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Right regular triples of semigroups

Csaba Tóth

Abstract. Let $\mathcal{M}(S;\Lambda;P)$ denote a Rees $I\times\Lambda$ matrix semigroup without zero over a semigroup S, where I is a singleton. If θ_S denotes the kernel of the right regular representation of a semigroup S, then a triple A,B,C of semigroups is said to be right regular, if there are mappings $A\stackrel{P}{\longleftarrow}B$ and $B\stackrel{P'}{\longrightarrow}C$ such that $\mathcal{M}(A;B;P)/\theta_{\mathcal{M}(A;B;P)}\cong\mathcal{M}(C;B;P')$. In this paper we examine right regular triples of semigroups.

1. Introduction and motivation

The notion of right regular triples of semigroups is defined in [19], where a special type of Rees matrix semigroups without zero over semigroups are examined. A triple A,B,C of semigroups is said to be right regular, if there are mappings

$$A\stackrel{P}{\longleftarrow}B\stackrel{P'}{\longrightarrow}C$$

such that the factor semigroup $\mathcal{M}(A;B;P)/\theta_{\mathcal{M}(A;B;P)}$ is isomorphic to the semigroup $\mathcal{M}(C;B;P')$, where $\theta_{\mathcal{M}(A;B;P)}$ is the kernel of the right regular representation of the semigroup $\mathcal{M}(A;B;P)$. In [19] it is proved that if A,B,C are semigroups such that $A/\theta_A\cong B$ and $B/\theta_B\cong C$, then the triple A,B,C is right regular. There is also an example given for a right regular triple A,B,C of semigroups such that none of the conditions $A/\theta_A\cong B$ and $B/\theta_B\cong C$ are fulfilled. These results motivate us to investigate right regular triples of semigroups. In this paper we examine the connection between the structure of semigroups belonging to a right regular triples of semigroups, and present quite a few examples of right regular triples of semigroups.

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2. Preliminaires

By a semigroup we mean a multiplicative semigroup, that is, a nonempty set endowed with an associative multiplication.

A nonempty subset L of a semigroup S is called a *left ideal* of S if $SL \subseteq L$. The concept of a right ideal of a semigroup is defined analogously. A semigroup S is said to be *left* (*resp., right*) *simple* if S itself is the only left (resp., right) ideal of S. A semigroup S is left (resp., right) simple if and only if $Sa = S$ (resp., $aS = S$) for every $a \in S$.

A semigroup S is called *left cancellative* if $xa = xb$ implies $a = b$ for every $x, a, b \in S$. A left cancellative and right simple semigroup is called a right group. A semigroup satisfying the identity $ab = b$ is called a *right zero semigroup.* By [2, Theorem 1.27.], a semigroup is a right group if and only if it is a direct product of a group and a right zero semigroup.

In [6, Theorem 1], it is shown that a semigroup S is embedded in an idempotent-free left simple semigroup if and only if S is idempotent-free and satisfies the condition: for all $a, b, x, y \in S$, $xa = xb$ implies $ya = yb$. Using the terminology of [16], a semigroup S satisfying this last condition is called a *left equalizer simple semigroup.* In other words, a semigroup S is left equalizer simple if, for arbitrary elements $a, b \in S$, the assumption that $xa = xb$ is satisfied for some $x \in S$ implies that $ya = yb$ is satisfied for all $y \in S$. By [16, Theorem 2.1], a semigroup S is left equalizer simple if and only if the factor semigroup S/θ_S is left cancellative.

A nonempty subset I of a semigroup S is called an *ideal* of S if I is a left ideal and a right ideal of S. A semigroup S is called *simple* if S itself is the only ideal of S. By [2, Lemma 2.28], a semigroup S is simple if and only if $SaS = S$ for every $a \in S$.

Let S be a semigroup and I be an ideal of S. We say that the homomorphism $\varphi : S \mapsto I$ is a *retract homomorphism* [13, Definition 1.44], if it leaves the elements of I fixed. In this case, I is called a *retract ideal* of S, and S is a *retract extension* of I by the Rees factor semigroup S/I.

