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A quasi-pseudometric on group-like Menger n-groupoids

Hamza Boujouf

Abstract. We introduce and investigate topologies on Menger n-groupoids. These topologies are defined by families of quasi-pseudometrics. We explore the relationship between the right X-closure property, continuity, and extension to an abelian binary group. Finally, we provide the necessary conditions for the topological embedding of group-like Menger n-groupoids in a locally compact binary group as an open subset.

1. Introduction and preliminaries

In the field of topological algebras, considerable attention has been devoted to the study of the properties of topological n-ary groups and n-ary semigroups. The properties of topological Menger n-groupoids have been recently explored in [2, 3, 4]. The generalization of some results is always interesting, and in this paper, we aim to extend some of the results from [1] to the case of Menger n-groupoids.

One of the generalized metric spaces is the pseudometric space introduced by Kuratowski. As the study of non-symmetric topology has gained renewed attention due to its application in various problems in applied physics, we have started utilizing quasi-pseudometric, which are another generalization of metric spaces introduced by Kelly J.C. in [11].

The question of describing families of quasi-pseudometrics that generate a topology on a Menger n-groupoid X, consistent with the n-ary operation and the operation resulting from the definition of Menger n-groupoid, is of interest. Notice that the topological Menger n-groupoid (X, g, τ) such that $g\colon X^n \to X\colon (x_1, \ldots, x_n) \mapsto g(x_1^n) = x_1$ is not uniformizable.

In this article we investigate the application of certain quasi-pseudometrics to define topologies on Menger n-groupoids, enabling the continuity of

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each translation within these structures and resulting in transformation into topological Menger n-groupoids. Specifically, we explore the use of invariant quasi-pseudometric families to generate topologies, examining their implications for the right X-closure property, continuity, and extension to an abelian binary group. By establishing compatibility conditions between these topologies and the n-ary operation, we emphasize the crucial contribution of invariant quasi-pseudometrics in defining and characterizing the topological properties of Menger n-groupoids. And at the end we gave the necessary conditions for the topological embeddable of group-like Menger n-groupoids in a locally compact binary group as an open set.

By a *Menger n-groupoid* (X,g) we mean the nonempty set X together with an n-ary operation $g\colon X^n \to X$ satisfying the superassociative law $g(g(x_1^n),y_1^{n-1}) = g(x_1, g(x_2,y_1^{n-1}),\dots,g(x_n,y_1^{n-1}))$. A Menger n-groupoid (X,g) is *i-solvable* if for all $a_1^{n-1}, b \in X$, the equation $g(a_1^{k-1},x,a_k^{n-1}) = b$, is uniquely solvable for the case $k=1$ and $k=i+1$. A Menger n-groupoid is called *$(1,j)$-commutative* if $g(x_1^{j-1},x_j,x_{j+1}^n) = g(x_j,x_2^{j-1},x_1,x_{j+1}^n)$, and *$(j,n)$-commutative* if $g(x_1^{j-1},x_j,x_{j+1}^n) = g(x_1^{j-1},x_n,x_{j+1}^{n-1},x_j)$ for $x_1^n \in X$. And (X,g) is *abelian* if $g(x_1^n) = g(x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)})$ for $x_1^n \in X$ and all permutations $\sigma \in \mathbb{S}_n$.

It should be noted (cf. [6]) that any Menger n-groupoid is isomorphic to some Menger algebra of full (n-1)-place functions. The necessary and sufficient conditions for partially commutative Menger n-groupoids to be isomorphic to Menger algebras of specific $(n-1)$-place functions are given in [7], [8] and [9]. Menger n-groupoids which are i-solvable are characterized in [5] (see also [6]).

A binary semigroup $(X,\cdot)$, where $x \cdot y = g(x, \overset{n-1}{y})$, is called a *diagonal semigroup* of a Menger n-groupoid (X,g). If a Menger n-groupoid (X,g) is i-solvable then its diagonal semigroup is a group (see [6]).

