a \cdot^F_{x_i} b = \overline{R}^n(a, x_1, ..., x_{i-1}, b, x_{i+1}, ..., x_n)$$

for all $a, b \in W\mathcal{F}_{(\tau,\tau')}(X_n)$.

Applying the fact that the partial operation $\overline{R}^n$ is partial superassociative over the set $W\mathcal{F}_{(\tau,\tau')}(X_n)$, we prove the following theorem.

Theorem 2.4. $(W\mathcal{F}_{(\tau,\tau')}(X_n), \cdot^F_{x_i})$ *is a partial semigroup.*

Proof. Let a, b and d be in $W\mathcal{F}_{(\tau,\tau')}(X_n)$. To prove that $(W\mathcal{F}_{(\tau,\tau')}(X_n), \cdot^F_{x_i})$ is a partial semigroup, we need to show that the equation $(a \cdot^F_{x_i} b) \cdot^F_{x_i} d = a \cdot^F_{x_i} (b \cdot^F_{x_i} d)$ is a weak identity for all $i = 1, ..., n$. For this, assume that $(a \cdot^F_{x_i} b) \cdot^F_{x_i} d$ is defined. Then we have that a belongs to the set $W\mathcal{F}_{(\tau,\tau')}(X_n)$ but b is an n-ary term of type τ. It follows directly from our hypothesis that another side, i.e., $a \cdot^F_{x_i} (b \cdot^F_{x_i} d)$ is also defined. Finally, we show that both sides are identical. We begin in the case when both a and b are terms of type τ. In fact,
$(a \cdot^F_{x_i} b) \cdot^F_{x_i} d = S^n(S^n(a, x_1, ..., x_{i-1}, b, x_{i+1}, ..., x_n), x_1, ..., x_{i-1}, d, x_{i+1}, ..., x_n)$
and
$a \cdot^F_{x_i} (b \cdot^F_{x_i} d) = S^n(a, x_1, ..., x_{i-1}, S^n(b, x_1, ..., x_{i-1}, d, x_{i+1}, ..., x_n), x_{i+1}, ..., x_n)$.
By the satisfaction of (C1) of S^n over $W_\tau(X_n)$, the equation $(a \cdot^F_{x_i} b) \cdot^F_{x_i} d = a \cdot^F_{x_i} (b \cdot^F_{x_i} d)$ is true. Otherwise, we obtain

$(a \cdot_{x_i}^F b) \cdot_{x_i}^F d = R^n(a, x_1, ..., x_{i-1}, b, x_{i+1}, ..., x_n) \cdot_{x_i}^F d =$
$R^n(R^n(a, x_1, ..., x_{i-1}, b, x_{i+1}, ..., x_n), x_1, ..., x_{i-1}, d, x_{i+1}, ..., x_n)$ and
$a \cdot_{x_i}^F (b \cdot_{x_i}^F d) = R^n(a, x_1, ..., x_{i-1}, S^n(b, x_1, ..., x_{i-1}, d, x_{i+1}, ..., x_n), x_{i+1}, ..., x_n)$. Because R^n satisfies (C1), in this case, we obtain that $(a \cdot_{x_i}^F b) \cdot_{x_i}^F d = a \cdot_{x_i}^F (b \cdot_{x_i}^F d)$. □

The partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X_n), \cdot_{x_i}^F)$ is said to be the *z-product semigroup* derived from the partial unitary Menger algebra $\mathcal{M}_u$. By the symbol $(a_j)_{j=1}^n$, we denote $(a_1, ..., a_n)$. On the product $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$ of $W\mathcal{F}_{(\tau,\tau')}(X_n)$, we define the partial mapping

$$*^F : (W\mathcal{F}_{(\tau,\tau')}(X_n))^n \times (W\mathcal{F}_{(\tau,\tau')}(X_n))^n \multimap\!\!\rightarrow (W\mathcal{F}_{(\tau,\tau')}(X_n))^n$$

by

$$(a_j)_{j=1}^n *^F (b_j)_{j=1}^n = (\overline{R}^n(a_j, b_1, ..., b_n))_{j=1}^n$$

for every $(a_j)_{j=1}^n, (b_j)_{j=1}^n \in (W\mathcal{F}_{(\tau,\tau')}(X_n))^n$.

The following theorem shows that the partial binary operation $*^F$ is associative over the Cartesian product $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$.

