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Probabilistic groupoids

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Abstract. Algebraic structures are commonly used as a tool in treatments of various processes. But their exactness reduces the opportunity of their application in nondeterministic environment. On the other hand, probability theory and fuzzy logic do not provide convenient means for expressing the result of combining elements in order to produce new ones. Moreover, these theories are not developed to "measure" algebraic properties. Therefore, we propose a new concept which relies both on universal algebra and probability theory.

We introduce probabilistic mappings, and by them we define the notion of a probabilistic algebra. Let A and B be non-empty sets, and let $\mathcal{D}_B$ be the set of all probability distributions on B. A probabilistic mapping from A to B is a mapping $h : A \to \mathcal{D}_B$. Let A be a set, $n \in \mathbb{N}$, and let $A^n = \{(a_1, a_2, \dots, a_n)|\ a_i \in A,\ i = 1, 2, \dots, n\}$ be the n-th power of A. Every probabilistic mapping from A^n to A is a probabilistic (n-ary) operation on A. A pair (A, F) of a set A and a family F of probabilistic operations on A is called a probabilistic algebra. When $F = \{f\}$ has one binary operation, then the probabilistic algebra (A, f) is a probabilistic groupoid. "Ordinary" groupoids are just a special type of probabilistic ones. Basic properties of probabilistic groupoids and some classes of probabilistic groupoids (with units, commutative, associative, idempotent, with cancellation, with inverses, quasigroups, groups) are treated in this paper. Here we consider only the finite case.

1. Probabilistic mappings

Let A and B be non-empty finite sets, and denote by $\mathcal{D}_B$ the set of all probability distributions on B, that is

$$\mathcal{D}_B = \{f|\ f : B \to \mathbb{R}, f(b) \geqslant 0 \text{ for } b \in B, \sum_{b\in B} f(b) = 1\}.$$

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When $B = \{b_1, b_2, \ldots, b_n\}$ is a finite set, a probability distribution $f : B \to \mathbb{R}$ can be also denoted, as usual, by the set of images $\{f(b_1), f(b_2), \ldots, f(b_n)\}$.

For every mapping h from A to $\mathcal{D}_B$ we say that it is a *probabilistic mapping* from A to B. We denote such a mapping by $h : A \looparrowright B$. If $h(a) = f$ for some $a \in A$, then we write $f = h_a$, and when $h_a(b) = p,\ p \in [0,1]$, we say that the probability of mapping the element $a \in A$ into $b \in B$ is p, or that b is an image of a with probability p. The element a is called a *pre-image* of b with probability $p = h_a(b)$. Given a fixed element $b \in B$, each element of A is a pre-image of b with some probability, but the set $h^{-1}\{b\} = \{h_a(b) |\ a \in A\}$ is not necessarily a probability distribution on A.

Example 1.1. $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $h : A \looparrowright B$:

$$h_1 = \begin{pmatrix} a & b & c & d \\ 0.3 & 0 & 0.7 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} a & b & c & d \\ 0.2 & 0 & 0.2 & 0.6 \end{pmatrix}.$$

In order to get the sets $\{h_a(b) |\ a \in A\}$, for every $b \in B$, to be probability distributions on A a necessary, but not sufficient, condition is to have the equality $|A| = |B|$. An example is given below.

Example 1.2. $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $s, h : A \looparrowright B$:

$$s_1 = \begin{pmatrix} a & b & c \\ 0.2 & 0.5 & 0.3 \end{pmatrix}, \quad s_2 = \begin{pmatrix} a & b & c \\ 0.6 & 0.4 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} a & b & c \\ 0.2 & 0.1 & 0.7 \end{pmatrix};$$

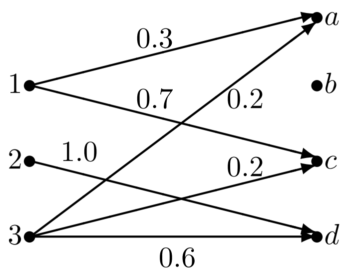
$$h_1 = \begin{pmatrix} a & b & c \\ 0.2 & 0.5 & 0.3 \end{pmatrix}, \quad h_2 = \begin{pmatrix} a & b & c \\ 0.2 & 0.5 & 0.3 \end{pmatrix}, \quad h_3 = \begin{pmatrix} a & b & c \\ 0.2 & 0.1 & 0.7 \end{pmatrix}.$$

The sets $s^{-1}\{a\} = \{0.2, 0.6, 0.2\}$, $s^{-1}\{b\} = \{0.5, 0.4, 0.1\}$, $s^{-1}\{c\} = \{0.3, 0, 0.7\}$ are probability distributions on A, while the set $h^{-1}\{a\} = \{0.2, 0.2, 0.2\}$ is not.

Note that every probabilistic mapping from A to B is actually a family of distributions on B indexed by the elements of A. In spite of the fact that this is a familiar notion (discrete stochastic process), the main idea is to consider some algebraic properties which are satisfied with certain "probability". Therefore, we start with this concept and appropriate new terminology.

