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On pseudo-ideals in partially ordered ternary semigroups

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Abstract. We study the properties of different types of pseudo-ideals of a partially ordered ternary semigroup and prove that the space of all strongly irreducible pseudo-ideals of a partially ordered ternary semigroup is a compact space.

1. Introduction

In [2], Hewitt and Zuckerman specified the method of construction of ternary semigroups from binary and specified various connections between such semigroups. Ternary semigroups are a special case of n-ary semigroups. So many results on ternary semigroups has an analogous version for n-ary semigroups. F.M. Sioson [5] proved some results on ideals in ternary semigroups. In [1], W.A. Dudek and I.M. Groździńska characterized some classes of regular ternary semigroups by ideals can be deduced from general results proved for n-ary semigroups. The notion of prime, semiprime and strongly prime bi-ideals in ternary semigroups was introduced by M. Shabir and M. Bano in [4]. The concept of ordered ternary semigroups was developed by A. Iampan in [3].

Our aim of this article is to introduce the concepts of prime pseudo-ideals and irreducible pseudo-ideals in a partially ordered ternary semigroup and to study their properties. We also prove that the space of all strongly irreducible pseudo-ideals of a partially ordered ternary semigroup is a compact space.

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2. Preliminaries

A non-empty set T with a ternary operation $[\ ] : T \times T \times T \longrightarrow T$ is called a *ternary semigroup* if $[\ ]$ satisfies the associative law, $[abcde] = [[abc]de] = [a[bcd]e] = [ab[cde]]$, for all $a, b, c, d, e \in T$.

For non-empty subsets X, Y and Z of a ternary semigroup T, $[XYZ] = \{[xyz] : x \in X, y \in Y \text{ and } z \in Z\}$. We write, $[XYZ]$ as XYZ, $[xyz] = xyz$ and $[XXX] = X^3$.

A ternary semigroup T is said to be a *partially ordered ternary semigroup* if there exist a partially ordered relation $\leqslant$ on T such that, $a \leqslant b \Rightarrow xya \leqslant xyb$, $xay \leqslant xby$, $axy \leqslant bxy$ for all $a, b, x, y \in T$. In this article, we write T for a partially ordered ternary semigroup, unless otherwise specified.

An element $e \in T$ is said to be an *identity element* of T if $exx = xxe = xex = x$ for all $x \in T$.

The set $\{t \in T : t \leqslant x, \text{for some} x \in X\}$ is denoted by $(X]$. A non-empty subset X of T is said to be a *partially ordered ternary subsemigroup* of T, if $[XXX] \subseteq X$ and $(X] = X$. A non-empty subset I of T is said to be a *partially ordered left* (respectively, *right, lateral*) *ideal* of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I$, $TIT \subseteq I$) and $(I] = I$.

A non-empty subset I of T is said to be *ideal* of T if it is a left ideal, a right ideal and a lateral ideal of T.

A partially ordered ternary subsemigroup I of T is called a *left* (respectively a *right*, a *lateral*) pseudo-ideal of T if $[xxxxI] \subseteq I$ (respectively, $[Ixxxx] \subseteq I$, $[xxIxx] \subseteq I$) for all $x \in T$. A pseudo-ideal I of T is said to be *proper pseudo-ideal* of T if it differs from T.

A non-empty subset I of T is said to be *two sided pseudo-ideal* of T, if it is both left and right pseudo-ideal of T. A non-empty subset I of T is said to be *pseudo-ideal* of T, if I is a left, a right and a lateral pseudo-ideal of T. Note that, the non-empty intersection of an arbitrary collection of pseudo-ideals of T is a pseudo-ideal of T.

Example 2.1. Let $\mathbb{N}$ be the set of all natural numbers. Define ternary operation $[\ ]$ on $\mathbb{N}$ by $[xyz] = xyz$ for all $x, y, z \in \mathbb{N}$, where $\cdot$ is a usual multiplication and a usual partial ordering relation $\leqslant$ on $\mathbb{N}$. Then $\mathbb{N}$ is a partially ordered ternary semigroup and $I = 3\mathbb{N}$ is a pseudo-ideal of $\mathbb{N}$.