Theorem 2.5. *$((W\mathcal{F}_{(\tau,\tau')}(X_n))^n, *^F)$ is a partial semigroup.*

Proof. Let $(a_j)_{j=1}^n, (b_j)_{j=1}^n$ and $(d_j)_{j=1}^n$ be n-tuples in $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$. We aim to show that the equation

$$((a_j)_{j=1}^n *^F (b_j)_{j=1}^n) *^F (d_j)_{j=1}^n = (a_j)_{j=1}^n *^F ((b_j)_{j=1}^n *^F (d_j)_{j=1}^n) \qquad (1)$$

holds as a weak identity. Assume that the left-hand side of the equation (1) is defined. Then we have that a tuple $(a_j)_{j=1}^n$ is in $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$ but $(b_j)_{j=1}^n$ and $(d_j)_{j=1}^n$ are in $(W_\tau(X_n))^n$. Furthermore, the right-hand side of (1) is defined. To show that both sides of the equation (1) coincide, we separate into three cases. If all $(a_j)_{j=1}^n, (b_j)_{j=1}^n)$ and $(d_j)_{j=1}^n$ belong to the Cartesian product $(W_\tau(X_n))^n$, we obtain

$$((a_j)_{j=1}^n *^F (b_j)_{j=1}^n) *^F (d_j)_{j=1}^n = (S^n(S^n(a_j, b_1, ..., b_n), d_1, ..., d_n))_{j=1}^n$$

and $(a_j)_{j=1}^n *^F ((b_j)_{j=1}^n *^F (d_j)_{j=1}^n) = (S^n(a_j, S^n(b_1, d_1, ..., d_n), ..., S^n(b_n, d_1, ..., d_n)))_{j=1}^n$. By the fact that S^n is superassociative over the set $W_\tau(X_n)$, we have that (1) is valid. If $(a_j)_{j=1}^n$ is an n-tuple of formulas of type (τ, τ') and both $(b_j)_{j=1}^n, (d_j)_{j=1}^n$ are n-tuples of n-ary terms of type τ, we obtain

$$((a_j)_{j=1}^n *^F (b_j)_{j=1}^n) *^F (d_j)_{j=1}^n = (R^n(R^n(a_j, b_1, ..., b_n), d_1, ..., d_n))_{j=1}^n$$

and $(a_j)_{j=1}^n *^F((b_j)_{j=1}^n *^F(d_j)_{j=1}^n) = (R^n(a_j, R^n(b_1, d_1, ..., d_n), ..., R^n(b_n, d_1, ..., d_n)))_{j=1}^n$. From the fact that the operation R^n satisfies (C1), we have that (1) is obtained as a weak identity.

In the case when there is a partition $\mathcal{P} = \{\{i_1, ..., i_k\}, \{i'_1, ..., i'_k\}\}$ on a set $\{1, ..., n\}$ such that a_{i_l} belongs to $W_\tau(X_n)$ and $a_{i'_l}$ is a formula in $\mathcal{F}_{(\tau,\tau')}(W_\tau(X_n))$ and both $(b_j)_{j=1}^n, (d_j)_{j=1}^n$ are n-tuples in $(W_\tau(X_n))^n$, we have that $((a_j)_{j=1}^n *^F (b_j)_{j=1}^n) *^F (d_j)_{j=1}^n$ equals to$(e_j)_{j=1}^n *^F (d_j)_{j=1}^n$ where $e_{i_l} = S^n(a_{i_l}, b_1, ..., b_n)$ and $e_{i'_l} = R^n(a_{i'_l}, b_1, ..., b_n)$ for all $l = 1, ..., k$, subsequently, $(p_j)_{j=1}^n$ where $p_{i_l} = S^n(S^n(a_{i_l}, b_1, ..., b_n), d_1, ..., d_n)$ and $p_{i'_l} = R^n(R^n(a_{i'_l}, b_1, ..., b_n), d_1, ..., d_n)$ for all $l = 1, ..., k$. On the other hand, we get $(a_j)_{j=1}^n *^F((b_j)_{j=1}^n *^F(d_j)_{j=1}^n)$ is equal to $(a_j)_{j=1}^n *^F(S^n(b_j, d_1, ..., d_n))_{j=1}^n$, subsequently, $(\overline{p}_j)_{j=1}^n$ where $\overline{p}_{i_l} = S^n(a_{i_l}, S^n(b_1, d_1, ..., d_n), ..., S^n(b_n, d_1, ..., d_n))$ and
$\overline{p}_{i'_l} = R^n(a_{i'_l}, S^n(b_1, d_1, ..., d_n), ..., S^n(b_n, d_1, ..., d_n))$ for all $l = 1, ..., k$. As a consequence, we have

$$((p_j)_{j=1}^n) = (\overline{p}_j)_{j=1}^n$$

where $p_{i_l} = \overline{p}_{i_l}$ and $p_{i'_l} = \overline{p}_{i'_l}$ for all $l = 1, ..., k$. This shows that the equation (1) holds as a weak identity. □