2. Representations of probabilistic mappings

Besides using the usual representations of mappings, in the case when the sets are finite (and not having many elements), weighted digraphs, stochastic matrices and tables are particularly convenient for expressing probabilistic mappings. In what follows, we give the graph, matrix and table representation of the probability mapping from Example 1.



$$\Pi = \begin{bmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & 0.2 & 0.6 \end{bmatrix}$$

h	h_1	h_2	h_3
a	0.3	0	0.2
b	0	0	0
c	0.7	0	0.2
d	0	1	0.6

3. Compositions of probabilistic mappings

Let $f : A \looparrowright B$ and $g : B \looparrowright C$ be probabilistic mappings. Define *composition* of f and g to be the mapping $h = g \bullet f$ which maps every element a of A into a real-valued function h_a on C, determined by the rule

$$h_a(c) = \sum_{b \in B} f_a(b) g_b(c),$$

for every $c \in C$.

Theorem 3.1. *A composition of probabilistic mappings is a probabilistic mapping.*

Proof. Let $f : A \looparrowright B$ and $g : B \looparrowright C$ be probabilistic mappings, and h be the composition of f and g. Then for the image h_a of an arbitrary element

a of A, we obtain

$$\sum_{c\in C} h_a(c) = \sum_{c\in C}\sum_{b\in B} f_a(b)g_b(c) = \sum_{b\in B}\Big(f_a(b)\sum_{c\in C} g_b(c)\Big) = \sum_{b\in B} f_a(b)\cdot 1 = 1.$$

Clearly $h_a(c) \geqslant 0$ for each $c \in C$, hence for every $a \in A$, h_a is a probability distribution on C, so h is a probabilistic mapping, $h : A \looparrowright C$. □

By the definition of the notion composition of probabilistic mappings and the matrix representation, we get the following result.

Theorem 3.2. *Let A, B and C be finite sets, $f : A \looparrowright B$ and $g : B \looparrowright C$. If Π_1 and Π_2 are the corresponding matrices of f and g, respectively, then their product $\Pi_1 \cdot \Pi_2$ is the matrix representation of the composition $g \bullet f$.*

Example 3.3. $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{u, v\}$:

$$\Pi_1(A \looparrowright B) = \begin{bmatrix} 0.3 & 0 & 0 & 0.7 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{bmatrix}, \qquad \Pi_2(B \looparrowright C) = \begin{bmatrix} 0.8 & 0.2 \\ 1 & 0 \\ 0 & 1 \\ 0.6 & 0.4 \end{bmatrix},$$

$$\Pi_1 \cdot \Pi_2(A \looparrowright C) = \begin{bmatrix} 0.66 & 0.34 \\ 0.6 & 0.4 \\ 0.44 & 0.56 \end{bmatrix}.$$

Theorem 3.4. *Let $f : A \looparrowright B, g : B \looparrowright C$ and $h : C \looparrowright D$. Then $h \bullet (g \bullet f) = (h \bullet g) \bullet f$.*

Proof. Let $a \in A$. For each $x \in D$ we have

$$\begin{aligned}(h \bullet (g \bullet f))_a(x) &= \sum_{c\in C}(g \bullet f)_a(c)h_c(x) = \sum_{c\in C}\Big(\sum_{b\in B} f_a(b)g_b(c)\Big)h_c(x) \\ &= \sum_{b\in B}\sum_{c\in C} f_a(b)g_b(c)h_c(x) = \sum_{b\in B} f_a(b)\Big(\sum_{c\in C} g_b(c)h_c(x)\Big) \\ &= \sum_{b\in B} f_a(b)(h \bullet g)_b(x) = ((h \bullet g) \bullet f)_a(x).\end{aligned}$$

□

4. Definition of probabilistic groupoids

Let $A \neq \emptyset$ and $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers. Then, for $n \in \mathbb{N}$, the n^{th} direct power of A is the set of ordered n-tuples $A^n = \{(a_1, a_2, \dots, a_n) | a_i \in A, i = 1, 2, \dots, n\}$. We take by definition $A^0 = \{\emptyset\}$.

Every probabilistic mapping $f : A^n \looparrowright A$, $n \in \mathbb{N} \cup \{0\}$, is said to be an *$n$-ary probabilistic operation on A*. The pair $(A, \mathcal{F})$ of a nonempty set A and a family $\mathcal{F}$ of probabilistic operations on A is called a *probabilistic algebra*. In the case when $\mathcal{F}$ consists of only one binary probabilistic operation $g : A \times A \looparrowright A$, we say that the probabilistic algebra is a *probabilistic groupoid*, denoted by (A, g), or just by A when g is known. We also use the notation $g_{a,b}$ for the probability distribution $g(a, b)$. If $g_{a,b}(c) = p$, then we say that the *probability the product of a and b to be c is p*.

The class of all "ordinary" groupoids can be considered as a subclass of the class of probabilistic groupoids. Namely, for $a \in A$, let $\epsilon_a \in \mathcal{D}_A$ be the probability distribution which is determined by

$$\epsilon_a(x) = \begin{cases} 1 : & x = a, \\ 0 : & x \neq a. \end{cases}$$

Denote by $\mathcal{D}_0$ the subset of $\mathcal{D}_A$ which consists of such functions, that is $\mathcal{D}_0 = \{\epsilon_a \in \mathcal{D}_A |\ a \in A\}$. Then an "ordinary" groupoid is the pair (A, g), where $g : A \times A \looparrowright \mathcal{D}_0$, under the identification $\epsilon_c \equiv c$.

For $A = \{a\}$ we have that $g : A \times A \to D_{\{a\}}$ is just $g_{a,a} = \epsilon_a$, so the probabilistic groupoid $(\{a\}, g)$ is in fact the (ordinary) trivial groupoid.

