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Weakly $g(x)$-invo-clean rings

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Abstract. An associative ring R with identity is a weakly $g(x)$-invo-clean ring if every $r \in R$ can be written as $r = v + s$ or $r = v - s$, where $v^2 = 1$ and s is a root of $g(x) \in C(R)[x]$. It is proved that an associtive ring R is weakly invo-clean and $2^2 = 1$ if and only if R is weakly $(x^2 - 2x)$-invo-clean if and only if R is weakly $(x^2 + 2x)$-invo-clean.

1. Introduction

Let R be an associative ring with identity. An element v of R is said to be an involution if $v^2 = 1$ and a quasi-involution if either v or $1 - v$ is an involution [14]. Let $U(R)$, $Id(R)$, $Nil(R)$, $C(R)$ and $Inv(R)$ will denote respectively the set of units, the set of idempotents, the set of nilpotents, the set of centrals and the set of involutions of R.

The ring R is said to be

• *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$ [3, 15],

• *weakly clean* if each $r \in R$ can be expressed as $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in Id(R)$ [1, 6, 7, 13].

• $g(x)$*-clean* if each its element is $g(x)$*-clean*, i.e. it can be written as the sum of an unit and a root of $g(x) \in C(R)[x]$ [12],

• $g(x)$*-clean* if each element is *weakly* $g(x)$*-clean*, i.e. it can be written as s either the sum or difference of an unit and a root of $g(x)$ [4],

• *invo-clean* if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ [8, 10],

• $g(x)$*-invo-clean* if for each $r \in R$ there exist $v \in Inv(R)$ and a root s of $g(x)$ such that $r = v + s$ [11],

• *weakly invo-clean* if for each $r \in R$ there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $r = v + e$ or $r = v - e$ [9].

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• *weakly $g(x)$-invo-clean* if for every $r \in R$ exist $v \in Inv(R)$ and a root s of $g(x)$ such that $r = v + s$ or $r = v - s$.

We study various properties of weakly $g(x)$-invo-clean rings as a proper generalization of invo-clean rings and a proper subclass of $g(x)$-invo-clean rings.

2. Main results

Simple examples of invo-clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$. Oppositely, $\mathbb{Z}_5$ is not invo-clean but however they are clean being finite [8].

It is evident that invo-clean rings are weakly invo-clean as this implication is extremely non-reversible by looking quickly at the field $\mathbb{Z}_5$.

Let $R = \mathbb{Z}_5$ and $g(x) = x^5 + 4x \in C(R)[x]$. Then R is $g(x)$-invo-clean.

It is clear that the $(x^2 - x)$-weakly invo-clean rings are precisely the weakly invo-clean rings. Obviously, $g(x)$-invo-clean rings are weakly $g(x)$-invo-clean and also if $g(-x) = -g(x)$ or $g(-x) = g(x)$, then the concepts $g(x)$-invo-clean and weakly $g(x)$-invo-clean coincide.

Every $g(x)$-invo-clean ring is weakly $g(x)$-invo-clean. The following example shows that a weakly $g(x)$-invo-clean ring is neither $g(x)$-invo-clean nor invo-clean ring, in general.

Example 2.1. (i) Let $R = \mathbb{Z}_5$ and $g(x) = x^5 + 4x \in C(R)[x]$. Then $Inv(R) = \{1, 4\}$, and $Root(g(x)) = \{0, 2, 3, 4\}$ and $Id(R) = \{0, 1\}$. Hence $\mathbb{Z}_5$ is a weakly $g(x)$-invo-clean ring which is not invo-clean.

(ii) Let $R = \mathbb{Z}_5$ and $g(x) = x^2 - x \in C(R)[x]$. Since R is weakly invo-clean but not invo-clean, R is weakly $g(x)$-invo-clean but not $g(x)$-invo-clean.

(iii) Let $R = \mathbb{Z}_7$ and $g(x) = x^7 + 6x \in C(R)[x]$. Then $Inv(\mathbb{Z}_7) = \{0, 1, 6\}$ and $Root(g(x)) = \{0, 2, 3, 5, 6\}$, $Id(\mathbb{Z}_7) = \{0, 1\}$. Hence $\mathbb{Z}_7$ is a weakly $g(x)$-invo-clean which is not weakly invo-clean.

