Quasigroups and Related Systems **32** (2024), 59 – 67

https://doi.org/10.56415/qrs.v32.06

On idempotent ordered semigroups

Susmita Mallick and Kalyan Hansda

Abstract. An element e of an ordered semigroup $(S, \cdot, \leqslant)$ is called an ordered idempotent if $e \leqslant e^2$. We call an ordered semigroup S idempotent ordered semigroup if every element of S is an ordered idempotent. Every idempotent semigroup is a complete semilattice of rectangular idempotent semigroups and in this way we arrive to many other important classes of idempotent ordered semigroups.

1. Introduction

Idempotents play an important role in different major subclasses of the regular semigroups S. A regular semigroup S is called orthodox if the set of all idempotents $E(S)$ forms a subsemigroup, and S is a band if $S = E(S)$.

T. Saito studied systematically the influence of order on idempotent semigroup [4]. In [1], Bhuniya and Hansda introduced the notion of ordered idempotents and studied different classes of regular ordered semigroups, such as, completely regular, Clifford and left Clifford ordered semigroups by their ordered idempotents. If T is a subsemigroup of S, then the set of ordered regular elements of T is denoted by $Reg_{\leqslant}(T)$ [2]. If $T =< E_{\leqslant}(S) >$ then $Reg_{\leqslant}(T) = T = Reg_{\leqslant}(S) \cap T$, in general. In [2], Hansda proved several equivalent conditions so that $Reg_{\leqslant}(T) = T = Reg_{\leqslant}(S) \cap T$ for $T = (Se], (eS]$ and $(eSf]$, where e, f are ordered idempotents. The purpose of this paper to study ordered semigroups in which every element is an ordered idempotent. Complete semilattice decomposition of these semigroups automatically suggests the looks of rectangular idempotent semigroups and in this way we arrive to many other important classes of idempotent ordered semigroups.

2010 Mathematics Subject Classification: 20M10, 06F05.

Keywords: ordered idempotent; idempotent ordered semigroup; rectangular idempotent ordered semigroup normal idempotent ordered semigroup.

2. Preliminaries

An ordered semigroup is a partially ordered set $(S, \leqslant)$, and at the same time a semigroup $(S, \cdot)$ such that for all $a, b, c \in S$; $a \leqslant b$ implies that $ca \leqslant cb$ and $ac \leqslant bc$. It is denoted by $(S, \cdot, \leqslant)$. Throughout this article, unless stated otherwise, S stands for an ordered semigroup. For every subset $H \subseteq S$, denote $(H] = \{t \in S : t \leqslant h, \text{ for some } h \in H\}$. Kehayopulu [3] defined Green's relations on a regular ordered semigroup S as follows:

$$a\mathcal{L}b \; if \; (S^1a] = (S^1b], \; a\mathcal{R}b \; if \; (aS^1] = (bS^1],$$

$$a\mathcal{J}b \; if \; (S^1aS^1] = (S^1bS^1], \text{ and } \mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

These four relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ are equivalence relations.

An equivalence relation ρ on S is called left (right) congruence if for every $a, b, c \in S$; $a\rho b$ implies $ca\rho cb$ $(ac\rho bc)$. By a congruence we mean both left and right congruence. A congruence ρ is called a semilattice congruence on S if for all $a, b \in S$, $a\rho a^2$ and $ab\rho ba$. By a complete semilattice congruence we mean a semilattice congruence σ on S such that for $a, b \in S$, $a \leqslant b$ implies that $a\sigma ab$. An element e of an ordered semigroup $(S, \cdot, \leqslant)$ is called an ordered idempotent [1] if $e \leqslant e^2$. An ordered semigroup S is called $\mathcal{H}-$commutative if for every $a, b \in S$, $ab \in (bSa]$.

If F is a semigroup, then the set $P_f(F)$ of all finite subsets of F is a semilattice ordered semigroup with respect to the product $\cdot$ and partial order relation $\leqslant$ given by: for $A, B \in P_f(F)$,

$$A \cdot B = \{ab \mid a \in A, b \in B\} \text{ and } A \leqslant B \text{ if and only if } A \subseteq B.$$

3. Idempotent ordered semigroups

In this section we characterize ordered semigroups of which every element is an ordered idempotent. We show that these ordered semigroups are analogous to bands.

We first make a natural analogy between band and idempotent ordered semigroup.