The triple (X,g,τ) is a *topological Menger n-groupoid* if g is continuous, in all variables together, in the topology τ defined on a Menger n-groupoid (X,g). Note that if the n-ay operation g is continuous on the topology τ defined on a Menger n-groupoid X, then the operation $g_{_{(2)}}$ defined by $g_{_{(2)}}(x_1^n,y_2^n) = g(g(x_1^n),y_2^n)$ will also be continuous in (X,τ). But the continuity of the operation $g_{_{(2)}}$ does not always imply the continuity of g or the binary operation $\cdot$. As an example, we take the Menger 3-groupoid (X,g) such that $X =]1,+\infty)$ with the sum of the usual topology on $]1,2] \cup [5,+\infty)$ and the discrete topology on the interval $[2,5]$, i.e. the topology τ is defined as the set of all sets representable as unions of elements of the usual topology on $]1, 2] \cup [5, +\infty)$ and of the discrete topology on $[2, 5]$ (see [3]).

Algebraic properties of the Menger n-groupoid are considered in detail in the monograph [6].

2. Results

A mapping $f : X \times X \to [0, +\infty)$ is called a *quasi-pseudometric* on X if for every x, y and z from X, the following conditions hold: $f(x, x) = 0$ and $f(x, y) \leqslant f(x, z) + f(z, y)$. If, in addition, $f(x, y) = f(y, x)$, then f is called a *pseudometric* (or *deviation*).

The maps $t_k : X \to X$, where $k \in N_n = \{1, 2, \dots, n\}$, defined by $t_k(x) = g(a_1^{k-1}, x, a_{k+1}^n)$, $a_1^n \in X$, are called the *translations.* A quasi-pseudometric on (X, g) is said to be *k-invariant*, if $f(t_k(x), t_k(y)) = f(x, y)$ for all $x, y, a_1^n \in X$. If f is k-invariant for each $k \in N_n$, then f is *invariant.* Furthermore, f is *right* (resp. *left*) *invariant* if $f(t_n(x), t_n(y)) = f(x, y)$ (resp. $f(t_1(x), t_1(y)) = f(x, y)$), for all $x, y \in X$.

Every family Φ of quasi-pseudometrics generates a topology on X in a standard way: the sets $B_f(x, \epsilon) = \{x \in X : f(x, y) < \epsilon\}$, where $y \in X$, $f \in \Phi$, $\epsilon > 0$, form a pre-base of such a topology.

Proposition 2.1. *If in a Menger n-groupoid (X, g) there are $c_1^j \in X$, $j < n$ and $i \in \{0, 1, \dots, j-1\}$ such that $g(c_1^i, \overset{n-j}{x}, c_{i+1}^j) = x$ for all $x \in X$, then every quasi-pseudometric f on X induces a new quasi-pseudometric $d_{a_1^k}$ defined by $d_{a_1^k}(x, y) = f(g(a_1^k, x, a_{k+1}^{n-1}), g(a_1^k, y, a_{k+1}^{n-1}))$. If f is additionally k-invariant, then $d_{a_1^k}$ is also k-invariant.*

Proof. Let f be a quasi-pseudometric on a Menger n-groupoid (X, g), and $x, y, z \in X$. Then $d_{a_1^k}(x, x) = f(g(a_1^k, x, a_{k+1}^{n-1}), g(a_1^k, x, a_{k+1}^{n-1})) = 0$ and $d_{a_1^k}(x, y) = f(g(a_1^k, x, a_{k+1}^{n-1}), g(a_1^k, y, a_{k+1}^{n-1})) \leqslant f(g(a_1^k, x, a_{k+1}^{n-1}), g(a_1^k, z, a_{k+1}^{n-1})) + f(g(a_1^k, z, a_{k+1}^{n-1}), g(a_1^k, y, a_{k+1}^{n-1})) = d_{a_1^k}(x, z) + d_{a_1^k}(z, y)$.