If $B \subseteq A$, we denote by $ext\mathcal{D}_B$ the subset of $\mathcal{D}_A$ determined by:

$$f \in ext\mathcal{D}_B \Leftrightarrow f(x) = 0 \text{ for every } x \in A \setminus B.$$

In the sequel we identify the distribution $ext\mathcal{D}_B$ on the set A and the distribution $\mathcal{D}_B$ on the set B. Clearly,

$$B_1 \subseteq B_2 \subseteq A \Rightarrow \mathcal{D}_{B_1} \subseteq \mathcal{D}_{B_2} \subseteq \mathcal{D}_A.$$

Unlike in the case of ordinary groupoids, for finite $|A| > 1$, there are infinitely many probabilistic groupoids. For instance, when $A = \{a, b\}$, one is given by

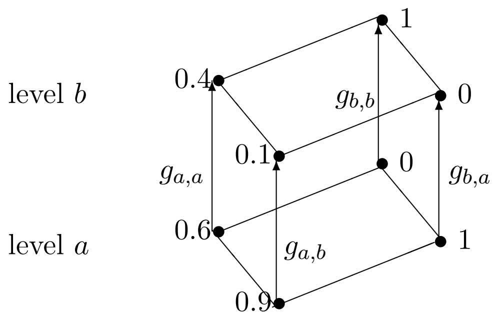
$$\begin{array}{c|cc} g & a & b \\ \hline a & g_{a,a} & g_{a,b} \\ b & g_{b,a} & g_{b,b} \end{array}, \quad \text{where}$$

$$g_{a,a} = \begin{pmatrix} a & b \\ 0.6 & 0.4 \end{pmatrix}, \qquad g_{b,a} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \epsilon_a,$$
$$g_{a,b} = \begin{pmatrix} a & b \\ 0.9 & 0.1 \end{pmatrix}, \qquad g_{b,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \epsilon_b.$$

This probabilistic groupoid can be presented in more convenient way by using only one table, as follows:

g	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,b}$
a	0.6	0.9	1	0
b	0.4	0.1	0	1

Finite probabilistic groupoids can be represented by "cubes" whose elements belong to $[0,1]$ and the sum of the elements along the vertical axes are equal to 1. The previous groupoid can be presented as follows.



5. Probabilistic subgroupoids

Let (A, g^A) and (B, g^B) be probabilistic groupoids, and $B \subseteq A$. If for every $a, b \in B$ we have that $g^B_{a,b} = g^A_{a,b}|_B$ ($g^A_{a,b}|_B$ is the restriction of $g^A_{a,b}$ on B, i.e., $g^B_{a,b} \in ext\mathcal{D}_B$), then we say that (B, g^B) is a *probabilistic subgroupoid* of (A, g^A).

Let (A, g) be a probabilistic groupoid and $B \subseteq A$. Then B is said to be a *closed subset* of A if $g_{a,b}(c) \neq 0$ implies $c \in B$, for every $a, b \in B$.

Theorem 5.1. *Let (A, g) be a probabilistic groupoid and $B \subseteq A$. Then B is a probabilistic subgroupoid of A if and only if B is a closed subset of A.*

Proof. Let B be a probabilistic subgroupoid of A, and $a, b \in B$ be arbitrary. Assume that there is $c \in A \backslash B$, such that $g_{a,b}(c) = p > 0$. Then

$$1 = \sum_{x \in A} g_{a,b}(x) = \sum_{x \in A \backslash B} g_{a,b}(x) + \sum_{x \in B} g_{a,b}(x) \geqslant p + \sum_{x \in B} g_{a,b}(x) = p + 1 > 1,$$

a contradiction.

If B is a closed subset of A then, for every $a, b \in B$, we have that

$$\sum_{x \in B} g_{a,b}(x) = 1,$$

since

$$\sum_{x \in A} g_{a,b}(x) = 1 \text{ and } x \notin B \text{ implies } g_{a,b}(x) = 0.$$

Hence, B is a probabilistic subgroupoid of A. □

6. Some classes of probabilistic groupoids

sectionSome classes of probabilistic groupoids Here we define several classes of probabilistic groupoids, corresponding to some classes of ordinary groupoids.

6.1 Probabilistic groupoids with units

Let (A, g) be a probabilistic grou-poid. An element $l \in A$ ($r \in A$) is said to be a *left* (*right*) unit if

$$(\forall x \in A)\ g_{l,x} = \epsilon_x \quad \Big((\forall x \in A)\ g_{x,r} = \epsilon_x\Big),$$

that is, the probability of the product of l and x to be x is 1 (the probability of the product of x and r to be x is 1), for every element $x \in A$. (Note that this implies $g_{l,x}(y) = 0$ ($g_{x,r}(y) = 0$), for each $y \neq x$.)

Let $a \in A$ be an arbitrary element, and consider the set

$$L_a = \{g_{a,x}(x) |\ x \in A\} \quad \Big(R_a = \{g_{x,a}(x) |\ x \in A\}\Big).$$

Let ${p_a}^L = inf\ L_a$ (${p_a}^R = inf\ R_a$). Then ${p_a}^L$ (${p_a}^R$) is called the *probability of the left* (*right*) *neutrality of a.* The following property is obvious.