Theorem 3.1. *Let B be a semigroup. Then $P_f(B)$ is idempotent ordered semigroup if and only if B is a band.*

Proof. Let B be a band and $U \in P_f(B)$. Choose $x \in U$. Then $x^2 \in U^2$ implies $x \in U^2$. Then $U \subseteq U^2$. So $P_f(B)$ is idempotent ordered semigroup.

Conversely, assume that B be a semigroup such that $P_f(B)$ is an idempotent ordered semigroup. Take $y \in B$. Then $Y = \{y\} \in P_f(B)$. Thus $Y \subseteq Y^2$, which implies $y = y^2$. Hence B is a band. □

Proposition 3.2. *Let B be a band, S be an idempotent ordered semigroup and $f : B \longrightarrow S$ be a semigroup homomorphism. Then there is an ordered semigroup homomorphism $\phi : P_f(B) \longrightarrow S$ such that the following diagram is commutative:*

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "l"'] & S \\
P_f(B) \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where $l : B \longrightarrow P_f(B)$ is given by $l(x) = \{x\}$.

Proof. Define $\phi : P_f(F) \longrightarrow S$ by: for $A \in P_f(F)$, $\phi(A) = \vee_{a \in A} f(a)$. Then for every $A, B \in P_f(F)$, $\phi(AB) = \vee_{a\in A, b\in B} f(ab) = \vee_{a\in A, b\in B} f(a)f(b) = (\vee_{a\in A} f(a))(\vee_{b\in B} f(b)) = \phi(A)\phi(B)$, and if $A \leqslant B$, then $\phi(A) = \vee_{a\in A} f(a) \leqslant \vee_{b\in B} f(b) = \phi(B)$ shows that ϕ is an ordered semigroup homomorphism. Also $\phi \circ l = f$. □

Lemma 3.3. *In an idempotent ordered semigroup S, $a^m \leqslant a^n$ for every $a \in S$ and $m, n \in \mathbb{N}$ with $m \leqslant n$.*

Every idempotent ordered semigroup S is completely regular and hence $\mathcal{J}$ is the least complete semilattice congruence on S, by [Lemma 4.13, [1]]. In an idempotent ordered semigroup S, the Green's relation $\mathcal{J}$ can equivalently be expressed as: for $a, b \in S$,

$$a\mathcal{J}b \text{ if there are } x, y, u, v \in S \text{ such that } a \leqslant axbya \text{ and } b \leqslant buavb.$$

Now we characterize the $\mathcal{J}-$class in an idempotent ordered semigroup.

Definition 3.4. An idempotent ordered semigroup S is called *rectangular* if for all $a, b \in S$, there are $x, y \in S$ such that $a \leqslant axbya$.

Example 3.5. $(\mathbb{N}, \cdot, \leqslant)$ is a rectangular idempotent ordered semigroup, whereas if we define $a\ o\ b = \min\{a, b\}$ *for all* $a, b \in \mathbb{N}$ then $(\mathbb{N}, \circ, \leqslant)$ is an idempotent ordered semigroup but not rectangular.

Also we have the following equivalent conditions.

Lemma 3.6. *Let S be an idempotent ordered semigroup. Then the following conditions are equivalent:*

1. *S is rectangular;*
2. *for all $a, b \in S$, there is $x \in S$ such that $a \leqslant axbxa$;*
3. *for all $a, b, c \in S$ there is $x \in S$ such that $ac \leqslant axbxc$.*

Proof. (1) $\Rightarrow$ (3): Let $a, b, c \in S$. Then there are $x, y \in S$ such that $a \leqslant axbya$. This implies $ac \leqslant axbyac \leqslant ax(bya)(bya)c \leqslant (axbyab)(axbyab)yac \leqslant a(axbyabya)b(axbyabya)c \leqslant atbtc$, where $t = axbyabya \in S$.

(3) $\Rightarrow$ (2): Let $a, b \in S$. Then there is $x \in S$ such that $a^2 \leqslant axbxa$. Then $a \leqslant a^2$ implies that $a \leqslant axbxa$.

(2) $\Rightarrow$ (1): This follows directly. □

As we can expect, we show that the equivalence classes in an idempotent ordered semigroup S determined by $\mathcal{J}$ are rectangular.