Thus, $d_{a_1^k}$ is a quasi-pseudometric on X. Moreover, if (X, g) satisfies the given condition and f is k-invariant. Then
$d_{a_1^k}(g(a_1^{k-1}, x, a_{k+1}^n), g(a_1^{k-1}, y, a_{k+1}^n)) =$
$f(g(a_1^k, g(c_1^{i-1}, x, c_{i+1}^n), a_{k+1}^{n-1}), g(a_1^k, g(c_1^{i-1}, y, c_{i+1}^n), a_{k+1}^{n-1})) = f(g(a_1^k, x, a_{k+1}^{n-1}),$
$g(a_1^k, y, a_{k+1}^{n-1})) = d_{a_1^k}(x, y)$. Therefore, $d_{a_1^k}$ is also k-invariant. □

Proposition 2.2. *If a topological Menger n-groupoid (X, g, τ) satisfies the assumption of Proposition 2.1, then the continuity of the operation $g_{(2)}$ implies the continuity of the operation g.*

Proof. Since $g(x_1^n) = g(g(c_1^i, \overset{n-j}{x}, c_{i+1}^j), x_2^n) = g_{(2)}(c_1^i, \overset{n-j}{x}, c_{i+1}^j, x_2^n)$, the continuity of the operation $g_{(2)}$ implies the continuity of the operation g. □

Theorem 2.3. *Let Φ be a family of k-invariant quasi-pseudometrics on a Menger n-groupoid (X, g). If the topology τ_f on X, is generated by the family Φ, then (X, g, τ_f) is a topological Menger n-groupoid.*

Proof. Let $f_1, \ldots, f_m \in \Phi$, and let ϵ and $x_1^n \in X$. The collection of sets $W = \{s \in X : f_i(s, g(x_1^n)) < \epsilon, i \in N_m\}$ forms a fundamental system of neighborhoods of the point $g(x_1^n)$ in the topology τ_f induced by Φ. The set $U_k = \{h \in X : f_i(h, x_k) < \epsilon, i \in N_m\}$ is a neighborhood of a point x_k, where $k \in N_n$, in the topology τ_f on X. If $h_k \in U_k$, then for each $i \in N_m$, we obtain

$$f_i(g(h_1^n), g(x_1^n)) \leqslant f_i(g(h_1^n), g(h_1^{n-1}, x_n)) + f_i(g(h_1^{n-1}, x_n), g(h_1^{n-2}, x_{n-1}, x_n)) + \ldots + f_i(g(h_1^2, x_3^n), g(h_1, x_2^n)) + f_i(g(h_1, x_2^n), g(x_1^n) = f_i(h_n, x_n) + f_i(h_{n-1}, x_{n-1}) + \ldots + f_i(h_2, x_2) + f_i(h_1, x_1) < n(\tfrac{\epsilon}{n}) = \epsilon.$$

Consequently, $g(h_1^n) \in W$ and therefore the operation g is continuous in τ_f. Thus, (X, g, τ_f) is a topological Menger n-groupoid. □

Corollary 2.4. *Let Φ be a family of k-invariant quasi-pseudometrics on a Menger n-groupoid (X, g). Then all translations t_k of X are continuous in the topology τ_f on X, generated by the family Φ.*

Proof. Theorem 2.3 establishes that g on (X, τ_f) is continuous Thus each translation $x \mapsto g(a_1^{k-1}, x, a_{k+1}^n)$ of X is continuous in the topology τ_f. □

Corollary 2.5. *Let Φ be a family of k-invariant quasi-pseudometrics on a Menger n-groupoid (X, g). If the topology τ_f on X, is generated by the family Φ, then the operation $g_{(2)}$ is continuous in τ_f.*

Theorem 2.6. *If a Menger n-groupoid (X, g) with a topology τ_f generated by the family Φ of quasi-pseudometrics invariant from the right is $(1, j)$-commutative for some $j \in N_n$, then (X, g, τ_f) is a topological Menger n-groupoid.*