Definition 2.2. A proper pseudo-ideal I of a partially ordered ternary semigroup T is called

(i) *prime pseudo-ideal* of T if $XYZ \subseteq I$ implies $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$ for any pseudo-ideals X, Y, Z of T, (*ii*) *strongly prime pseudo-ideal* of T if $XYZ \cap YZX \cap ZXY \subseteq I$ implies $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$ for any pseudo-ideals X, Y, Z of T,

(*iii*) *semiprime pseudo-ideal* of T if X is a pseudo-ideal of T and $X^n \subseteq I$ implies $X \subseteq I$ for some odd natural number n.

3. Main Results

Definition 3.1. A proper pseudo-ideal I of T is said to be *irreducible* (respectively, *strongly irreducible*) pseudo-ideal of T if $X \cap Y \cap Z = I$ (respectively $X \cap Y \cap Z \subseteq I$) implies $X = I$ or $Y = I$ or $Z = I$ (respectively $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$) for all pseudo-ideals X, Y, Z of T.

Remark 3.2. Every strongly irreducible pseudo-ideal of T is an irreducible pseudo-ideal of T but converse is not true in general.

Theorem 3.3. *Let X be a proper pseudo-ideal of T. For any $t(\neq 0) \in T \setminus X$ there exists an irreducible pseudo-ideal I of T such that $X \subseteq I$ and $t \notin I$.*

Proof. Let $\mathcal{I} = \{X_\alpha : X_\alpha \text{ is a pseudo-ideal of } T, X \subseteq X_\alpha, t \notin X_\alpha\}$, where $\alpha \in \Delta$ is any indexing set. As X is a pseudo-ideal of T and $t \notin X$, we have $X \in \mathcal{I}$, so $\mathcal{I} \neq \emptyset$. Evidently $\mathcal{I}$ is partially ordered set under the inclusion of sets. If $\{X_i : i \in \Delta\}$ is a totally ordered subset (chain) of $\mathcal{I}$ then $\bigcup_{i \in \Delta} X_i$ is a pseudo-ideal of T containing X and $t \notin \bigcup_{i \in \Delta} X_i$. Therefore $\bigcup_{i \in \Delta} X_i$ is an upper bound of $\{X_i : i \in \Delta\}$. Thus every chain in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. Hence by Zorn's Lemma, there exists a maximal element say I in the collection $\mathcal{I}$. This shows that I is a pseudo-ideal of T such that $X \subseteq I$ and $t \notin I$.

Now we show that I is an irreducible pseudo-ideal of T. Let I_1, I_2 and I_3 be any three pseudo-ideals of T such that $I = I_1 \cap I_2 \cap I_3$ then $I \subseteq I_1, I \subseteq I_2$ and $I \subseteq I_3$. If I_1, I_2 and I_3 properly contain I, then according to hypothesis $t \in I_1, t \in I_2$ and $t \in I_3$. Thus $t \in I_1 \cap I_2 \cap I_3 = I$. Which contradicts to the fact that $t \notin I$. Therefore either $I = I_1$ or $I = I_2$ or $I = I_3$. Hence I is an irreducible. □

Theorem 3.4. *Any proper pseudo-ideal of T is the intersection of all irreducible pseudo-ideals containing it.*

Proof. Let X be the any proper pseudo-ideal of T and $\{X_i : i \in \Delta\}$ be the family of all irreducible pseudo-ideals of T containing X. Then $X \subseteq \bigcap_{i\in\Delta} X_i$. If $X \subsetneq \bigcap_{i\in\Delta} X_i$ then there exists $t(\neq 0) \in \bigcap_{i\in\Delta} X_i$ such that $t \notin X$. This implies $t \in X_i \ \forall i \in \Delta$. Since $t \notin X$, then by Theorem 3.3, there exists an irreducible pseudo-ideal say Y of T containing X but not containing t. This is a contradiction to $t \in X_i \ \forall i \in \Delta$. Thus $\bigcap_{i\in\Delta} X_i \subseteq X$. Hence $X = \bigcap_{i\in\Delta} X_i$. □

Theorem 3.5. *Every strongly irreducible semiprime pseudo-ideal of T is a strongly prime pseudo-ideal of T.*