Theorem 3.7. *Every idempotent ordered semigroup is a complete semilattice of rectangular idempotent ordered semigroups.*

Proof. Let S be an idempotent ordered semigroup. Then $\mathcal{J}$ is the least complete semilattice congruence on S. Now consider a $\mathcal{J}$-class $(c)_{\mathcal{J}}$ for some $c \in S$. Since $\mathcal{J}$ is a complete semilattice congruence, $(c)_{\mathcal{J}}$ is a subsemigroup of S. Let $a, b \in (c)_{\mathcal{J}}$. Then there is $x \in S$ such that $a \leqslant axbxa$, which implies that $a \leqslant a(axb)b(bxa)a$, that is, $a \leqslant aubva$ where $u = axb$ and $v = bxa$. Also the completeness of $\mathcal{J}$ implies that $(a)_{\mathcal{J}} = (a^2xbxa)_{\mathcal{J}} = (axb)_{\mathcal{J}} = (bxa)_{\mathcal{J}}$, and $u, v \in (c)_{\mathcal{J}}$. Thus $(c)_{\mathcal{J}}$ is a rectangular idempotent ordered semigroup. □

Definition 3.8. An idempotent ordered semigroup S is called *left* (*right*) *zero* if for every $a, b \in S$, there exists $x \in S$ such that $a \leqslant axb$ ($a \leqslant bxa$).

Proposition 3.9. *An idempotent ordered semigroup is left zero if and only if it is left simple.*

Proof. First suppose that S is a left zero idempotent ordered semigroup and $a \in S$. Then for any $b \in S$, there is $x \in S$ such that $b \leqslant bxa$, which shows that $b \in (Sa]$. Thus $S = (Sa]$ and hence S is left simple.

Conversely, assume that S is left simple. So for every $a, b \in S$, there is $s \in S$ such that $a \leqslant sb$. Then $a \leqslant a^2$ gives that $a^2 \leqslant asb$. Thus S is a left zero idempotent ordered semigroup. □

Lemma 3.10. *In an idempotent ordered semigroup S, the following conditions are equivalent:*

1. *For all $a, b \in S$, there is $x \in S$ such that $ab \leqslant abxba$.*
2. *For all $a, b \in S$, there is $x \in S$ such that $ab \leqslant axbxa$.*

3 *For all $a, b \in S$, there is $x, y \in S$ such that $ab \leqslant axbya$.*

Proof. (1) ⇒ (3): This follows directly.

(3) ⇒ (2): This is similar to the Lemma 3.6.

(2) ⇒ (1): Let $a, b \in S$. Then there is $x \in S$ such that $bab \leqslant baxbxba$. Now since $ab \leqslant ababab$, we have $ab \leqslant ab(abaxbx)ba$. □

Definition 3.11. An idempotent ordered semigroup S is called *left regular* if for every $a, b \in S$ there is $x \in S$ such that $ab \leqslant axbxa$.

Theorem 3.12. *An idempotent ordered semigroup S is left regular if and only if $\mathcal{L} = \mathcal{J}$ is the least complete semilattice congruence on S.*

Proof. First we assume that S is left regular. Let $a, b \in S$ be such that $a\mathcal{J}b$. Then there are $u, v, x, y \in S$ such that $a \leqslant ubv$ and $b \leqslant xay$. Since S is left regular, there are $s, t \in S$ such that $bv \leqslant bsvsb$ and $ay \leqslant atyta$. Then $a \leqslant ubsvsb$ and $b \leqslant xatyta$; which shows that $a\mathcal{L}b$. Thus $\mathcal{J} \subseteq \mathcal{L}$. Again $\mathcal{L} \subseteq \mathcal{J}$ on every ordered semigroup and hence $\mathcal{L} = \mathcal{J}$. Since every idempotent ordered semigroup is completely regular, it follows that $\mathcal{L}$ is the least complete semilattice congruence on S, by [Theorem 5.10, [1]]

Conversely, let $\mathcal{L}$ is the least complete semilattice congruence on S. Consider $a, b \in S$. Then $ab\mathcal{L}ba$ implies that $ab \leqslant xba$ for some $x \in S$. This implies that

$$ab \leqslant abab \leqslant abxba.$$

Thus S is a left regular idempotent ordered semigroup, by Lemma 3.10. □

Theorem 3.13. *Let S be an idempotent ordered semigroup. Then the following conditions are equivalent:*

1. *S is left regular;*
2. *S is a complete semilattice of left zero idempotent ordered semigroups;*
3. *S is a semilattice of left zero idempotent ordered semigroups.*

Proof. (1) ⇒ (2): In view of Theorem 3.12, it is sufficient to show that each $\mathcal{L}$-class is a left zero idempotent ordered semigroup. Let L be an $\mathcal{L}$-class and $a, b \in L$. Then L is a subsemigroup, since $\mathcal{L}$ is a semilattice congruence. Since $a\mathcal{L}b$ there is $x \in S$ such that $a \leqslant xb$. This implies that $a \leqslant a^3 \leqslant a^2xb \leqslant a^2xb^2 \leqslant aub$, where $u = axb$.