A transformation ϱ of a semigroup S is called a *right translation* of S if $(xy)\varrho = x(y\varrho)$ is satisfied for every $x, y \in S$. For an arbitrary element a of a semigroup S, $\varrho_a : x \mapsto xa$ $(x \in S)$ is a right translation of S which is called an *inner right translation* of S corresponding to the element a. For an arbitrary semigroup S, the mapping $\Phi_S : a \mapsto \varrho_a$ is a homomorphism of S into the semigroup of all right translations of S. The homomorphism Φ_S is called the *right regular representation* of S. For an arbitrary semigroup

S, let θ_S denote the kernel of Φ_S. It is clear that $(a, b) \in \theta_S$ for elements $a, b \in S$ if and only if $xa = xb$ for all $x \in S$. A semigroup S is called *left reductive* if θ_S is the identity relation on S. Thus θ_S is faithful if and only if S is left reductive. The congruence θ_S plays an important role in the investigation of the structure of the semigroup S. In [4], the author characterizes semigroups S for which the factor semigroup S/θ_S is a group. In [5], semigroups S are characterized for which the factor semigroup S/θ_S is a right group. In [15, Theorem 2], a construction is given which shows that every semigroup S can be obtained from the factor semigroup S/θ_S by using this construction. In [18], the authors study the probability that two elements which are selected at random with replacement from a finite semigroup have the same right matrix.

If S is a semigroup, I and Λ are nonempty sets, and P is a $\Lambda \times I$ matrix with entries $P(\lambda, i)$, then the set $\mathcal{M}(S; I, \Lambda; P)$ of all triples $(i, s, \lambda) \in I \times S \times \Lambda$ is a semigroup under the multiplication $(i, s, \lambda)(j, t, \mu) = (i, sP(\lambda, j)t, \mu)$. According to the terminology in [2, §3.1], this semigroup is called a *Rees* $I \times \Lambda$ *matrix semigroup without zero over the semigroup* S *with* $\Lambda \times I$ *sandwich matrix* P. In [19], Rees matrix semigroups $\mathcal{M}(S; I, \Lambda; P)$ without zero over semigroups S satisfying $|I| = 1$ are in the focus. In our present paper we also use such type of Rees matrix semigroups, which will be denoted by $\mathcal{M}(S; \Lambda; P)$. In this case the matrix P can be considered as a mapping of Λ into S, and so the entries of P will be denoted by $P(\lambda)$. If the element of I is denoted by 1, then the element $(1, s, \lambda)$ of $\mathcal{M}(S; \Lambda; P)$ can be considered in the form (s, λ); the operation on $\mathcal{M}(S; \Lambda; P)$ is $(s, \lambda)(t, \mu) = (sP(\lambda)t, \mu)$.

For notations and notions not defined but used in this paper, we refer the reader to books [2], [9], and [13].

3. Results

Theorem 3.1. *If A, B, C is a right regular triple of semigroups such that A is right simple, then C is also right simple.*

Proof. Assume that A, B, C is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C; B; P').$$

Assume that A is right simple. Let $(a_1, b_1), (a_2, b_2) \in \mathcal{M}(A; B; P)$ be arbitrary elements. Since A is right simple, we have $aP(b_1)A = A$, and so there is an element $\xi \in A$ such that $a_1P(b_1)\xi = a_2$ and $(a_1, b_1)(\xi, b_2) = (a_2, b_2)$. Hence the Rees matrix semigroup $\mathcal{M}(A; B; P)$ is right simple. As every homomorphic image of a right simple semigroup is right simple, the Rees matrix semigroup $\mathcal{M}(C; B; P')$ is right simple. Let $c, \eta \in C$ be an arbitrary elements. Then, for any $b \in B$, $(c, b)\mathcal{M}(C; B; P') = \mathcal{M}(C; B; P')$, and so

$$(c, b)(u, v) = (\eta, b)$$

for some $(u, v) \in \mathcal{M}(C; B; P')$. Hence $cP'(b)u = \eta$. Thus $cC = C$ for every $c \in C$. Then C is right simple. □

Theorem 3.2. *If A, B, C is a right regular triple of semigroups such that A is a right group, then C is also a right group.*

Proof. Assume that A, B, C is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C; B; P').$$