Proof. Let I be a strongly irreducible semiprime pseudo-ideal of T. If X, Y and Z are three pseudo-ideals of T such that $[XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. Then $(X\cap Y\cap Z)^3 = [(X\cap Y\cap Z)(X\cap Y\cap Z)(X\cap Y\cap Z)] \subseteq [XYZ]$. Similarly $(X \cap Y \cap Z)^3 \subseteq [YZX]$ and $(X \cap Y \cap Z)^3 \subseteq [ZXY]$. This proves that $(X\cap Y\cap Z)^3 \subseteq [XYZ]\cap[YZX]\cap[ZXY] \subseteq I$. Therefore $(X\cap Y\cap Z)^3 \subseteq I$. Since I is a semiprime pseudo-ideal, $(X \cap Y \cap Z) \subseteq I$. Also since I is a strongly irreducible pseudo-ideal of T. Therefore, by definition of strongly irreducible pseudo-ideal, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. Hence I is a strongly prime pseudo-ideal of T. □

Corollary 3.6. *Every strongly irreducible semiprime pseudo-ideal of T is prime pseudo-ideal of T.*

Definition 3.7. A pseudo-ideal X of partially ordered ternary semigroup T is called idempotent if $X^3 = X$.

Theorem 3.8. *The following assertions for a partially ordered ternary semigroup T with identity are equivalent.*

(i) *Every pseudo-ideal of T is idempotent.*

(ii) *For every three pseudo-ideals X, Y, Z of T,*
$X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY]$.

(iii) *Every proper pseudo-ideal of T is semiprime.*

(iv) *Each proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which contain it.*

Proof. $(i) \Rightarrow (ii)$: Suppose that, every pseudo-ideal of T is idempotent. Let X, Y and Z be three pseudo-ideals of T. Then $X \cap Y \cap Z$ is a pseudo-ideal of T, so $X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq [XYZ]$. Similarly $X \cap Y \cap Z \subseteq [YZX]$ and $X \cap Y \cap Z \subseteq [ZXY]$. Therefore $X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY]$.

$(ii) \Rightarrow (i)$: Let X be a pseudo-ideal of T. Then from $(ii), X = X \cap X \cap X \subseteq [XXX] \cap [XXX] \cap [XXX] = [XXX] = X^3 \Rightarrow X \subseteq X^3$. As X be a pseudo-ideal of T, so $X^3 \subseteq X$. Thus $X^3 = X$. This shows that every pseudo-ideal of T is idempotent.

$(i) \Rightarrow (iii)$: Suppose that, every pseudo-ideal of T is idempotent. Let X be a proper pseudo-ideal of T. Let Y be a pseudo-ideal of T such that $Y^3 \subseteq X$, then by hypothesis $Y^3 = Y$. Thus $Y \subseteq X$. This shows that X is semiprime pseudo-ideal of T. Hence every pseudo-ideal of T is semiprime.

$(iii) \Rightarrow (iv)$: Suppose that each proper pseudo-ideal of T is semiprime. By Theorem 3.4, any proper pseudo-ideal X of T is the intersection of all irreducible pseudo-ideals of T containing it. By (iii), every proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which containing it.

$(iv) \Rightarrow (i)$: Suppose that each proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which contain it. Let X be a pseudo-ideal of T. Therefore it is the intersection of all irreducible semiprime pseudo-ideals of T which contain it. Therefore X is a semiprime pseudo-ideal of T. As $X^3 \subseteq X^3 \Rightarrow X \subseteq X^3$ but $X^3 \subseteq X$ always. This shows that $X = X^3$. Hence every pseudo-ideal of T is idempotent. □

Theorem 3.9. *If every pseudo-ideal of T is strongly prime pseudo-ideal of T then each pseudo-ideal of T is idempotent.*

Proof. Suppose that, each pseudo-ideal of T is strongly prime, then each pseudo-ideal of T is semiprime. Thus by Theorem 3.8, every pseudo-ideal of T is idempotent. □

Theorem 3.10. *If every pseudo-ideal of T is idempotent and the set of pseudo-ideals of T is totally ordered under set inclusion then each pseudo-ideal of T is strongly prime pseudo-ideal of T.*

Proof. Suppose that every pseudo-ideal of T is idempotent and the set of pseudo-ideals of T is totally ordered under set inclusion. Let I, X, Y and Z be pseudo-ideals of T such that $[XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. As every pseudo-ideal of T is idempotent so, $X \cap Y \cap Z$ is idempotent. Then

$X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq [XYZ]$. Similarly $X \cap Y \cap Z \subseteq [YZX]$ and $X \cap Y \cap Z \subseteq [ZXY]$. Therefore $X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. As the set of all pseudo-ideal of T is totally ordered under set inclusion, therefore for pseudo-ideals X, Y, Z of T, we have the following six possibilities,

1) $X \subseteq Y \subseteq Z$, 2) $X \subseteq Z \subseteq Y$, 3) $Y \subseteq X \subseteq Z$
4) $Y \subseteq Z \subseteq X$, 5) $Z \subseteq X \subseteq Y$, 6) $Z \subseteq Y \subseteq X$.