Proposition 6.1. *An element l is a left unit* (*a right unit*) *if and only if the probability of its left neutrality* (*right neutrality*) *is one.*

Proposition 6.2. *Let (A,g) be a probabilistic groupoid and let $a \in A$. Then the probability ${p_b}^R$ $({p_b}^L)$ of the right neutrality (left neutrality) of an arbitrary element $b \in A$, $b \neq a$, does not exceed $1-{p_a}^L$ $(1-{p_a}^R)$.* Proof. *Let $a \in A$ be fixed element and let $b \neq a \in A$ be arbitrary element. Then we have:*

$$\begin{aligned}{p_b}^R &= inf\{g_{x,b}(x)|x \in A\} \leqslant g_{a,b}(a) = 1 - \sum_{\substack{x\in A\\ x\neq a}} g_{a,b}(x)\\ &\leqslant 1 - g_{a,b}(b) \leqslant 1 - inf\{g_{a,x}(x)|x \in A\} = 1 - {p_a}^L. \qquad \square\end{aligned}$$

As a consequence of Proposition 6.2, we obtain the following statement.

Corollary 6.3. *Let l (r) be a left unit (a right unit) of a probabilistic groupoid (A,g). Then the probability of the right neutrality (left neutrality) of any other element of A is 0.*

It is clear that a probabilistic groupoid does not have to possess a left unit, but if it has one, then it does not need to be a unique one; the same holds for the right units. However, like in the case of ordinary groupoids, a probabilistic groupoid can not have distinct left and right units.

Theorem 6.4. *Let (A,g) be a probabilistic groupoid and let l be its left unit and let r be its right unit. Then $l = r$.*

Proof. Assume that $l \neq r$. Since l is a left unit, we have that $g_{l,r}(r) = \epsilon_r(r) = 1$, and since r is a right unit, $g_{l,r}(l) = \epsilon_l(l) = 1$ also holds. But then

$$1 = \sum_{x\in A} g_{l,r}(x) \geqslant g_{l,r}(r) + g_{l,r}(l) = 2,$$

a contradiction. □

An element $e \in A$ which is both left and right unit is said to be a *unit* of a probabilistic groupoid (A,g).

Having in mind the Corollary 6.3, we have the following property.

Corollary 6.5. *Let e be the unit of a probabilistic groupoid (A,g). Then the probability of both left and right neutrality of any element of A which is distinct of e is 0.*

6.2 Idempotent probabilistic groupoids

Let (A, g) be a probabilistic groupoid and $a \in A$. Then the number $p = g_{a,a}(a)$ is called the *probability of the idempotence of* a. The element a is said to be *idempotent* if $p = 1$.

Proposition 6.6. *Let e be the unit of a probabilistic groupoid (A, g). Then e is an idempotent element.*

Let $I = \{g_{x,x}(x) | x \in A\}$ be the set of the probabilities of idempotence of the elements of (A, g). Then $p^I = inf I$ is called the *probability of the idempotence of the probabilistic groupoid* (A, g). Hence, the probability of the idempotence of any particular element is at least p^I. Probabilistic groupoid (A, g) is said to be *idempotent* if $p^I = 1$ (i.e., if all of its elements are idempotent ones).

6.3 Commutative probabilistic groupoids

Let $a, b \in A$, and for every $z \in A$ let $p^z_{a,b} = min\{g_{a,b}(z), g_{b,a}(z)\}$. Let

$$p_{a,b} = \sum_{z \in A} p^z_{a,b}.$$

Then we say that the elements a and b *commute with probability* $p_{a,b}$. The value of $p^{com} = inf\{p_{a,b} | a, b \in A\}$ is said to be the *probability of the commutativity of the probabilistic groupoid* (A, g). (A, g) is called a *commutative probabilistic groupoid* if all of its elements commute with probability one, that is if $p^{com} = 1$.

Theorem 6.7. *A probabilistic groupoid (A, g) is commutative if and only if*

$$(\forall a, b \in A) \ g_{a,b} = g_{b,a}.$$

Proof. Let (A, g) be commutative and $a, b \in A$. Then $p^{com} = 1$ implies

$$\sum_{z \in A} min\{g_{a,b}(z), g_{b,a}(z)\} = 1.$$

Let us assume that $g_{a,b} \neq g_{b,a}$. It means that $g_{a,b}(u) \neq g_{b,a}(u)$, for some $u \in A$. Without loss of generality we can take that $g_{a,b}(u) < g_{b,a}(u)$. Then we obtain

$$1=\sum_{z\in A} min\{g_{a,b}(z), g_{b,a}(z)\}=\sum_{\substack{z\in A\\ z\neq u}} min\{g_{a,b}(z), g_{b,a}(z)\}+min\{g_{a,b}(u), g_{b,a}(u)\}$$

$$\leqslant \sum_{\substack{z\in A\\ z\neq u}} g_{b,a}(z) + g_{a,b}(u) < \sum_{\substack{z\in A\\ z\neq u}} g_{b,a}(z) + g_{b,a}(u) = \sum_{z\in A} g_{b,a}(z) = 1,$$

a contradiction.