By the completeness of $\mathcal{L}$, $a \leqslant xb$ implies that $(a)_{\mathcal{L}} = (axb)_{\mathcal{L}}$, and hence $u \in L$. Thus S is left zero idempotent ordered semigroup.

(2) $\Rightarrow$ (3): This implication is trivial.

(3) $\Rightarrow$ (1): Let ρ be a semilattice congruence on S such that each ρ-class is a left zero idempotent ordered semigroup. Consider $a, b \in S$. Then $ab,\ ba \in (ab)_\rho$ shows that there is $s \in (ab)_\rho$ such that $ab \leqslant absba \leqslant absbsba \leqslant a(bsb)b(bsb)a$. Hence S is left regular. ☐

Lemma 3.14. *Let S be an idempotent ordered semigroup. Then the following conditions are equivalent:*

1. *S is $\mathcal{H}$-commutative;*
2. *for all $a, b \in S$, $ab \in (baS] \cap (Sba]$;*
3. *S is a complete semilattice of t-simple idempotent ordered semigroups;*
4. *S is a semilattice of t-simple idempotent ordered semigroups.*

Proof. (1) $\Rightarrow$ (2): Consider $a, b \in S$. Since S is $\mathcal{H}-$ commutative, there is $u \in S$ such that $ab \leqslant bua$. Also for $u, a \in S$, $ua \leqslant asu$ for some $s \in S$. Thus $ab \leqslant basu$, which shows that $ab \in (baS]$. Similarly $ab \in (Sba]$. Hence $ab \in (baS] \cap (Sba]$.

(2) $\Rightarrow$ (3): Suppose that J be an $\mathcal{J}$-class in S and $a, b \in J$. Since J is rectangular there is $x \in J$ such that $a \leqslant axbxa$. Also by the given condition (2) there is $u \in J$ such that $bxa \leqslant xaub$. So $a \leqslant ax^2aub \leqslant vb$, where $v = ax^2au$. Since $\mathcal{J}$ is a complete semilattice congruence on S, $(a)_{\mathcal{J}} = (a^2x^2aub)_{\mathcal{J}} = (ax^2au)_{\mathcal{J}} = (v)_{\mathcal{J}}$. So $v \in J$. This shows that J is left simple. Similarly it can be shown that J is also right simple. Thus S is a complete semilattice of t-simple idempotent ordered semigroups.

(3) $\Rightarrow$ (4): This follows trivially.

(4) $\Rightarrow$ (1): Let S be the semilattice Y of t-simple idempotent ordered semigroups $\{S_\alpha\}_{\alpha \in Y}$ and ρ be the corresponding semilattice congruence on S. Then there are $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and $b \in S_\beta$. Then $ba, ab \in S_{\alpha\beta}$. Since $S_{\alpha\beta}$ is t-simple, $ab \leqslant xba$ for some $x \in S_{\alpha\beta}$. Now for $x, ba \in S_{\alpha\beta}$ there is $y \in S_{\alpha\beta}$ such that $x \leqslant bay$. This finally gives $ab \leqslant bta$, where $t = ayb$. ☐

Definition 3.15. An idempotent ordered semigroup $(S, ., \leqslant)$ is called *weakly commutative* if for any $a, b \in S$ there exists $u \in S$ such that $ab \leqslant bua$.

Theorem 3.16. *For an idempotent ordered semigroup S, the followings are equivalent:*

1. *S is weakly commutative;*
2. *for any $a, b \in S,\ ab \in (baS] \cap (sba]$;*
3. *S is complete semilattice of left and right simple idempotent ordered semigroups.*

Proof. (1) $\Rightarrow$ (2): Let $a, b \in S$. Then there exists $u \in S$ such that $ab \leqslant bua$, also for $u, a \in S$, there exists $z \in S$ such that $ua \leqslant azu$. Thus $ab \leqslant bua \leqslant baza\ for\ za \in S$. So $ab \leqslant (baS]$. Similarly $ab \in (Sba]$. Hence $ab \in (baS] \cap (sba]$.