In such cases, we have respectively,

1) $X \cap Y \cap Z = X$, 2) $X \cap Y \cap Z = X$, 3) $X \cap Y \cap Z = Y$,
4) $X \cap Y \cap Z = Y$, 5) $X \cap Y \cap Z = Z$, 6) $X \cap Y \cap Z = Z$.

Therefore $X \cap Y \cap Z = X$ or $X \cap Y \cap Z = Y$ or $X \cap Y \cap Z = Z$. Thus from $X \cap Y \cap Z \subseteq I$, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. This shows that I is a strongly prime pseudo-ideal of T. □

Theorem 3.11. *If the set of pseudo-ideals of T is totally ordered under set inclusion then every pseudo-ideal of T is idempotent if and only if each pseudo-ideal of T is prime.*

Proof. Suppose that every pseudo-ideal of T is idempotent. Let I, X, Y and Z be pseudo-ideals of T such that $XYZ \subseteq I$. As every pseudo-ideal of T is idempotent so, $X \cap Y \cap Z$ is idempotent. Then $X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq XYZ \subseteq I$. Therefore $X \cap Y \cap Z \subseteq I$. As in the proof of the Theorem 3.10 we get $X \cap Y \cap Z = X$ or $X \cap Y \cap Z = Y$ or $X \cap Y \cap Z = Z$. Thus from $X \cap Y \cap Z \subseteq I$, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. This shows that I is a prime pseudo-ideal of T.

Conversely, suppose that every pseudo-ideal of T is a prime pseudo-ideal of T. Since the set of pseudo-ideals of T is totally ordered under set inclusion, therefore the concepts of primeness and strongly primeness coincide. Hence by Theorem 3.9, every pseudo-ideal of T is idempotent. □

Definition 3.12. An proper pseudo-ideal X of T is said to be *maximal pseudo-ideal* of T if X is not properly contained in any proper pseudo-ideal of T.

Theorem 3.13. *Every maximal pseudo-ideal X of T is irreducible pseudo-ideal of T.*

Proof. Let X be a maximal pseudo-ideal of T. Suppose X is not irreducible pseudo-ideal of T. i.e. for any three pseudo-ideals A, B and C of T such that $A \cap B \cap C = X \Rightarrow A \neq X, B \neq X$ and $C \neq X \Rightarrow X \subset A \subset T, X \subset$

$B \subset T, X \subset C \subset T$. Which is contradiction to X be a maximal pseudo-ideal of T. Hence X is an irreducible pseudo-ideal of T. □

Definition 3.14. Let X be the non-empty subset of T. Then the intersection of all pseudo-ideals of T containing X is the smallest pseudo-ideal of T containing X. This pseudo-ideal of T is called the pseudo-ideal of T generated by X and it is denoted by $(X)_{pi}$. A pseudo-ideal I of T is said to be the principal pseudo-ideal generated by an element x if I is a pseudo-ideal generated by $\{x\}$ for some $x \in T$ and is denoted by $(x)_{pi}$.

Let $\mathfrak{A}$ be the set of all pseudo-ideals of T and $\mathfrak{B}$ be the set of all strongly irreducible pseudo-ideals of T. For each $X \in \mathfrak{A}$, we define $\Psi_X = \{Y \in \mathfrak{B} : X \nsubseteq Y\}$

Theorem 3.15. *The family,* $\mathfrak{J}(\mathfrak{B}) = \{\Psi_X : X \in \mathfrak{A}\}$ *forms a topology on the set* $\mathfrak{B}$.

Proof. (i) As $\{0\} \in \mathfrak{A}$, so $\Psi_{\{0\}} = \{Y \in \mathfrak{B} : \{0\} \nsubseteq Y\} = \emptyset$. Thus $\emptyset \in \mathfrak{J}(\mathfrak{B})$.