Let $g_{a,b} = g_{b,a}$, for all $a, b \in A$. Hence, $g_{a,b}(z) = g_{b,a}(z)$, for every $z \in A$. Then

$$p_{a,b} = \sum_{z\in A} p^z_{a,b} = \sum_{z\in A} min\{g_{a,b}(z), g_{b,a}(z)\} = \sum_{z\in A} g_{a,b}(z) = 1.$$

By $p_{a,b} = 1$ for all $a, b \in A$, we get $p^{com} = inf\{p_{a,b}|a, b \in A\} = 1$, that is, (A, g) is a commutative probabilistic groupoid. □

6.4 Composite products of probabilistic groupoids

Given a set $A = \{a_1, a_2, \ldots, a_n\}$, we define inductively *terms* over the set A as follows. Each element $x \in A$ is a term of length 1, the terms of length 2 are (xy), where $x, y \in A$, and if T_1 and T_2 are already defined terms of lengths l_1 and l_2, then (T_1T_2) is a term of length $l_1 + l_2$. For instance, given $x, y, z, t \in A$, $x(yz), (xy)z$ are terms of length 3 (and also $z(tz), (tz)y, \ldots$), terms of length 4 are $t(x(yz)), t((xy)z), (x(yz))t, ((xy)z)t, (xy)(zt)$ (and also $t(x(xx)), y((xt)t), (t(yz))x, \ldots$). (Here, we avoided the non-necessary outside brackets.)

For a probabilistic groupoid (A, g), to each term T over the set A of length at least 2, we associate a probability distribution g_T in an inductive way as follows. To each term ab, $a, b \in A$, of length 2 we associate the probability distribution $g_{a,b}$ (the product of a and b in the probabilistic groupoid (A, g)). To the terms $T = T_1T_2$ of length $l \geqslant 3$ we associate inductively a probability distribution $g_T = g_{T_1,T_2}$ over A as follows.

(1) If $T_1 \in A$ then $g_{T_1,T_2}(z) = \sum_{u\in A} g_{T_1,u}(z)g_{T_2}(u)$.

(2) If $T_2 \in A$ then $g_{T_1,T_2}(z) = \sum_{u\in A} g_{T_1}(u)g_{u,T_2}(z)$.

(3) If $T_1, T_2 \notin A$ then $g_{T_1,T_2}(z) = \sum_{u\in A} g_{T_1,u}(z)g_{T_2}(u)$

$$= \sum_{u\in A} \big(\sum_{v\in A} g_{T_1}(v) g_{v,u}(z) \big) g_{T_2}(u).$$

Note that $g_{T_1,u}$ and g_{u,T_2} are probability distributions and that, by the inductive hypothesis, when T_1 (or T_2) is of length $\geqslant 2$, the probability distribution g_{T_1} (or g_{T_2}) is defined.

Theorem 6.8. *Let (A, g) be a probabilistic groupoid and let T be a term of length at least 2. Then g_T is a probability distribution on A.*

Proof. The claim is trivial when the length of T is 2. Let T be of length at least 3, i.e., $T = T_1T_2$. We use an induction of the length of the terms.

By the definition of g_T we have to consider three cases.

(1) Let $T_1 \in A$. Then we have

$$\sum_{z\in A} g_{T_1,T_2}(z) = \sum_{z\in A}\sum_{u\in A} g_{T_1,u}(z) g_{T_2}(u) = $$
$$= \sum_{u\in A} g_{T_2}(u) \sum_{z\in A} g_{T_1,u}(z) = \sum_{u\in A} g_{T_2}(u)\cdot 1 = 1.$$

(2) The case $T_2 \in A$ follows the steps of the case (1).

(3) Let $T_1, T_2 \notin A$. Then we have

$$\sum_{z\in A} g_{T_1,T_2}(z) = \sum_{z\in A} \big(\sum_{u\in A} g_{T_1,u}(z) g_{T_2}(u) \big)$$
$$= \sum_{u\in A} g_{T_2}(u) \big(\sum_{z\in A} g_{T_1,u}(z) \big) = \text{(by case (2), since } u\in A)$$
$$= \sum_{u\in A} g_{T_2}(u)\cdot 1 = 1.$$

□

Example 6.9. Let (A, g), where $A = \{a, b\}$, be a probabilistic groupoid given by the table

g	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,b}$
a	0.3	0.8	1	0.4
b	0.7	0.2	0	0.6

.

We have $g_{a,(a,a)} = \begin{pmatrix} a & b \\ 0.65 & 0.35 \end{pmatrix}$, since $g_{a,(a,a)}(z) = \sum_{u\in A} g_{a,u}(z) g_{a,a}(u)$ and then $g_{a,(a,a)}(a) = \sum_{u\in A} g_{a,u}(a) g_{a,a}(u) = 0.3\cdot 0.3 + 0.8\cdot 0.7 = 0.65$, $g_{a,(a,a)}(b) = \sum_{u\in A} g_{a,u}(b) g_{a,a}(u) = 0.7\cdot 0.3 + 0.2\cdot 0.7 = 0.35$.

One can also compute that $g_{(a,a),a} = \begin{pmatrix} a & b \\ 0.79 & 0.21 \end{pmatrix}$, $g_{(b,a),(a,b)} = \begin{pmatrix} a & b \\ 0.4 & 0.6 \end{pmatrix}$, and so on.