Proof. Let a Menger n-groupoid (X, g) be $(1, j)$-commutative. Then for any $f_1, \ldots, f_m \in \Phi$, $\epsilon > 0$, $x_1^n \in X$ the collection of sets

$$W = \{s \in X : f_i(s, g(x_1^n)) < \epsilon, i \in N_m\}$$

forms a basis for the topology τ_f induced by Φ on X. Consider the set $U_k = \{h \in X : f_i(h, x_k) < \epsilon, i \in N_m\}$, which is a neighborhood of a point

x_k, where $k \in N_n$, in the topology τ_f. If $h_k \in U_k$ for $k \in N_n$, then for each $i \in N_m$, we have:

$$\begin{aligned} f_i(g(h_1^n), g(x_1^n)) \leqslant\, & f_i(g(h_1, h_2^{n-1}, h_n), g(h_1, h_2^{n-1}, x_n)) + \\ & f_i(g(h_1, h_2^{n-1}, x_n), g(h_1, h_2^{n-2}, x_{n-1}, x_n)) + \ldots + \\ & f_i(g(h_1, h_2, x_3^n), g(h_1, x_2^n)) + f_i(g(h_1, x_2^n), g(x_1^n)) \\ = & f_i(g(h_n, h_2^{n-1}, h_1), g(x_n, h_2^{n-1}, h_1)) + \\ & f_i(g(h_{n-1}, h_2^{n-2}, h_1, x_n), g(x_{n-1}, h_2^{n-2}, h_1, x_n)) \\ & + \ldots + f_i(g(h_2, h_1, x_3^n), g(x_2, h_1, x_3^n)) + f_i(g(h_1, h_2^n), g(x_1^n)) \\ = & f_i(h_n, x_n) + f_i(h_{n-1}, x_{n-1}) + \ldots + f_i(h_2, x_2) + f_i(h_1, x_1) \\ < & n(\tfrac{\epsilon}{n}) = \epsilon. \end{aligned}$$

Consequently, we can conclude that $g(h_1^n) \in W$, and therefore the operation g is continuous in τ_f. Hence, (X, g, τ_f) is a topological Menger n-groupoid. □

In a similar manner, we can prove

Theorem 2.7. *If a Menger n-groupoid (X, g) with a topology τ_f generated by the family Φ of quasi-pseudometrics invariant from the left is (j, n)-commutative for some $j \in N_n$, then (X, g, τ_f) is a topological Menger n-groupoid.*

Corollary 2.8. *If an abelian Menger n-groupoid (X, g) with a topology τ_f generated by the family Φ of quasi-pseudometrics invariant either from the right or from the left, then (X, g, τ_f) is a topological Menger n-groupoid.*

Remark 2.9. Proposition 2.2 and the above theorems also are valid in the case of topologies generated by a family of pseudometrics.

Any i-solvable Menger n-groupoid is a commutative n-group derived from its diagonal group (see [5]). Then there exists a binary group $(G, \cdot)$ such that $G \supset X$ for which $A = \{a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} : a_i \in X, i \in N_{n-1}\}$ is a normal subgroup, and the quotient group of G/A is cyclic of order $n-1$ (see for example [12]). For all $y \in X$, $X = yA = Ay$, and $g(a_1^n) = a_1 \cdot a_2 \cdot \ldots \cdot a_n$, where $a_1 \cdot a_2 \cdot \ldots \cdot a_k = a_1^k$ is the product calculated in thee group $(G, \cdot)$. Such defined group $(G, \cdot)$ is called the *covering group* for (X, g).

Based on these findings, we can prove the following result.

Proposition 2.10. *Let (X, g) be i-solvable Menger n-groupoid and let f be a left invariant quasi-pseudometric on X such that for each $x, y \in X$, $f(x, y) \leqslant 1$. If f_G is an extension of f such that*

$$f_G(y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) = f(g(y, a_1^{n-1}), g(y, b_1^{n-1}))$$

if $k \in N_{n-1}$, $a_1^{n-1}, b_1^{n-1} \in X$, *and* $f_G(z,s) = 1$ *if* z *and* s *belong to different cosets, then* f_G *is a left-invariant quasi-pseudometric on* G.