(ii) Since $T \in \mathfrak{A}$, we have $\Psi_T = \{Y \in \mathfrak{B} : T \nsubseteq Y\} = \mathfrak{B}$ because $\mathfrak{B}$ is the collection of all proper strongly irreducible pseudo-ideals of T. Thus $\mathfrak{B} \in \mathfrak{J}(\mathfrak{B})$.

(iii) Let $\Psi_{X_1}, \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$. We show that $\Psi_{X_1} \cap \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$. Let $Y \in \Psi_{X_1} \cap \Psi_{X_2}$ then $Y \in \mathfrak{B}$ such that $X_1 \nsubseteq Y$ and $X_2 \nsubseteq Y$. Suppose that $X_1 \cap X_2 \subseteq Y$. Now, we have $X_1 \cap X_2 \cap T = X_1 \cap X_2 \subseteq Y$. Since Y is a strongly irreducible pseudo-ideal of T, therefore either $X_1 \subseteq Y$ or $X_2 \subseteq Y$ or $T \subseteq Y$. But $T \nsubseteq Y$ (since Y is proper). Therefore $X_1 \subseteq Y$ or $X_2 \subseteq Y$, which is a contradiction. Hence $X_1 \cap X_2 \nsubseteq Y$. Therefore $Y \in \Psi_{X_1 \cap X_2}$. Thus $\Psi_{X_1} \cap \Psi_{X_2} \subseteq \Psi_{X_1 \cap X_2}$. On the other hand if $Y \in \Psi_{X_1 \cap X_2}$ then $Y \in \mathfrak{B}$ and $X_1 \cap X_2 \nsubseteq Y$. This implies that $X_1 \nsubseteq Y$ and $X_2 \nsubseteq Y$. Therefore $Y \in \Psi_{X_1}$ and $Y \in \Psi_{X_2} \Rightarrow Y \in \Psi_{X_1} \cap \Psi_{X_2}$. Hence $\Psi_{X_1 \cap X_2} \subseteq \Psi_{X_1} \cap \Psi_{X_2}$.This shows that $\Psi_{X_1} \cap \Psi_{X_2} = \Psi_{X_1 \cap X_2}$. Thus $\Psi_{X_1} \cap \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$.

(iv) Let $\{X_\alpha\}_{\alpha \in \Delta}$ (where Δ is any indexing set.) be family of pseudo-ideals of T and $\{\Psi_{X_\alpha} : \alpha \in \Delta\} \subseteq \mathfrak{J}(\mathfrak{B})$. Then $\bigcup_{\alpha \in \Delta} \Psi_{X_\alpha} = \{Y \in \mathfrak{B} : X_\alpha \nsubseteq Y$ for some $\alpha \in \Delta\} = \{Y \in \mathfrak{B} : \left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi} \nsubseteq Y\} = \Phi_{\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi}} \in \mathfrak{J}(\mathfrak{B})$, where $\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi}$ is the pseudo-ideal of T generated by $\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)$. Therefore from $(i), (ii), (iii)$ and (iv), we get the set $\mathfrak{J}(\mathfrak{B})$ forms a topology on $\mathfrak{B}$. □

Theorem 3.16. *If T is partially ordered ternary semigroup with identity then $\mathfrak{B}$ is a compact space.*

Proof. Suppose that $\{\Psi_{X_k} : k \in \Delta\}$ is an open covering of $\mathfrak{B}$, where Δ is an indexing set. That is $\mathfrak{B} = \bigcup_{k\in\Delta} \Psi_{X_k}$. By Theorem 3.15, $\Psi_T = \mathfrak{B}$, therefore $\Psi_T = \bigcup_{k\in\Delta} \Psi_{X_k} \Rightarrow \Psi_T = \Psi_{\left(\bigcup_{k\in\Delta} X_k\right)_{pi}} \Rightarrow T = \left(\bigcup_{k\in\Delta} X_k\right)_{pi}$. As $e \in T, e \in \left(\bigcup_{k\in\Delta} X_k\right)_{pi}$. Hence $e \in (\bigcup_{i=1}^{n} X_i)_{pi} \Rightarrow T = (\bigcup_{i=1}^{n} X_i)_{pi} \Rightarrow \mathfrak{B} = \bigcup_{k=1}^{n} \Psi_{X_k}$. This shows that every open cover of $\mathfrak{B}$ has finite subcover. Hence $\mathfrak{B}$ is compact space. □

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