Assume that A is a right group, that is, right simple and left cancellative. By the proof of Theorem 3.1, the semigroups $\mathcal{M}(A; B; P)$ and C are right simple. Let $(a, b), (a_1, b_1), (a_2, b_2) \in \mathcal{M}(A; B; P)$ be arbitrary elements with

$$(a, b)(a_1, b_1) = (a, b)(a_2, b_2).$$

Then

$$(aP(b)a_1, b_1) = (aP(b)a_2, b_2),$$

that is,

$$aP(b)a_1 = aP(b)a_2 \quad \text{and} \quad b_1 = b_2.$$

As A is left cancellative, we get $a_1 = a_2$, and so

$$(a_1, b_1) = (a_2, b_2).$$

Hence the semigroup $\mathcal{M}(A; B; P)$ is left cancellative. As $\mathcal{M}(A; B; P)$ is also right simple, it is a right group. From the left cancellativity of $\mathcal{M}(A; B; P)$ it follows that $\theta_{\mathcal{M}(A;B;P)} = \iota_{\mathcal{M}(A;B;P)}$. Thus the semigroup $\mathcal{M}(C; B; P')$ is left cancellative. Assume $xc_1 = xc_2$ for elements $x, c_1, c_2 \in C$. Let $b \in B$ be arbitrary. As C is right simple, there are elements $u, v \in C$ such that $P(b)u = c_1$ and $P(b)v = c_2$. Thus

$$xP(b)u = xP(b)v.$$

Then, for an arbitrary $b' \in B$,

$$(x,b)(u,b') = (x,b)(v,b')$$

is satisfied in $\mathcal{M}(C;B;P)$. As $\mathcal{M}(C;B;P)$ is left cancellative, we get $u = v$, from which it follows that $c_1 = c_2$. Hence C is left cancellative. By the above, C is right simple. Consequently C is a right group. □

Theorem 3.3. *If A, B, C is a right regular triple of semigroups such that A is simple, then C is also simple.*

Proof. Assume that A, B, C is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A;B;P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C;B;P').$$

Assume that A is simple. Let $(a,b),(u,v) \in \mathcal{M}(A;B;P)$ and $z \in B$ be an arbitrary elements. Then $AP(z)aP(b)A = A$ implies that there are elements $\xi, \eta \in A$ such that $\xi P(z)aP(b)\eta = u$ and so $(\xi,z)(a,b)(\eta,v) = (u,v)$. Hence the Rees matrix semigroup $\mathcal{M}(A;B;P)$ is simple. As every homomorphic image of a simple semigroup is simple, the Rees matrix semigroup $\mathcal{M}(C;B;P')$ is simple.

Let $c_1, c_2 \in C$ and $b_1, b_2 \in B$ be arbitrary elements. Then

$$\mathcal{M}(C;B;P')(c_1,b_1)\mathcal{M}(C;B;P') = \mathcal{M}(C;B;P'),$$

and so there are elements $(x,\xi),(y,\eta) \in \mathcal{M}(C;B;P')$ such that

$$(xP(\xi)c_1P(b_1)y,\eta) = (x,\xi)(c_1,b_1)(y,\eta) = (c_2,b_2).$$

Hence

$$xP(\xi)c_1P(b_1)y = c_2.$$

Thus

$$Cc_1C = C$$

for every $c_1 \in C$. Then C is simple. □

The next proposition is used in the proof of Theorem 3.5.

Proposition 3.4. *Let A be a semigroup, Λ be an arbitrary nonempty set and $P : \Lambda \mapsto A$ is an arbitrary mapping. If A is left equalizer simple, then the Rees matrix semigroup $\mathcal{M}(A;\Lambda;P)$ is also left equalizer simple.*

Proof. Suppose that A is a left equalizer simple semigroup, Λ is a nonempty set and $P : \Lambda \mapsto A$ is a mapping. Take $(a_1, b_1), (a_2, b_2), (a, b) \in \mathcal{M}(A; \Lambda; P)$. Suppose that

$$(a, b)(a_1, b_1) = (a, b)(a_2, b_2).$$

This means that

$$(aP(b)a_1, b_1) = (aP(b)a_2, b_2) \quad \Longleftrightarrow \quad aP(b)a_1 = aP(b)a_2 \text{ and } b_1 = b_2.$$