Proof. Let a Megner n-groupoid (X, g) be i-solvalbe and let $(G, \cdot)$ be its covering group. It's clear that f_G is well-defined on $G \times G$, does not depend on the choice of $y \in X$, is non-negative, and it is a quasi-pseudometric on G. Moreover, if x, z belong to different cosets, then for any $t \in G$, the elements tx, tz also belong to different cosets. Then $f_G(tx, tz) = 1 = f_G(x, z)$. Now, if $x = y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}$, $z = y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}$, $t = y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}$, where $1 \leqslant k \leqslant n-1$, $1 \leqslant m \leqslant n-1$, $a_1^{n-1}, b_1^{n-1}, c_1^{n-1} \in X$, then $tx = y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1} y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}$. Since $c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1} y^k = y^k d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1}$, for some $d_1^{n-1} \in X$, then then $tx = y^{m+k} d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}$.

Similarly, we obtain $tz = y^{m+k} d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}$.

Therefore,

$$\begin{aligned}
&f_G(tx, tz) = \\
&f_G(y^{m+k} d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, y^{m+k} d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) = \\
&f(g(y d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = \\
&f(g(p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = \\
&f(g(y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = \\
&f_G(g(y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = f_G(x, z),
\end{aligned}$$

where $d_1^{n-1} \in X^{n-1}$ such that $y d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} = p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y$, with $m + k \leqslant n-1$.

If $m + k > n-1$, then

$$\begin{aligned}
&f_G(tx, tz) = \\
&f_G(y^{m+k-(n-1)} y^{n-1} d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, y^{m+k-(n-1)} y^{n-1} d_1 \cdot d_2 \cdot \\
&\qquad \ldots \cdot d_{n-1} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) = \\
&f(g(y^n d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y^n d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = \\
&f(g(y^{n-1} \cdot g(p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1})), g(y^{n-1} \cdot g(p_1 \cdot p_2 \cdot \ldots \cdot \\
&\qquad p_{n-1} y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}))) = \\
&f(g(p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) \\
&\qquad = f(g(y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) \\
&= f_G(y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) = f_G(x, z),
\end{aligned}$$

where $d_1^{n-1} \in X$ such that $y d_1 \cdot d_2 \cdot \ldots \cdot d_{n-1} = p_1 \cdot p_2 \cdot \ldots \cdot p_{n-1} y$.
Thus, f_G is a left invariant quasi-pseudometric on G. □

We will say that the family of quasi-pseudometrics Φ on a Menger n-groupoid (X, g) is *right* X*-closed*, if for all $f \in \Phi$, $a_1^{n-1} \in X$ the map

$d_{a_1^{n-1}}$ defined by $d_{a_1^{n-1}}(x,y) = f(g(x,a_1^{n-1}),g(y,a_1^{n-1}))$ for all $x,y \in X$, is a pseudo-metric on (X,g).

Theorem 2.11. *Let Φ be a right X-closed family of left-invariant quasi-pseudometrics on a Menger n-groupoid (X,g) such that for some $a \in X$, $f \in \Phi$, $k = 2,3,\dots,n-1$, the map $d_{a,k}$ defined by*

$$d_{a,k}(x,y) = f(g(\overset{k}{a},x,\overset{n-1-k}{a}),g(\overset{k}{a},y,\overset{n-1-k}{a}))$$

is a quasi-pseudometric on (X,g). Then g is continuous in the topology τ generated by Φ. Moreover, if (X,g) is associative and i-solvable, then on the group $(G,\cdot)$ there exists τ_G consistent with the semigroup structure, X is an open subset of G and τ is a restriction topology on X from G.

Proof. According to Theorem 2.3 the operation g is continuous in (X,τ). If (X,g) is an associative and i-solvable Menger n-groupoid, then there exists an abelian binary group $(G,\cdot)$, such that $G \supset X$. If $\Phi_G = \{f_G\}$ is a family of quasi-pseudometrics on $(G,\cdot)$, generated by Φ, then Φ_G is right G-closed. Let's show it.