The partial semigroup $((W\mathcal{F}_{(\tau,\tau')}(X_n))^n, *^F)$ in Theorem 2.5 is called the *binary partial comitant* of the partial algebra $\mathcal{M}_u$. More advanced topics in the binary comitanat induced by any Menger algebra, we refer the readers to the monograph [6].

Recall from [1, 12] that the concept of homomorphism for partial algebras is different from a total algebra. In fact, if $\mathcal{A}, \mathcal{B}$ are partial algebras of the same type with indexed sets $\{f_i^{\mathcal{A}} \mid i \in I\}$ and $\{f_i^{\mathcal{B}} \mid i \in I\}$ of partial operations on A and B, respectively, then by a *weak homomorphism* we mean a mapping $\phi : A \to B$ satisfying: if $(a_1, ..., a_{n_i}) \in \text{dom} f_i^{\mathcal{A}}$, then $(\phi(a_1), ..., \phi(a_{n_i})) \in \text{dom} f_i^{\mathcal{B}}$ and then, for all $i \in I$,

$$\phi(f_i^{\mathcal{A}}(a_1, ..., a_{n_i})) = f_i^{\mathcal{B}}(\phi(a_1), ..., \phi(a_{n_i})).$$

If a weak homomorphism ϕ is injective, we call ϕ a *weak monomorphism.* In this case, we say that $\mathcal{A}$ is *weak embeddable* into $\mathcal{B}$.

Theorem 2.6. *The partial diagonal semigroup* $(W\mathcal{F}_{(\tau,\tau')}(X_n), +^F)$ *is weak embeddable into the binary partial comitant* $((W\mathcal{F}_{(\tau,\tau')}(X_n))^n, *^F)$.

Proof. For any a in the partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X_n), +^F)$, we define the mapping from $W\mathcal{F}_{(\tau,\tau')}(X_n)$ to the Cartesian set $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$ by $\varphi(a) =$

$(a, ..., a)$. To show that φ is a weak monomorphism, suppose that $(y_1, y_2) \in \mathrm{dom}+^F$. It follows that $y_1 \in W\mathcal{F}_{(\tau,\tau')}(X_n)$ and $y_2 \in W_\tau(X_n)$. Then by the definition of φ, we have $(\varphi(y_1), \varphi(y_2)) = ((y_1, ..., y_1), (y_2, ..., y_2)) \in \mathrm{dom} *^F$. From this, we obtain two cases: $y_1, y_2 \in W_\tau(X_n)$ and $y_1 \in \mathcal{F}_{(\tau,\tau')}(W_\tau(X_n)), y_2 \in W_\tau(X_n)$. If y_1, y_2 are terms in $W_\tau(X_n)$, we have $\varphi(y_1 +^F y_2) = \varphi(S^n(y_1, y_2, ..., y_2)) = (S^n(y_1, y_2, ..., y_2), ..., S^n(y_1, y_2, ..., y_2)) = (y_1, ..., y_1) *^F (y_2, ..., y_2) = \varphi(y_1) *^F \varphi(y_2)$. If y_1 is an n-ary formula of type (τ, τ') and y_2 is an n-ary term of type τ, we obtain $\varphi(y_1 +^F y_2) = \varphi(R^n(y_1, y_2, ..., y_2))$.

On the other hand, we have $\varphi(y_1) *^F \varphi(y_2) = (y_1, ..., y_1) *^F (y_2, ..., y_2) = (R^n(y_1, y_2, ..., y_2), ..., R^n(y_1, y_2, ..., y_2))$. In this case, we get $\varphi(y_1 +^F y_2) = \varphi(y_1) *^F \varphi(y_2)$. Clearly, φ is injective. Therefore, the partial semigroup $(W\mathcal{F}_{(\tau,\tau')}(X_n), +^F)$ is weak embeddable into $((W\mathcal{F}_{(\tau,\tau')}(X_n))^n, *^F)$. □