Since A is left equalizer simple we have that, for all $x \in A$ and $y \in \Lambda$:

$$xP(y)a_1 = xP(y)a_2,$$

hence,

$$(x, y)(a_1, b_1) = (x, y)(a_2, b_2).$$

Thus, $\mathcal{M}(A; \Lambda; P)$ is a left equalizer simple semigroup. □

Theorem 3.5. *Let A, B, C be a right regular triple of semigroups such that $P' : B \mapsto C$ is surjective. If A is left equalizer simple, then C is left cancellative.*

Proof. Assume that A, B, C is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C; B; P').$$

From Proposition 3.4, we have that $\mathcal{M}(A; B; P)$ is a left equalizer simple semigroup, and hence $\mathcal{M}(C; B; P')$ is left cancellative by [16, Theorem 2.1]. Now, take $x, c_1, c_2 \in C$ such that $xc_1 = xc_2$. Since P' is surjective, there exists $b \in B$ such that $P'(b) = x$. Then $P'(b)c_1 = P'(b)c_2$. Let $c \in C$ be arbitrary, then

$$(c, b)(c_1, b) = (cP'(b)c_1, b) = (cP'(b)c_2, b) = (c, b)(c_2, b).$$

Since $\mathcal{M}(C; B; P')$ is left cancellative, $(c_1, b) = (c_2, b)$, hence $c_1 = c_2$. Thus C is left cancellative. □

Theorem 3.6. *Let A, B, C be a right regular triple of semigroups such that C is left commutative. If A is left equalizer simple, then C is left cancellative.*

Proof. From the proof of Theorem 3.5, we know that $\mathcal{M}(C;B;P')$ is left cancellative. Again, take $x, c_1, c_2 \in C$ such that $xc_1 = xc_2$. Then for arbitrary $b \in B$,

$$P'(b)xc_1 = P'(b)xc_2.$$

Since C is left commutative,

$$xP'(b)c_1 = xP'(b)c_2,$$

and then

$$(x,b)(c_1,b) = (x,b)(c_2,b).$$

$\mathcal{M}(C;B;P')$ is left cancellative, thus we get $c_1 = c_2$, and that C is left cancellative. □

Theorem 3.7. *Let A, B, C be a right regular triple of semigroups such that $P : B \mapsto A$ is surjective. If A is left reductive, then C is also left reductive.*

Proof. Assume that A, B, C is a right regular triple of semigroups. Then there are mappings $P : B \mapsto A$ and $P' : B \mapsto C$ such that

$$\mathcal{M}(A;B;P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C;B;P').$$

Assume, that A is a left reductive semigroup, $(a_1,b_1),(a_2,b_2) \in \mathcal{M}(A;B;P)$ are elements such that

$$\forall (x,y) \in \mathcal{M}(A;B;P) : \; (x,y)(a_1,b_1) = (x,y)(a_2,b_2).$$

This means that

$$xP(y)a_1 = xP(y)a_2 \quad \text{and} \quad b_1 = b_2.$$

Since A is left reductive, we get that

$$\forall y \in B : \; P(y)a_1 = P(y)a_2.$$

In this case, P is a surjective mapping, hence using again that A is left reductive, we have $a_1 = a_2$. We conclude that $(a_1,b_1) = (a_2,b_2)$, and thus $\mathcal{M}(A;B;P)$ is left reductive.

We know, that if S is a left reductive semigroup, then $\theta_S = \iota_S$. This means, that $\mathcal{M}(A;B;P) \cong \mathcal{M}(C;B;P')$, hence $\mathcal{M}(C;B;P')$ is also left reductive.

Now suppose that $c_1, c_2 \in C$ are such elements, that

$$\forall c \in C: \ cc_1 = cc_2.$$

Take two elements, $(c_1, b), (c_2, b)$ from $\mathcal{M}(C; B; P')$. For arbitrary $(x, y) \in \mathcal{M}(C; B; P')$ we have:

$$(x, y)(c_1, b) = (xP'(y)c_1, b) = (xP'(y)c_2, b) = (x, y)(c_2, b).$$

In the second equality, we used the assumption that $\forall c \in C: \ cc_1 = cc_2$. Since $\mathcal{M}(C; B; P')$ is left reductive, we have $(c_1, b) = (c_2, b)$, and thus $c_1 = c_2$. We conclude that C is left reductive. □

Let A be a semigroup and B be a nonempty set. For a mapping P of B into A, let α_P denote the following relation on A:

$$\alpha_P = \{(a_1, a_2) \in A \times A: \ (\forall a \in A)(\forall b \in B) \ aP(b)a_1 = aP(b)a_2\}.$$

It is clear that α_P is a right congruence on A.