Proposition 2.2. *Let R be a Boolean ring, $|R| > 2$, $c \in R \setminus \{0, 1\}$ and $g(x) = (x + 1)(x + c)$. Then R is not weakly $g(x)$-invo-clean.*

Proof. Suppose that R is not weakly $g(x)$-invo-clean. Hence $c = v + s$ or $c = v - s$ such that $v \in Inv(R)$ and $g(s) = 0$. Since $v \in Inv(R)$ and R is a Boolean ring, $v = 1$. So $s = c - 1$ or $s = 1 - c$. But $g(c - 1) \neq 0$ and $g(1 - c) \neq 0$. a contradiction. Then R is not weakly $g(x)$-invo-clean. □

Let R and S be two rings. Suppose $\phi : C(R) \to C(S)$ is a ring homomorphism with $\phi(1_R) = 1_S$. If $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$, we let $g_\phi(x) := \sum_{i=0}^{n} \phi(r_i) x^i \in C(S)[x]$.

Lemma 2.3. *Let R and S be two rings, $\phi : R \to S$ be a ring epimorphism and $g(x) = \sum_{i=0}^{n} r_i x^i \in C(R)[x]$. If R is weakly $g(x)$-invo-clean, then S is weakly $g_\phi(x)$-invo-clean.*

Proof. Suppose that $t \in S$. Since ϕ is a ring epimorphism, there exists $r \in R$ such that $t = \phi(r)$. Since R is weakly $g(x)$-invo-clean, $r = v + s$ or $r = v - s$ such that $v \in Inv(R)$ and $g(s) = 0$. Hence $t = \phi(r) = \phi(v) + \phi(s)$ or $t = \phi(r) = \phi(v) - \phi(s)$. Since $v \in Inv(R)$, $v^2 = 1$. Hence $(\phi(v))^2 = 1_S$, and so $\phi(v) \in Inv(S)$. Since $g(s) = 0$, $g_\phi(\phi(s)) = 0$. Therefore S is weakly $g_\phi(x)$-invo-clean. □

Definition 2.4. Let R and S be two rings such that R is weakly $g(x)$-invo-clean. If there is an epimorphism $\phi : R \to S$, then S is called a *weakly $\overline{g}(x)$-invo-clean.*

Corollary 2.5. *Let R and S be two rings. Then the following statements hold.*

- (i) *Let I be an ideal of a weakly $g(x)$-invo-clean ring R. Then R/I is weakly $\overline{g}(x)$-invo-clean.*
- (ii) *Let the upper triangular matrix ring $T_n(R)$ is weakly $g(x)$-invo-clean. Then R is weakly $\overline{g}(x)$-invo-clean.*

Proof. (i). Suppose that R is weakly $g(x)$-invo-clean. Since $\phi : R \to R/I$ is an epimorphism, R/I is weakly $\overline{g}(x)$-invo-clean, by Lemma 2.3.

(ii). Let the upper triangular matrix ring $T_n(R)$ is weakly $g(x)$-invo-clean. Since there exists an epimorphism $\phi : T_n(R) \to R$, R is weakly $\overline{g}(x)$-invo-clean, by Lemma 2.3. □

Lemma 2.6. *Let $\{R_i\}_{i=1}^{n}$ be rings, $R = \prod_{i=1}^{n} R_i$ and $g(x) \in \mathbb{Z}[x]$. Then R is weakly $g(x)$-invo-clean if and only if there exist $1 \leqslant l \leqslant n$ such that R_l is weakly $g(x)$-invo-clean and R_j is $g(x)$-invo-clean for all $j \neq l$.*

Proof. Suppose that $i \in \{1, 2, \ldots, n\}$. Hence R_i is weakly $g(x)$-invo-clean, by Lemma 2.3. Assume that neither R_1 nor R_2 are $g(x)$-invo-clean. Hence there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that r_1 is not a sum of an involution and a root of $g(x)$ and r_2 is not a difference of an involution and a root of $g(x)$. Then $(r_1, r_2) \in R_1 \times R_2$ is not weakly $g(x)$-invo-clean, a contradiction.

Conversely, Suppose that there exist $1 \leqslant l \leqslant n$ such that R_l is weakly $g(x)$-invo-clean and R_j is $g(x)$-invo-clean for all $j \neq l$. Let $r = (r_i) \in R$. Then there exist $v_l \in Inv(R)$ and a root s_l of $g(x)$ such that $r_l = v_l + s_l$ or $r_l = v_l - s_l$. If $r_l = v_l + s_l$, then for each $i \neq l$, $r_i = v_i + s_i$ such that $v_i \in Inv(R)$ and $g(s_i) = 0$. Then $r = (v_i) + (s_i)$ such that $(v_i) \in Inv(R)$ and $g((s_i)) = 0$. If $r_l = v_l - s_l$, then for each $i \neq l$, $r_i = v_i - s_i$ such that $v_i \in Inv(R)$ and $g(s_i) = 0$. Then $r = (v_i) - (s_i)$ such that $(v_i) \in Inv(R)$ and $g((s_i)) = 0$. Therefore R is weakly $g(x)$-invo-clean. □