3. Weak monomorphisms of formulas

Let A be a nonempty set and I a nonempty set of positive integers. On the set $F_n(A)$ of all n-ary partial functions on A, an $(n+1)$-ary operation (also called *composition of partial functions*) $\mathcal{O}$ can be defined by the following way:

If $f, g_1, ..., g_n \in F_n(A)$ and $(a_1, ..., a_n) \in A^n$, then by $\mathcal{O}(f, g_1, ..., g_n)$ we denote the partial function

$$\mathcal{O} : (F_n(A))^{n+1} \multimap F_n(A)$$

defined by

$$\begin{aligned} \mathrm{dom}\ (\mathcal{O}) = \{(a_1, ..., a_n) \in A^n | (a_1, ..., a_n) \in \textstyle\bigcap_{i=1}^n \mathrm{dom}(g_i)\} \\ \text{and } (g_1(a_1, ..., a_n), ..., g_n(a_1, ..., a_n)) \in \mathrm{dom}(f) \\ \text{and } \mathcal{O}(f, g_1, ..., g_n)(a_1, ..., a_n) = f(g_1(a_1, ..., a_n), ..., g_n(a_1, ..., a_n)). \end{aligned}$$

It is clear that this composition satisfies the superassociativity, i.e., for any $n \in I$, $f, g_1, ..., g_n, h_1, ..., h_n \in F_n(A)$, we have

$$\mathcal{O}(\mathcal{O}(f, g_1, ..., g_n), h_1, ..., h_n) = \mathcal{O}(f, \mathcal{O}(g_1, h_1, ..., h_n), ..., \mathcal{O}(g_n, h_1, ..., h_n)).$$

The algebra $(F_n(A), \mathcal{O})$ is called the *Menger algebra of rank n of partial n-ary functions*. Obviously, a semigroup of partial transformations is a particular case of this algebra if $n = 1$.

The solution to the problem "Can the partial unitary Menger algebra $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n, (x_j)_{j\leqslant n, n\in\mathbb{N}^+})$ be weak embeddable into the algebra of functions defined on some set?" is now proposed.

Theorem 3.1. *The partial unitary Menger algebra* $\mathcal{M}_u$ *of type* $(n+1, 0, ..., 0)$ *is weak embeddable into unitary Menger algebra of the same type of partial* n*-ary functions such that its selectors correspond to the projection operations of this set.*

Proof. We begin the proof of this theorem by constructing the partial n-ary function with respect to each element of the partial algebra $\mathcal{M}_u$ of type $(n+1, 0, ..., 0)$. For each $n \in \mathbb{N}^+$ and each element $a \in W\mathcal{F}_{(\tau,\tau')}(X_n)$, the partial n-ary function

$$\lambda_a : (W\mathcal{F}_{(\tau,\tau')}(X_n))^n \multimap W\mathcal{F}_{(\tau,\tau')}(X_n)$$

can be defined by

$$\lambda_a(b_1, ..., b_n) = \overline{R}^n(a, b_1, ..., b_n)$$

for all $b_1, ..., b_n \in W\mathcal{F}_{(\tau,\tau')}(X_n)$.

It is clear that the set $\overline{F_n(W\mathcal{F}_{(\tau,\tau')}(X_n))} = \{\lambda_a \mid a \in W\mathcal{F}_{(\tau,\tau')}(X_n)\}$ is closed with respect to the partial composition of functions. Hence the algebra

$$(\overline{F_n(W\mathcal{F}_{(\tau,\tau')}(X_n))}, \mathcal{O}, (\mathrm{pr}_i^n)_{i \leqslant n, n \in \mathbb{N}^+})$$

of type $(n+1, 0, ..., 0)$ is obtained where pr_i^n is the projection operation defined by $\mathrm{pr}_i^n(b_1, ..., b_n) = b_i$ for all $i = 1, ..., n$.

Since $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n, (x_j)_{j \leqslant n, n \in \mathbb{N}^+})$ is a partial algebra, we actually have to consider the mapping

$$\phi : W\mathcal{F}_{(\tau,\tau')}(X_n) \to \overline{F_n(W\mathcal{F}_{(\tau,\tau')}(X_n))},$$

defined by

$$\phi(a) = \lambda_a$$

for all $a \in W\mathcal{F}_{(\tau,\tau')}(X_n)$.