Remark 3.8. It is clear that if P is a mapping of a semigroup B into a semigroup A such that α_P is the identity relation on A, then $\theta_{\mathcal{M}(A;B;P)}$ is the identity relation on $\mathcal{M}(A; B; P)$, and hence the triple A, B, A is right regular.

Let A, B, C be semigroups and $P: B \to A$, $P': B \to C$ be arbitrary mappings. We shall say that the triple A, B, C is right regular with respect to the couple (P, P') if $\mathcal{M}(A; B; P)/\theta_{\mathcal{M}(A;B;P)} \cong \mathcal{M}(C; B; P')$.

Theorem 3.9. *Let A and B be arbitrary semigroups, and P be a mapping of B into A such that α_P is a congruence on A. Then the triple $A, B, A/\alpha_P$ is right regular with respect to (P, P'), where P' is defined by $P': b \mapsto [P(b)]_{\alpha_P}$ for every $b \in B$.*

Proof. Let Φ be the mapping of the Rees matrix semigroup $M = \mathcal{M}(A; B; P)$ onto the Rees matrix semigroup $\mathcal{M}(A/\alpha_P; B; P')$ defined by

$$\Phi: (a, b) \mapsto ([a]_{\alpha_P}, b).$$

For arbitrary elements $(a_1, b_1), (a_2, b_2)$ of M, we have

$$\Phi((a_1, b_1)(a_2, b_2)) = \Phi((a_1P(b_1)a_2, b_2)) = ([a_1P(b_1)a_2]_{\alpha_P}, b_2) =$$

$$= ([a_1]_{\alpha_P}[P(b_1)]_{\alpha_P}[a_2]_{\alpha_P}, b_2) = ([a_1]_{\alpha_P}P'(b_1)[a_2]_{\alpha_P}, b_2) =$$
$$= ([a_1]_{\alpha_P}, b_1)([a_2]_{\alpha_P}, b_2) = \Phi((a_1, b_1))\Phi((a_2, b_2)).$$

Hence, Φ is a homomorphism. It is clear that Φ is surjective. We show that the kernel $ker\Phi$ of Φ is the kernel of the right regular representation of M. For elements (a_1, b_1) and (a_2, b_2) of M, the equation

$$(a, b)(a_1, b_1) = (a, b)(a_2, b_2)$$

is satisfied for every $a \in A$ and every $b \in B$ if and only if

$$(aP(b)a_1, b_1) = (aP(b)a_2, b_2),$$

that is

$$\Phi((a_1, b_1)) = \Phi((a_2, b_2)).$$

Thus, $ker\Phi = \theta_M$ which proves our theorem. □

A semigroup satisfying the identity $axyb = ayxb$ is called a medial semigroup. It is easy to see that if A is a medial semigroup, then, for an arbitrary semigroup B and an arbirtary mapping of B into A, the right congruence α_P is a congruence on A. Thus we have the following corollary.

Corollary 3.10. *Let A be a medial semigroup. Then, for an arbitrary semigroup B and an arbitrary mapping P of B into A, the triple $A, B, A/\alpha_P$ is right regular, where P' is defined in Theorem 3.9.*

If ϱ is an arbitrary congruence on a semigroup S, then $\varrho^* = \{(a, b) \in S \times S : (\forall s \in S)(sa, sb) \in \varrho\}$ (defined in [16]) is also a congruence on S which is called the *right colon congruence of ϱ*.

Remark 3.11. If P is a mapping of a nonempty set B onto a semigroup A, then $\alpha_P \supseteq \theta_A^*$. If P is surjective, then $\alpha_P = \theta_A^*$.

Remark 3.11 and Theorem 3.9 imply the following corollary.

Corollary 3.12. *Let A be an ideal of a semigroup B such that there is a surjective homomorphism P of B onto A. Let P' denote the mapping of B onto A/θ_A^* defined in the following way: $P' : b \mapsto [P(b)]_{\theta_A^*}$ for every $b \in B$. Then the triple $A, B, A/\theta_A^*$ is right regular with respect to (P, P').*

Since the projective homomorphism $P_A : (a, b) \mapsto a$ of the direct product $A \times B$ of semigroups A and B is surjective, Remark 3.11 and Theorem 3.9 imply the following corollary.