(2) $\Rightarrow$ (3): Since S is an idempotent ordered semigroup, by Theorem 3.7 we have ρ is a complete semilattice congruence. We now have to show that, for each $z \in S, J = (z)_\rho$ is left and right simple. For this let us choose $a, b \in J$. Then there exists $x, y \in S$ such that $a \leqslant axbya$. So from the given condition $bya \in (syab]$ and therefore there is $s_1 \in S$ such that $bya \leqslant s_1yab$. Therefore $a \leqslant axs_1yab$. Now since ρ is complete semilattice congruence on S, we have $(a)_\rho = (a^2xs_1yab)_\rho = (axs_1yab)_\rho = (axs_1ya)_\rho$. Thus $a \leqslant ub$, where $u = axs_1ya \in J$. Hence J is left simple and similarly it is right simple.

(3) $\Rightarrow$ (1): Let S is complete semilattice Y of left and right simple idempotent ordered semigroups $\{S_\alpha\}_{\alpha \in Y}$. Thus $S = \{S_\alpha\}_{\alpha \in Y}$. Take $a, b \in S$. Then there are $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and $b \in S_\beta$. Thus $ab \in S_{\alpha\beta}$. So $ab, ba \leqslant S_{\alpha\beta}$. Then there are $u, v \in S_{\alpha\beta}$ such that $ab \leqslant uba$ and $ab \leqslant bav$ implies $ab \leqslant ab^2 \leqslant bta$, where $t = avub$. Hence S is weakly commutative. This completes the proof. □

Definition 3.17. An idempotent ordered semigroup $(S, \cdot, \leqslant)$ is called *normal* if for any $a, b, c \in S$, there exists $x \in S$ such that $abca \leqslant acxba$.

Theorem 3.18. *For an idempotent ordered semigroup S, the followings are equivalent:*

1. *S is normal;*
2. *aSb is weakly commutative, for any $a, b \in S$;*
3. *aSa is weakly commutative, for any $a \in S$.*

Proof. (1) $\Rightarrow$ (2): Consider $axb, ayb \in aSb\ for\ x, y \in S$. As S is normal, $\exists u, v \in S$ such that $(axb)(ayb) \leqslant (axb)(ayb)(axb)(ayb) \leqslant aybuxba^2xbayb$, for $xba,yb\in S \leqslant (ayb)uxb(bay)v(a^2x)b$, for $a^2x,bay\in S \leqslant (ayb)(uxb^2ayva)(axb)$. This implies $(axb)(ayb)\leqslant (ayb)t(axb)\leqslant (ayb)(ayb)t(axb)(axb)$, $t = uxb^2ayva$ and thus $(axb)(ayb) \leqslant aybsaxb$, where $s = aybtaxb \in aSb$. Thus aSb is weakly commutative.

(2) $\Rightarrow$ (3): This is obvious by taking $b = a$.

(3) $\Rightarrow$ (1): Let $a, b, c \in S$. Then $abca, aca \in aSa$. Since aSa is weakly commutative. Then there is $s \in aSa$ such that $(abca)aca \leqslant acasaabca$. Now for $aba, abca \in aSa$, there is $t \in aSa$ such that $abaabca \leqslant abcataba$. Thus $abca \leqslant (abca)(abca) \leqslant abca^2ca^2bca \leqslant abca^2ca^2ba^2bca = (abcaaca)(abaabca)$ $\leqslant (acasa^2ca)(abcataba) \leqslant acuba$; where $u = asa^2bca^2bcata \in S$. Hence S is normal. □

Definition 3.19. An idempotent ordered semigroup $(S, \cdot, \leqslant)$ is called *left normal* (*right normal*) if for any $a, b, c \in S$, there exists $x \in S$ such that $abc \leqslant acxb$ ($abc \leqslant bxac$).

Theorem 3.20. *Let S be a left normal idempotent ordered semigroup, then*

1. *$\mathcal{L}$ is the least complete semilattice congruence on S;*
2. *S is a complete semilattice of LZ-idempotent ordered semigroups.*

Proof. (1): Let $a, b \in S$ such that $a\rho b$. Then there are $x, y, u, v \in S$ such that

$$a \leqslant a(xbya), b \leqslant b(uavb). \tag{1}$$

Since S is left normal, we have for $x, b, ya \in S, xbya \leqslant xyatb$ for some $t \in S$. Similarly there is $s \in S$ such that $uavb \leqslant uvbsa$. So from (1), $a \leqslant (axyat)b$ and $b \leqslant (buvbs)a$. Hence $a\mathcal{L}b$. Thus $\rho \subseteq \mathcal{L}$.