First, note that for any $f_G \in \Phi_G$, the function $\frac{f_G}{1+f_G} \in \Phi_G$ and satisfies $|\frac{f_G}{1+f_G}| \leqslant 1$. Therefore, without loss of generality, we can assume that every quasi-pseudometric $f_G \in \Phi_G$ satisfies the inequality $|f_G| \leqslant 1$. Let $x,z,t \in G$. Then $x = y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}$, $z = y^l b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}$, $t = y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}$, where $1 \leqslant k \leqslant n-1$, $1 \leqslant l \leqslant n-1$, $1 \leqslant m \leqslant n-1$, $a_1^{n-1}, b_1^{n-1}, c_1^{n-1} \in X$. Thus $y \in X$.

If x,z belong to different cosets $y^k A$, then xt and zt belong to different cosets as well, and therefore $f_G(xt,zt) = 1$.

If $l = k$, then x and z belong to some coset. In this case,

$$\begin{aligned}
&f_G(xt,zt) = \\
&f_G(y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}, y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1} y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}) = \\
&f_G(y^{n-m} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}, y^{n-m} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1} y^m c_1 \cdot c_2 \cdot \\
&\qquad \ldots \cdot c_{n-1}) = \\
&f(g(y^{n-m} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}), g(y^{n-m} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1} y^m c_1 \cdot \\
&\qquad c_2 \cdot \ldots \cdot c_{n-1})) = \\
&d_{c_1^{n-1}}(g(y^{n-m} a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1} y^m c_1 \cdot c_2 \cdot \ldots \cdot c_{n-1}), g(y^{n-m} b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1} y^m c_1 \cdot \\
&\qquad c_2 \cdot \ldots \cdot c_{n-1})) = \\
&(d_{c_1^{n-1}})_{y^{n-m-1}}(g(y a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}), g(y b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1})) = \\
&((d_{c_1^{n-1}})_{y^{n-m-1}})_G(y^k a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1}, y^k b_1 \cdot b_2 \cdot \ldots \cdot b_{n-1}) = ((d_{c_1^{n-1}})_{y^{n-m-1}})_G(x,z),
\end{aligned}$$

which belongs to Φ_G since $(d_{c_1^{n-1}})_{y^{n-m-1}} \in \Phi$. Hence, Φ_G is right G-closed.

Since f_G is an extension of the quasi-pseudometric f, the topology τ_G on $(G,\cdot)$, generated by the family Φ_G induces a topology on X that coincides with the topology generated by the family Φ.

Since $f_G(x,z) = 1$ when x and z belong to different cosets y^kA for $1 \leqslant k \leqslant n-1$, each y^kA is an open subset of G, and in particular, X is an open subset of G.

The continuity of the multiplication follows from the Theorem 2.3 by considering $n = 2$ and $k \in \{1,2\}$. Therefore, $(G,\cdot,\tau_G)$ is a topological semigroup. □

We say that a Menger n-groupoid (X,g) is *weakly left* (respectively, *right*)*-invertible* if for all elements $a,b \in X$ there exist $c_1^{n-2} \in X$ such that $g(c_1^{n-2},a,X) \cap g(c_1^{n-2},b,X) \neq \emptyset$ (respectively, $g(X,a,c_1^{n-2}) \cap g(X,b,c_1^{n-2}) \neq \emptyset$).

Theorem 2.12. *An associative i-solvable Menger n-groupoid* (X,g) *with a locally compact topology* τ *is a topological semigroup if all translations are injective, open, and continuous. Additionally, if* (X,g) *is weakly left* (*or weakly right*)*-invertible, then* (X,g,τ) *is topologically embeddable in a locally compact binary group as an open set.*

Proof. Let (X,g) be an associative i-solvable Menger n-groupoid. Then it is a commutative Menger n-group derived from its diagonal group $(X,\cdot\,)$ (cf. [5]). Let τ be a locally compact topology on X such that the translations are injective, open, and continuous. Then, by Ellis's theorem [10], the binary operation is continuous, and sequentially, g is continuous. Therefore, we can conclude that $(X,\cdot\,,\tau)$ is a topological semigroup, and in particular (X,g,τ) is a topological group.