Corollary 3.13. *For arbitrary semigroups A and B, the triple $A, A \times B, A/\theta_A^*$ is right regular with respect to the couple (P_A, P'), where P_A denotes the projection homomorphism $P_A : (a, b) \mapsto a$ and $P' : A \times B \to A/\theta_A^*$ is defined by $P' : (a, b) \mapsto [a]_{\theta_A^*}$.*

Theorem 3.14. *Let A and B be arbitrary semigroups, and φ be a mapping of A into B such that α_φ is a congruence on B. Then the triple $A \times B, A, A/\theta_A^* \times B/\alpha_\varphi$ is right regular with respect to the couple (P_A, P'), where P_A is defined by $P_A : a \mapsto (a, \varphi(a))$ and P' is defined by $P' : a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\alpha_\varphi})$.*

Proof. Suppose that $(((a_1, b_1), a_2), ((a_3, b_3), a_4)) \in \theta_M$, where $M = \mathcal{M}(A \times B; A; P_A)$. This means that, for every $x, x' \in A$ and $y \in B$,

$$((x, y), x')((a_1, b_1), a_2) = ((x, y), x')((a_3, b_3), a_4) \iff$$

$$\iff ((xx'a_1, y\varphi(x')b_1), a_2) = ((xx'a_3, y\varphi(x')b_3), a_4).$$

The equality holds if and only if

$$xx'a_1 = xx'a_3, \quad y\varphi(x')b_1 = y\varphi(x')b_3, \quad a_2 = a_4,$$

that is

$$(a_1, a_3) \in \theta_A^*, \quad (b_1, b_3) \in \alpha_\varphi, \quad a_2 = a_4 \tag{1}$$

Let Φ be the mapping of $\mathcal{M}(A \times B; A; P_A)$ into $\mathcal{M}(A/\theta_A^* \times B/\alpha_\varphi; P')$ defined by $\Phi : ((a, b), a') \mapsto (([a]_{\theta_A^*}, [b]_{\alpha_\varphi}), a')$ for every $a, a' \in A$ and every $b \in B$. Since

$$\Phi(((a_1, b_1), a_2)((a_3, b_3), a_4)) = \Phi((a_1a_2a_3, b_1\varphi(a_2)b_3), a_4) =$$

$$= (([a_1a_2a_3]_{\theta_A^*}, [b_1\varphi(a_2)b_3]_{\alpha_P}), a_4) = (([a_1]_{\theta_A^*}, [b_1]_{\alpha_\varphi}), a_2)(([a_3]_{\theta_A^*}, [b_3]_{\alpha_\varphi}), a_4) =$$

$$= \Phi(((a_1, b_1), a_2))\Phi(((a_3, b_3), a_4))$$

for every $a_1, a_2, a_3, a_4 \in A$ and $b_1, b_3 \in B$, Φ is a homomorphism. It is clear that Φ is a surjective.
Since $(((a_1, b_1), a_2), ((a_3, b_3), a_4)) \in ker\Phi$ if and only if all three conditions in (1) are satisfied, we have $ker\Phi = \theta_M$ and this proves our theorem. □

If $\varphi : A \mapsto B$ defined in Theorem 3.14 is surjective, then $\alpha_\varphi = \theta_B^*$ by Remark 3.11, and thus we have the following corollaries:

Corollary 3.15. *Let A and B be semigroups, and φ be a surjective mapping of A onto B. Then the triple $A \times B, A, A/\theta_A^* \times B/\theta_B^*$ is right regular with respect to the couple (P_A, P'), where P_A is defined by $P_A : a \mapsto (a, \varphi(a))$ and P' is defined by $P' : a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\theta_B^*})$.*

Corollary 3.16. *Let A be a semigroup, and B be a retract ideal of A. Let φ be a retract homomorphism of A onto B. Then the triple $A \times B, A, A/\theta_A^* \times B/\theta_B^*$ is right regular with respect to the couple (P_A, P'), where P_A is defined by $P_A : a \mapsto (a, \varphi(a))$ and P' is defined by $P' : a \mapsto ([a]_{\theta_A^*}, [\varphi(a)]_{\theta_B^*})$.*

If B is an ideal of a semigroup A such that B is a group, then $\varphi_B : A \to B$ defined by $\varphi_B(a) = ae$ $(a \in A)$ is a retract homomorphism of A onto B, where e denotes the identity element of the group B.