Again, let $a, b \in S$ such that $a\mathcal{L}b$. Thus there are $u, v \in S$ such that $a \leqslant ub$ and $b \leqslant va$. Also we have $a \leqslant a^3 = aaa \leqslant auba \leqslant aubba$ for some $u, b \in S$. Therefore $a\rho b$. Thus $\mathcal{L} \subseteq \rho$. Thus $\mathcal{L} = \rho$.

(2): Here we are only to proof that each $\mathcal{L}$-class is a left zero. For this let $\mathcal{L}$-class $(x)_\mathcal{L} = L$, (say) for some $x \in S$. Clearly $(x)_\mathcal{L}$ is a subsemigroup of S. Take $a, b \in L$. Then $y, z \in S$ such that $a \leqslant yb, b \leqslant za$. Since S is left normal, there is $t \in S$ such that $a \leqslant yb \leqslant (yb)b \leqslant yzab$.

This implies $a \leqslant a^2 \leqslant a(ayzb)b$. Thus $(a)_\mathcal{L} = (a^2yzb)_\mathcal{L} = (ayzb)_\mathcal{L}$. Therefore L is left zero. Hence S is a complete semilattice of left zero idempotent ordered semigroups. □

Theorem 3.21. *Let S be a idempotent ordered semigroup, then S is normal if and only if $\mathcal{L}$ is right normal band congruence and $\mathcal{R}$ is left normal band congruence.*

Proof. First we shall see that $\mathcal{L}$ is left congruence on S. For this let us take $a, b \in S$ such that $a\mathcal{L}b$ and $c \in S$. Then there is $x, y \in S$ such that $a \leqslant xb, b \leqslant ya$. Now as S is normal idempotent ordered semigroup, $ca \leqslant cxb \leqslant cxbcxb \leqslant cxbx(s_1)cb$ for some $s_1 \in S$. Thus $a \leqslant s_2cb$, where $s_2 = cxbxs_1 \in S$. Again $cb \leqslant s_4ca$ where $s_4 = cyays_3 \in S$. So $ca\mathcal{L}cb$. It finally shows that $\mathcal{L}$ is congruence on S. Similarly it can be shown that $\mathcal{R}$ is congruence on S.

Next consider that $a, b, c \in S$ are arbitrary. Then since S is a normal idempotent ordered semigroup, $abc \leqslant abcabc \leqslant abcbt_1ac \leqslant acb(t_1t_2bac)$ for some $acbt_1t_2 \in S$. Also $bac \leqslant bacbac \leqslant bacat_3bc \leqslant (bct_3t_4abc)$ for some $bct_3t_4 \in S$. So $abc\mathcal{L}bac$. Similarly $abc\mathcal{R}acb$. This two relations respectively shows that $\mathcal{L}$ is right normal band congruence and $\mathcal{R}$ is left normal band congruence.

Conversely, suppose that $\mathcal{L}$ is a right normal band congruence and $\mathcal{R}$ is a left normal band congruence. Consider $a, b,$ and $c \in S$. Then $abc\mathcal{R}acb$ *and* $bca\mathcal{L}cba$. Then $\exists x_1, x_2 \in S$ such that

$$abc \leqslant (acb)x_1 \text{ and } bca \leqslant x_2cba.$$

Now then $abc \leqslant (abc)bca \leqslant (acb)x_1bca \leqslant ac(bx_1x_2c)ba$ for some $bx_1x_2c \in S$. Hence S is an idempotent ordered semigroup. ☐

References

[1] **A.K. Bhuniya and K. Hansda**, *On completely regular and Clifford ordered semigroups*, Afrika Math., **31** (2020), 1029 – 1045.

[2] **K. Hansda**, *Regularity of subsemigroups generated by ordered idempotents*, Quasigroups and Related Systems, **22** (2014), 217 – 222.

[3] **N. Kehayopulu**, *Note on Green's relation in ordered semigroups*, Math. Japonica, **36** (1991), 211 – 214.

[4] **T. Saito**, *Ordered idempotent semigroups*, J. Math. Soc. Japan., **14** (1962), 150 – 169.

Received July 01, 2023

Department of Mathematics, Visva Bharati University, Santiniketan, Bolpur - 731235, West Bengal, India
E-mails: susmita.mallick3@gmail.com, kalyan.hansda@visva-bharati.ac.in