We now show that the mapping ϕ is a weak homomorphism from the partial Menger algebra $\mathcal{M}_u$ to $(\overline{F_n(W\mathcal{F}_{(\tau,\tau')}(X_n))}, \mathcal{O}, (\mathrm{pr}_i^n)_{i \leqslant n, n \in \mathbb{N}^+})$. Indeed, for all $a, b_1, ..., b_n \in W\mathcal{F}_{(\tau,\tau')}(X_n)$, we prove that the equation

$$\phi(\overline{R}^n(a, b_1, ..., b_n)) = \mathcal{O}(\phi(a), \phi(b_1), ..., \phi(b_n))$$

is satisfied as a weak identity. To do this, assume that $(a, b_1, ..., b_n) \in \mathrm{dom}(\overline{R}^n)$. This implies that a belongs to $W\mathcal{F}_{(\tau,\tau')}(X_n)$ and $b_1, ..., b_n$ belong to $W_\tau(X_n)$. Applying the definition of ϕ, we obtain $(\phi(a), \phi(b_1), ..., \phi(b_n)) \in \mathrm{dom}(\mathcal{O})$. Thus the above equation is equivalent to

$$\lambda_{\overline{R}^n(a,b_1,...,b_n)} = \mathcal{O}(\lambda_a, \lambda_{b_1}, ..., \lambda_{b_n}).$$

In order to prove that this equality is satisfied, we suppose that $(y_1, ..., y_n)$ are elements in $(W\mathcal{F}_{(\tau,\tau')}(X_n))^n$. If an n-tuple $(y_1, ..., y_n)$ comes from the Cartesian product $(W_\tau(X_n))^n$, then $\lambda_{\overline{R}^n(a,b_1,...,b_n)}(y_1, ..., y_n)$ is defined and equals to $\overline{R}^n(\overline{R}^n(a, b_1, ..., b_n), y_1, ..., y_n)$. Since we known from Theorem 2.1 that the partial operation satisfies the superassociativity as a weak identity, by the definition of $\mathcal{O}$, we have

$$\begin{aligned}
\lambda_{\overline{R}^n(a,b_1,...,b_n)}(y_1, ..., y_n) &= \overline{R}^n(\overline{R}^n(a, b_1, ..., b_n), y_1, ..., y_n) \\
&= \overline{R}^n(a, \overline{R}^n(b_1, y_1, ..., y_n), ..., \overline{R}^n(b_n, y_1, ..., y_n)) \\
&= \lambda_a(\overline{R}^n(b_1, y_1, ..., y_n), ..., \overline{R}^n(b_n, y_1, ..., y_n)) \\
&= \lambda_a(\lambda_{b_1}(y_1, ..., y_n), ..., \lambda_{b_n}(y_1, ..., y_n)) \\
&= \mathcal{O}(\lambda_a, \lambda_{b_1}, ..., \lambda_{b_n})(y_1, ..., y_n).
\end{aligned}$$

Furthermore, it is also an injection. In fact, assume that $\lambda_{a_1} = \lambda_{a_2}$ for some $a_1, a_2 \in W\mathcal{F}_{(\tau,\tau')}(X_n), n \in \mathbb{N}^+$. Then

$$\lambda_{a_1}(y_1, ..., y_n) = \lambda_{a_2}(y_1, ..., y_n).$$

Thus, in particular, we have

$$\overline{R}^n(a_1, y_1, ..., y_n) = \overline{R}^n(a_2, y_1, ..., y_n).$$

Replacing each element y_j in this equation by variables x_j in $W_\tau(X_n)$ for all $j = 1, ..., n$, we obtain

$$\overline{R}^n(a_1, x_1, ..., x_n) = \overline{R}^n(a_2, x_1, ..., x_n).$$

According to Theorem 2.2, we conclude $a_1 = a_2$. This shows that ϕ is a weak monomorphism.

Finally, for each $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$, and $x_i \in W\mathcal{F}_{(\tau,\tau')}(X_n)$, we have

$$\lambda_{x_i}(y_1, ..., y_n) = R^n(x_i, y_1, ..., y_n) = y_i$$

for all $(y_1, ..., y_n) \in (W_\tau(X_n))^n$. Thus $\lambda_{x_i} = \mathrm{pr}_i^n$, which means that selectors are transformed into projection operations. Thus, the proof is finished. □

With out variables $x_1, ..., x_n$ acting as the nullary operations, we have the following theorem.