Let R be a ring with an identity and S be a ring which is an R-R-bimodule such that $(s_1 s_2)r = s_1(s_2 r)$, $(s_1 r)s_2 = s_1(r s_2)$ and $(r s_1)s_2 = r(s_1 s_2)$ hold for all $s_1, s_2 \in S$ and $r \in R$. The *ideal extension* of R by S is defined to be the additive abelian group $I(R, S) = R \oplus S$ with multiplication $(r, s_1)(r', s_2) = (rr', rs_2 + s_1 r' + s_1 s_2)$. If $g(x) = (r_0, s_0) + (r_1, s_1)x + \cdots + (r_n, s_n)x^n \in C(I(R, S))[x]$, then $g_R(x) = r_0 + r_1 x + \cdots + r_n x^n \in C(R)[x]$.

Lemma 2.7. *Let R be a ring with an identity and S be a ring which is an R-R-bimodule. If $I(R, S)$ is weakly $g(x)$-invo-clean, then R is weakly $g_R(x)$-invo-clean.*

Proof. Suppose that $\phi_R : I(R, S) \to R$ by $\phi_R(r, s) = r$. Since ϕ_R is a ring epimorphism, R is weakly $g_R(x)$-invo-clean by Lemma 2.3. □

Let R be a ring and $\alpha : R \longrightarrow R$ be a ring endomorphism. The ring $R[[x, \alpha]]$ of skew formal power series over R; that is all formal power series in x with coefficients from R with multiplication defined by $xr = \alpha(r_x)$ for all $r \in R$. It is clear that $R[[x]] = R[[x, 1_R]]$ and $R[[x, \alpha]] \cong I(R, \langle x \rangle)$ where $\langle x \rangle$ is the ideal generated by x.

Proposition 2.8. *Let R be a ring and $\alpha : R \to R$ be a ring endomorphism. If $R[[x, \alpha]]$ is weakly $g(x)$-invo-clean, then R is weakly $g_\phi(x)$-invo-clean such that $\phi : R[[x, \alpha]] \to R$ is defined by $\phi(f) = f(0)$.*

Proof. Suppose that the skew formal power series $R[[x, \alpha]]$ over R is weakly $g(x)$-invo-clean. Since $\phi : R[[x, \alpha]] \longrightarrow R$ is defined by $\phi(f) = f(0)$ is an epimorphism, R is weakly $\overline{g}(x)$-invo-clean, by Lemma 2.3. □

Theorem 2.9. *Let R be a ring, k be an even positive integer and $a, b \in R$. Then R is weakly $(ax^k - bx)$-invo-clean if and only if R is weakly $(ax^k + bx)$-invo-clean.*

Proof. Assume taht R is weakly $(ax^k - bx)$-invo-clean and $r \in R$. Hence $-r = v \pm s$ where $v \in Inv(R)$ and $as^k - bs = 0$. Then $r = (-v) \pm (-s)$ where $-v \in Inv(R)$ and $a(-s)^k + b(-s) = 0$. Therefore R is weakly $(ax^k + bx)$-invo-clean.

Conversely, assume that R is weakly $(ax^k + bx)$-invo-clean and $r \in R$. Hence $-r = v \pm s$ where $v \in Inv(R)$ and $as^k + bs = 0$. Then $r = (-v) \pm (-s)$ where $-v \in Inv(R)$ and $a(-s)^k - bs = 0$. Therefore R is weakly $(ax^k - bx)$-invo-clean. □

Theorem 2.9 does not hold for odd powers.

Example 2.10. Let $R = \mathbb{Z}_5$. Since $Inv(\mathbb{Z}_5) = \{1, 2\}$, $Root(x^5 + 4x) = \{0, 2, 3, 4\}$ and $Root(x^5 - 4x) = \{0, 2, 3, 4\}$. Then the ring $\mathbb{Z}_5$ is a weakly $(x^5 + 4x)$-invo-clean ring which is not weakly $(x^5 - 4x)$-invo-clean.

Theorem 2.11. *Let R be a ring and $a \in C(R)$. Then R is weakly invo-clean and $a \in Inv(R)$ if and only if R is weakly $x(x - a)$-invo-clean.*

Proof. Suppose that R is weakly invo-clean and $a \in Inv(R)$. Let $r \in R$. Then $ra = v + e$ or $ra = v - e$ for some $v \in Inv(R)$ and $e \in Id(R)$. Hence $r = va + ea$ or $r = va - ea$. It is clear that $va \in Inv(R)$ and ea is a root of $x(x - a)$. Therefore R is weakly $x(x - a)$-invo-clean.