6.5 Associative probabilistic groupoids

Consider a probabilistic grou-poid (A, g). Let $a, b, c \in A$ and let $p^z_{a,b,c} = min\{g_{(a,b),c}(z), g_{a,(b,c)}(z)\}$, where

$$g_{(a,b),c}(z) = \sum_{u \in A} g_{(a,b)}(u) g_{u,c}(z), \quad g_{a,(b,c)}(z) = \sum_{u \in A} g_{a,u}(z) g_{(b,c)}(u).$$

Define

$$p_{a,b,c} = \sum_{z \in A} p^z_{a,b,c}$$

to be the *probability of the associativity of the elements a,b and c*, while the probability $p^{ass} = inf\{p_{a,b,c} | a, b, c \in A\}$ is referred to be the *probability of the associativity of the probabilistic groupoid* (A, g). A probabilistic groupoid is said to be *associative* (or a *probabilistic semigroup*) if $p^{ass} = 1$.

We prove the following statement in the same manner as Theorem 6.7.

Theorem 6.10. *A probabilistic groupoid* (A, g) *is associative if and only if*

$$(\forall a, b, c \in A) g_{a,(b,c)} = g_{(a,b),c}.$$

Proof. Let (A, g) be associative probabilistic groupoid, and assume that $g_{a,(b,c)} \neq g_{(a,b),c}$ for some $a, b, c \in A$. Consequently, there is a $u \in A$ such that $g_{a,(b,c)}(u) < g_{(a,b),c}(u)$ (the assumption $g_{a,(b,c)}(u) > g_{(a,b),c}(u)$ would cause negligible changes of the proof). Since $1 = p^{ass} = inf\{p_{x,y,z} | x, y, z \in A\}$, we obtain that $p_{a,b,c} = 1$. Then we have:

$$\begin{aligned} 1 = p_{a,b,c} &= \sum_{z \in A} p^z_{a,b,c} = \sum_{z \in A} min\{g_{(a,b),c}(z), g_{a,(b,c)}(z)\} \\ &= \sum_{z \neq u} min\{g_{(a,b),c}(z), g_{a,(b,c)}(z)\} + min\{g_{(a,b),c}(u), g_{a,(b,c)}(u)\} \\ &\leqslant \sum_{z \neq u} g_{(a,b),c}(z) + min\{g_{(a,b),c}(u), g_{a,(b,c)}(u)\} \\ &< \sum_{z \neq u} g_{(a,b),c}(z) + g_{(a,b),c}(u) = \sum_{z \in A} g_{(a,b),c}(z) = 1, \end{aligned}$$

a contradiction. Hence, $g_{a,(b,c)} = g_{(a,b),c}$ for all $a, b, c \in A$.

On the other hand, if $(\forall a,b,c \in A) g_{a,(b,c)} = g_{(a,b),c}$ holds in a probabilistic groupoid (A,g), then $p^z_{a,b,c} = g_{(a,b),c}(z) = g_{a,(b,c)}(z)$, for all $a,b,c \in A$, and every $z \in A$. Therefore, $\sum_{z\in A} p^z_{a,b,c} = \sum_{z\in A} p_{a,(b,c)}(z) = 1$, that is $p_{a,b,c} = 1$, for every $a,b,c \in A$. This implies $p^{ass} = inf\{p_{a,b,c} | a,b,c \in A\} = 1$, which means that (A,g) is an associative probabilistic groupoid. □

Example 6.11. We will find all probabilistic semigroups of order 2. Let $A = \{a,b\}$ and

g	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,b}$
a	α_1	α_2	α_3	α_4
b	β_1	β_2	β_3	β_4

,

where $\alpha_i \geqslant 0,\ \beta_i \geqslant 0,\ \alpha_i + \beta_i = 1$. Since we want the associativity to be satisfied, i.e., $g_{(a,a),a}(z) = g_{a,(a,a)}(z),\ g_{(a,a),b}(z) = g_{a,(a,b)}(z),\ g_{(a,b),a}(z) = g_{a,(b,a)}(z), \ldots \ \ldots, g_{(b,b),b}(z) = g_{b,(b,b)}(z)$, for $z \in \{a,b\}$, we obtain the following equations with unknowns α_i and β_i:

$$
\begin{array}{ll}
\alpha_1\alpha_1 + \beta_1\alpha_3 = \alpha_1\alpha_1 + \alpha_2\beta_1, & \alpha_1\beta_1 + \beta_1\beta_3 = \beta_1\alpha_1 + \beta_2\beta_1, \\
\alpha_1\alpha_2 + \beta_1\alpha_4 = \alpha_1\alpha_2 + \alpha_2\beta_2, & \alpha_1\beta_2 + \beta_1\beta_4 = \beta_1\alpha_2 + \beta_2\beta_2, \\
\alpha_2\alpha_1 + \beta_2\alpha_3 = \alpha_1\alpha_3 + \alpha_2\beta_3, & \alpha_2\beta_1 + \beta_2\beta_3 = \beta_1\alpha_3 + \beta_2\beta_3, \\
\alpha_2\alpha_2 + \beta_2\alpha_4 = \alpha_1\alpha_4 + \alpha_2\beta_4, & \alpha_2\beta_2 + \beta_2\beta_4 = \beta_1\alpha_4 + \beta_2\beta_4, \\
\alpha_3\alpha_1 + \beta_3\alpha_3 = \alpha_3\alpha_1 + \alpha_4\beta_1, & \alpha_3\beta_1 + \beta_3\beta_3 = \beta_3\alpha_1 + \beta_4\beta_1, \\
\alpha_3\alpha_2 + \beta_3\alpha_4 = \alpha_3\alpha_2 + \alpha_4\beta_2, & \alpha_3\beta_2 + \beta_3\beta_4 = \beta_3\alpha_2 + \beta_4\beta_2, \\
\alpha_4\alpha_1 + \beta_4\alpha_3 = \alpha_3\alpha_3 + \alpha_4\beta_3, & \alpha_4\beta_1 + \beta_4\beta_3 = \beta_3\alpha_3 + \beta_4\beta_3, \\
\alpha_4\alpha_2 + \beta_4\alpha_4 = \alpha_3\alpha_4 + \alpha_4\beta_4, & \alpha_4\beta_2 + \beta_4\beta_4 = \beta_3\alpha_4 + \beta_4\beta_4.
\end{array}
$$