Corollary 3.17. *Let A be a semigroup and B be an ideal of A such that B is a group. Then the triple $A \times B, A, A/\theta_A^* \times B$ is right regular with respect to the couple (P_A, P'), where P_A is defined by $P_A : a \mapsto (a, \varphi_B(a))$ and P' is defined by $P' : a \mapsto ([a]_{\theta_A^*}, \varphi_B(a))$; here φ_B denotes the above surjective homomorphism of A onto B.*

References

[1] **H. Ayik, N. Ruškuc**, *Generators and relations of Rees matrix semigroups*, Proc. Edinburgh Math. Soc., **42** (1999), 481 – 495.

[2] **A.H. Clifford, G.B. Preston**, *The Algebraic Theory of Semigroups I*, Amer. Math. Soc. Providence R.I. (1961).

[3] **A.H. Clifford, G.B. Preston**, *The Algebraic Theory of Semigroups II*, Amer. Math. Soc. Providence R.I. (1967).

[4] **J.L. Chrislock**, *Semigroups whose regular representation is a group*, Proc. Japan Acad., **40** (1964), 799 – 800.

[5] **J.L. Chrislock**, *Semigroups whose regular representation is a right group*, Amer. Math. Monthly, **74** (1967), 1097 – 1100.

[6] **Cohn, P.M.**, *Embeddings in semigroups with one-sided division*, J. London Math. Soc., **31-2** (1956), 169 – 181.

[7] **L. Descalco, N. Ruškuc**, *On automatic Rees matrix semigroups*, Comm. Algebra, **30** (2002), 1207 – 1226.

[8] **J.B. Hickey**, *Semigroups under a sandwich operation*, Proc. Edinburgh Math. Soc., **26(3)** (1983), 371 – 382.

[9] **J.M. Howie**, *An Introduction to Semigroup Theory*, Academic Press, London (1976)

[10] **M. Kambites**, *The loop problem for Rees matrix semigroups*, Semigroup Forum, **76** (2008), 204 – 216.

[11] **M.V. Lawson**, *Rees matrix semigroups*, Proc. Edinburgh Math. Soc., **33** (1990), 23 – 37.

[12] **D.B. McAlister**, *Rees matrix covers for locally inverse semigroups*, Trans. Amer. Math. Soc., **227** (1983), 27 – 738.

[13] **A. Nagy**, *Special Classes of Semigroups*, Kluwer Academic Publishers, Dordrecht/Boston/London (2001).

[14] **A. Nagy**, *Left reductive congruences on semigroups*, Semigroup Forum, **87** (2013), 129 – 148.

[15] **A. Nagy**, *Remarks on the paper "M. Kolibiar, On a construction of semigroups"*, Period. Math. Hungar., **71** (2015), 261 – 264.

[16] **A. Nagy**, *Left equalizer simple semigroups*, Acta Math. Hungar., **148-2** (2016), 300 – 311.

[17] **A. Nagy**, *A construction of left equalizer simple medial semigroups*, Period. Math. Hungar., **86** (2023), no.1, 37 – 42.

[18] **A. Nagy, Cs. Tóth**, *On the probability that two elements of a finite semigroup have the same right matrix*, Comment. Math. Univ. Carolin., **63(1)** (2022), 21 – 31.

[19] **A. Nagy, Cs. Tóth**, *On special Rees matrix semigroups over semigroups*, (preprint), https://doi.org/10.48550/arXiv.1609.09821

[20] **M. Petrich**, *Embedding semigroups into idempotent generated ones*, Monatsh. Math., **141(4)** (2004), $315_3 22$.

[21] **D. Rees**, *On semi-groups*, Proc. Cambridge Philosophical Soc., **36** (1940), 387 – 400.

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Department of Algebra, Institute of Mathematics,
Budapest University of Technology and Economics
Műegyetem rkp. 3, Budapest, 1111, Hungary
E-mail: tcsaba94@gmail.com