Now, consider a weakly right-invertible Menger n-groupoid (X,g). Therefore, for any elements $a,b \in X$, and for certain sequence $c_1^{n-2} \in X$ the relation $g(X,a,c_1^{n-2}) \cap g(X,b,c_1^{n-2}) \neq \emptyset$ holds. Thus, for some $x,y \in X$, we have $g(x,a,c_1^{n-2}) = g(y,a,c_1^{n-2})$. Consequently, $xac_1^{n-2} = xac_1^{n-2}$ in $(G,\cdot)$. Invoking the injectivity of the translations of X we obtain $xa = yb$ or $Xa \cap Xb \neq \emptyset$. Hence, by [14], (X,g,τ) is topologically embeddable in a locally compact binary group as an open set. □

This theorem can be considered an extension of Ellis's theorem in [10] to the case of Menger n-groupoids with locally compact topologies.

Corollary 2.13. *Let (X, g) be an associative, weakly left (or weakly right)-invertible Menger n-quasigroup with a locally compact topology τ is topologically embeddable in a locally compact binary group as an open set if all translations are injective, open, and continuous in τ.*

Theorem 2.14. *An associative i-solvable Menger n-groupoid (X, g) with a locally compact topology τ is a topological semigroup if all translations are injective, open, and the operation $g_{(2)}$ is continuous. Additionally, if for every $x, y \in X$ and for every neighborhood V of point x there exists a neighborhood V' of point y such that $g(x, \overset{n-1}{y}) \in \bigcap_{y' \in V'} g(V, \overset{n-1}{y'})$. Then (X, g, τ) is topologically embeddable in a locally compact binary group as an open set.*

Proof. Let (X, g) be an associative i-solvable Menger n-groupoid. Again, from [5], it follows that (X, g) is a commutative Menger n-group derived from its diagonal group $(X, \cdot)$. Let τ be a locally compact topology on X such that the translations are injective, open, and $g_{(2)}$ is continuous. Then g is continuous, and according again to Ellis's theorem the binary operation is also continuous. Therefore, we can conclude that $(X, \cdot, \tau)$ is a topological semigroup, and in particular, (X, g, τ) is topological group.

If for every $x, y \in X$ and for every neighborhood V of point x there exists a neighborhood V' of point y such that $g(x, \overset{n-1}{y}) \in \bigcap_{y' \in V'} g(V, \overset{n-1}{y'})$. Then $xy = g(x, \overset{n-1}{y}) \in \bigcap_{y' \in V'} g(V, \overset{n-1}{y'}) = \bigcap_{y' \in V'} V \cdot y'$. As the diagonal-topological semigroup $(X, \cdot, \tau)$ is commutative, then $yx = g(x, \overset{n-1}{y}) \in \bigcap_{y' \in V'} y' \cdot V$. Consequently, $(X, \cdot, \tau)$ verifies the condition F of [13]. Thus, (X, g, τ) is topologically embeddable in a locally compact binary group as an open set. □

Corollary 2.15. *An associative Menger n-group (X, g) with a locally compact topology τ is topologically embeddable in a locally compact binary group as an open set if all translations are injective, open, the operation $g_{(2)}$ is continuous, and the following condition is satisfied: For every $x, y \in X$ and for every neighborhood V of point x there exists a neighborhood V' of point y such that $g(x, \overset{n-1}{y}) \in \bigcap_{y' \in V'} g(V, \overset{n-1}{y'})$.*

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Sebha University, Departement of Mathematics at Faculty of Science
P.O.Box:18758
Sebha, Libya
E-mail: hamzaan14@gmail.com