After simplification of the above equalities, two cases remain to be considered.

Case 1: $\alpha_4 \neq 0$ or $\beta_1 \neq 0$. Then we have $\alpha_2 = \alpha_3$ and $\beta_2 = \beta_3$, and the above system reduces to

$$
\begin{aligned}
&\beta_1\alpha_4 = \alpha_2\beta_2, \\
&\alpha_1\beta_2 + \beta_1\beta_4 = \beta_1\alpha_2 + \beta_2\beta_2, \\
&\alpha_2\alpha_2 + \beta_2\alpha_4 = \alpha_1\alpha_4 + \alpha_2\beta_4.
\end{aligned}
$$

After replacing β_i by $1 - \alpha_i$ we get that the last system reduces to one equation

$$\alpha_4(1 - \alpha_1) = \alpha_2(1 - \alpha_2).$$

It follows that in the case $\alpha_1 \neq 1$ we can choose arbitrary value for $\alpha_1 \in [0,1)$ and then we have the solution

g	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,b}$
a	α_1	α_2	α_2	$\alpha_2\frac{1-\alpha_2}{1-\alpha_1}$
b	$1-\alpha_1$	$1-\alpha_2$	$1-\alpha_2$	$1-\alpha_2\frac{1-\alpha_2}{1-\alpha_1}$

,

for any α_2 such that $0 \leqslant \alpha_2\frac{1-\alpha_2}{1-\alpha_1} \leqslant 1$. In the case $\alpha_4 \neq 0$ we can choose arbitrary value for $\alpha_4 \in (0,1]$ and then we have the solution

g	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,b}$
a	$1-\alpha_2\frac{1-\alpha_2}{\alpha_4}$	α_2	α_2	α_4
b	$\alpha_2\frac{1-\alpha_2}{\alpha_4}$	$1-\alpha_2$	$1-\alpha_2$	$1-\alpha_4$

,

for any α_2 such that $0 \leqslant \alpha_2\frac{1-\alpha_2}{\alpha_4} \leqslant 1$.

We notice that in this case all probabilistic semigroups are commutative, since $g_{a,b} = g_{b,a}$.

Case 2: $\alpha_4 = 0$ and $\beta_1 = 0$. Then $\alpha_1 = 1$ and $\beta_4 = 1$ and the starting system of equations reduces to

$$\begin{array}{llll}\alpha_2+\beta_2\alpha_3=\alpha_3+\alpha_2\beta_3, & \alpha_2\beta_2=0, & \beta_2\beta_2=\beta_2, & \alpha_2\alpha_2=\alpha_2,\\ \alpha_3\beta_2+\beta_3=\beta_2\alpha_2+\beta_2, & \alpha_3\beta_3=0, & \alpha_3\alpha_3=\alpha_3, & \beta_3\beta_3=\beta_3.\end{array}$$

There are only three solutions in this case:

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in \{(1,0,0,0),(1,0,1,0),\ (1,1,1,0)\},$$

and only for $(\alpha_1,\alpha_2,\ \alpha_3,\alpha_4) = (1,0,1,0)$ we have non-commutative (ordinary) semigroup.

6.6 Probabilistic quasigroups

An ordinary groupoid $(Q,\cdot)$ is said to be a quasigroup if

$$(\forall a,b \in Q)(\exists! x,y \in Q)(ax = b \ \&\ ya = b).$$

We say that a probabilistic groupoid (Q,g) is a *probabilistic quasigroup with probability* p (or a p-*quasigroup*) if

$$(\forall a,b \in Q)(\exists x,y \in Q)(g_{a,x}(b) \geqslant p \ \&\ g_{y,a}(b) \geqslant p).$$

Note that for $0 \leqslant q < p \leqslant 1$, every $p-quasigroup$ is a $q-quasigroup$ as well. It is also clear that every probabilistic groupoid is a 0-quasigroup.

In the case of p-quasigroups, depending of the value of p, for some $a, b \in Q$ may exist several $x, y \in Q$ such that $g_{a,x}(b) \geqslant p$ and/or $g_{y,a}(b) \geqslant p$. Since in any distribution $g_{\alpha,\beta}$, when $p > 1/2$, may exist (if any) a unique element b such that $g_{\alpha,\beta}(b) = p$, we have the following.