Theorem 3.2. *The partial Menger algebra* $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n)$ *of type* $(n+1)$ *is weak embeddable into a Menger algebra of the same type of partial* n*-ary functions.*

Proof. let e, c be constant elements not belong to $W\mathcal{F}_{(\tau,\tau')}(X_n)$ and $e \neq c$. We extend a set $W\mathcal{F}_{(\tau,\tau')}(X_n)$ to $\overline{W\mathcal{F}_{(\tau,\tau')}(X_n)} := W\mathcal{F}_{(\tau,\tau')}(X_n) \cup \{e, c\}$. For each a in $\overline{W\mathcal{F}_{(\tau,\tau')}(X_n)}$, an n-ary partial function on $\overline{W\mathcal{F}_{(\tau,\tau')}(X_n)}$ can be defined by the following

$$\lambda'_a(b_1, ..., b_n) := \begin{cases} \overline{R}^n(a, b_1, ..., b_n) & \text{if } b_j \in W\mathcal{F}_{(\tau,\tau')}(X_n) \text{ for all } j = 1, ..., n, \\ e & \text{if } b_j = e \text{ for all } j = 1, ..., n, \\ c & \text{otherwise.} \end{cases}$$

We prove that

$$\lambda'_{\overline{R}^n(a,b_1,...,b_n)} = \mathcal{O}(\lambda'_a, \lambda'_{b_1}, ..., \lambda'_{b_n})$$

is a weak identity for any $a, b_1, ..., b_n \in \overline{W\mathcal{F}_{(\tau,\tau')}(X_n)}$.

Let $y_1, ..., y_n \in W\mathcal{F}_{(\tau,\tau')}(X_n)$. It follows immediately from Theorem 3.1 that

$$\lambda'_{\overline{R}^n(a,b_1,...,b_n)}(y_1, ..., y_n) = \mathcal{O}(\lambda'_a, \lambda'_{b_1}, ..., \lambda'_{b_n})(y_1, ..., y_n).$$

If $y_1, ..., y_n \in \{e\}$, then by the definition, we have

$$\lambda'_{\overline{R}^n(a,b_1,...,b_n)}(e, ..., e) = \overline{R}^n(a, b_1, ..., b_n)$$

and $\mathcal{O}(\lambda'_a, \lambda'_{b_1}, ..., \lambda'_{b_n})(e, ..., e) = \lambda'_a(\lambda'_{b_1}(e, ..., e), ..., \lambda'_{b_n}(e, ..., e)) = \lambda'_a(b_1, ..., b_1)$ $= \overline{R}^n(a, b_1, ..., b_n)$, which implies

$$\lambda'_{\overline{R}^n(a,b_1,...,b_n)}(e, ..., e) = \mathcal{O}(\lambda'_a, \lambda'_{b_1}, ..., \lambda'_{b_n})(e, ..., e).$$

In all other cases, we have

$$\lambda'_{\overline{R}^n(a,b_1,...,b_n)}(y_1, ..., y_n) = c = \mathcal{O}(\lambda'_a, \lambda'_{b_1}, ..., \lambda'_{b_n})(y_1, ..., y_n).$$

This finishes the proof of a weak homomorphism. Assume that $\lambda'_{a_1} = \lambda'_{a_2}$. Due to the existence of e in $\overline{W\mathcal{F}_{(\tau,\tau')}(X_n)}$, it implies that $\lambda'_{a_1}(e, ..., e) = \lambda'_{a_2}(e, ..., e)$, as a result $a_1 = a_2$. Thus, the mapping $\varphi : a \mapsto \lambda'_a$ is a weak monomorphism from the partial Menger algebra $(W\mathcal{F}_{(\tau,\tau')}(X_n), \overline{R}^n)$ to some algebra of partial n-ary functions. □

4. Conclusions

In this work, the partial operation of type $(n+1)$ defined on the set of terms and formulas which satisfies the superassociative law as a weak identity is seeked. The main result shows that the set $W\mathcal{F}_{(\tau,\tau')}(X_n)$ equipped with one partial operation $\overline{R}^n$ and n elements of nullary operations forms a unitary Menger algebra. Three binary operations, $+^F$, $\cdot_{x_i}^F$ and $*^F$ defined on $W\mathcal{F}_{(\tau,\tau')}(X_n)$ and derived from $\overline{R}^n$, are weak associative and their corresponding partial structures are obtained. Finally, we show that there is a weak monomorphism from the partial algebra $\mathcal{M}_u$ into the algebra of partial functions. Another direction of the future research in this line should be devoted to the study of partial superassociative operations on formula languages.

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