Conversely, assume that R is weakly $x(x-a)$-invo-clean. Write $0 = v+s$ or $0 = v - s$ where $v \in Inv(R)$ and $s(s - a) = 0$. Hence $s = \pm v \in Inv(R)$ and $s - a = 0$, and so $a \in Inv(R)$. Suppose that $r \in R$ and write $ra = v+s$ or $ra = v - s$ where $v \in Inv(R)$ and $s(s - a) = 0$. Hence $r = va + sa$ or $r = va - sa$ such that $va \in Inv(R)$ and

$$(sa)^2 = s(s - a + a)a^2 = s(s - a)a^2 + saa^2 = sa.$$

Then R is weakly invo-clean. □

Corollary 2.12. *Let R be a ring and $n \in \mathbb{N}$. Then*

(*i*) *R is weakly invo-clean and $2 \in Inv(R)$,*

(*ii*) *R is weakly $(x^2 - 2x)$-invo-clean,*

(*iii*) *R is weakly $(x^2 + 2x)$-invo-clean.*

Proof. Follows from Theorems 2.9 and 2.11. □

Lemma 2.13. *Let R be a commutative ring. Then $R[x]$ is not weaklly $(x^2 - x)$-invo-clean.*

Proof. Suppose that $R[x]$ is weaklly $(x^2 - x)$-invo-clean. Hence $x = v \pm s$ where $v \in Inv(R[x])$ and s is a root of $x^2 - x$. Then $x \mp s \in Inv(R[x])$. So $1 \in Nil(R)$ by [11, Lemma 3.6], a contradiction. $\square$

A *Morita context* is a 6-tuple $\mathcal{M}(R, M, K, S, \phi, \psi)$, where R and S are rings, M is an (R, S)-bimodule, K is a (S, R)-bimodule, and $\phi : M \otimes_S K \to R$ and $\psi : K \otimes_R M \to S$ are bimodule homomorphisms such that $T(\mathcal{M}) = \begin{pmatrix} R & M \\ K & S \end{pmatrix}$ is an associative ring with the obvious matrix operations. The ring $T(\mathcal{M})$ is the Morita context ring associated with $\mathcal{M}$. For more on Morita context rings see [2, 5, 16, 17]. If $g(x) = \begin{pmatrix} r_0 & m_0 \\ k_0 & s_0 \end{pmatrix} + \begin{pmatrix} r_1 & m_1 \\ k_1 & s_1 \end{pmatrix} x + \cdots + \begin{pmatrix} r_n & m_n \\ k_n & s_n \end{pmatrix} x^n \in C(T(\mathcal{M}))[x]$, then $g_R(x) = r_0 + r_1 x + \cdots + r_n x^n \in C(R)[x]$ and $g_S(x) = rs_0 + s_1 x + \cdots + s_n x^n \in C(S)[x]$.

Theorem 2.14. *Let the Morita context ring $T(\mathcal{M}) = \begin{pmatrix} R & M \\ K & S \end{pmatrix}$ is weakly $g(x)$-invo-clean with $\phi, \psi = 0$. Then R is weakly $g_R(x)$-invo-clean and S is weakly $g_S(x)$-invo-clean.*

Proof. Suppose that $T(\mathcal{M})$ is weakly $g(x)$-invo-clean with $\phi, \psi = 0$. Hence $I = \begin{pmatrix} 0 & M \\ K & S \end{pmatrix}$ and $J = \begin{pmatrix} R & M \\ K & 0 \end{pmatrix}$ are two ideals of $T(\mathcal{M})$. Since $T(\mathcal{M})/I \cong R$ and $T(\mathcal{M})/J \cong S$, the assertion holds by Lemma 2.3. $\square$

Corollary 2.15. *Let R and S be two rings and M be an (R, S)-bimodule. Let $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be the formal triangular matrix ring. If T is weakly $g(x)$-invo clean, then R is weakly $g_R(x)$-invo-clean and S is weakly $g_S(x)$-invo-clean.*

Proof. Follows from Theorem 2.14. $\square$

Corollary 2.16. *Let R be a commutative ring and M be an (R, R)-bimodule such that $2M = 0$. Then $T = \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$ is weakly $g(x)$-invo clean if and only if R is weakly $g(x)$-invo-clean.*

Proof. Follows from Lemma 2.3 and Corollary 2.15. □

We close the article with the following two problems.

Problem 2.17. *What is the behaviour of the matrix rings over weakly $g(x)$-invo clean rings?*

Problem 2.18. *Let R be a weakly $g(x)$-invo clean ring and $e \in Id(R)$. What is the behaviour of the corner ring eRe?*

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