Proposition 6.12. *If $p > 1/2$, then for any finite p-quasigroup it is true that*

$$(\forall a, b \in Q)(\exists! x, y \in Q)(g_{a,x}(b) \geqslant p \ \&\ g_{y,a}(b) \geqslant p).$$

Proof. The proof follows by the Pigeonhole Principal. Let $Q = \{q_1, \ldots, q_n\}$ be a p-quasigroup and $p > 1/2$. If $g_{a,x_1}(b) \geqslant p$ and $g_{a,x_2}(b) \geqslant p$ for some $a, b, x_1 \neq x_2 \in Q$, then we have for each of the rest $n-1$ elements $c \in Q \setminus \{b\}$ to find some $x \in Q \setminus \{x_1, x_2\}$ such that $g_{a,x}(c) \geqslant p$. □

Corollary 6.13. *1-quasigroups are ordinary quasigroups.*

A probabilistic groupoid (A, g) is said to be with *left* (*right*) *cancellation* if for every $a, b, c \in A$ we have

$$g_{a,b} = g_{a,c} \Rightarrow b = c \quad (g_{a,b} = g_{c,b} \Rightarrow a = c).$$

A probabilistic groupoid is said to be *cancellative* if it is with left and right cancellation.

Proposition 6.14. *If $p > 1/2$, then a p-quasigroup is a cancellative probabilistic groupoid.*

Proof. Let $p > 1/2$ and let (Q, g) be a p-quasigroup. If $g_{a,x} = g_{a,y}$, then for the distribution $g_{a,x}$ there is a unique $b \in Q$ such that $g_{a,x}(b) = g_{a,y}(b) \geqslant p$. Now, by Proposition 6.12, we have $x = y$. □

Example 6.15. A 0.5-quasigroup (Q, g), where $Q = \{1, 2, 3, 4\}$, is presented by the distributions given in Table 1. We can see there that $g_{2,1}(2) \geqslant 0.5$, $g_{1,4}(2) \geqslant 0.5$, $g_{4,4}(2) \geqslant 0.5$, etc.

6.7 Inverse elements

Let (A, g) be a probabilistic groupoid which possess a unit e, and let $a, b \in A$. If $g_{a,b}(e) = p$, then we say that a is a *left inverse of b with probability p* and that b is a *right inverse of a with probability p*. It is obvious that left/right p-inverses of an element do not have to exist, but if so, then there might be more than one. If an element a is both left p-inverse and right

	$g_{1,1}$	$g_{1,2}$	$g_{1,3}$	$g_{1,4}$	$g_{2,1}$	$g_{2,2}$	$g_{2,3}$	$g_{2,4}$
1	0	0.7	0.5	0.1	0	0.3	0.5	0.04
2	0.5	0.1	0	0.6	0.5	0.1	0.5	0.36
3	0.5	0.2	0	0	0.4	0.1	0	0.5
4	0	0	0.5	0.3	0.1	0.5	0	0.1

	$g_{3,1}$	$g_{3,2}$	$g_{3,3}$	$g_{3,4}$	$g_{4,1}$	$g_{4,2}$	$g_{4,3}$	$g_{4,4}$
1	0.5	0.2	0	0.1	0.1	0	0.2	0.5
2	0.4	0.55	0	0.1	0.4	0	0.5	0.5
3	0.1	0.25	0.5	0.3	0	0.5	0.13	0
4	0	0	0.5	0.5	0.5	0.5	0.17	0

Table 1: A probabilistic 0.5-quasigroup of order 4.

p-inverse of an element b, then the elements a and b are referred as *mutually p-inverse* or *p-inverse to each other.*

If e is a unit of (A, g), then $g_{a,e}(e) = \epsilon_a(e) = \begin{cases} 1 : e = a, \\ 0 : e \neq a. \end{cases}$ Hence, the only left p-inverse of e is e itself, and it can be only a 1-inverse as well. So, the next property holds.

Proposition 6.16. *Let e be the unit of a probabilistic groupoid (A, g). Then e is left and right 1-inverse element to itself.*

Further on, instead of a 1-inverse element, we will say simply an inverse element.

We will prove that an inverse element in a probabilistic semigroup is unique.

Theorem 6.17. *Let the element a of a probabilistic semigroup $A = (A, g)$ have left inverse b and right inverse c. Then $b = c$.*

Proof. Given that b is a left inverse and c is a right inverse of a, we will prove that $\epsilon_b = \epsilon_c$, that implies $b = c$. Denote by e the unit of A. We have

$$g_{b,(a,c)}(z) = \sum_{u \in A} g_{b,u}(z) g_{a,c}(u) = g_{b,e}(z) \cdot 1 = g_{b,e}(z) = \epsilon_b(z),$$

since $g_{a,c}(e) = 1$ and $g_{a,c}(u) = 0$ when $u \neq e$. In the same way

$$g_{(b,a),c}(z) = \sum_{u \in A} g_{b,a}(u) g_{u,c}(z) = 1 \cdot g_{e,c}(z) = g_{e,c}(z) = \epsilon_c(z).$$

Now, $g_{b,(a,c)}(z) = g_{(b,a),c}(z)$ implies $\epsilon_b(z) = \epsilon_c(z)$ for every $z \in A$, i.e., $\epsilon_b = \epsilon_c$. □

The unique left and right inverse of an element $a \in A$ is called an inverse of a and is denoted by a^{-1}.

Proposition 6.18. *Let (A, g) be a probabilistic semigroup with unit e and let an element $b \in A$ has a left (right) inverse. Then for every $c, d \in A$ we have*

$$g_{b,c} = g_{b,d} \implies c = d \qquad (g_{c,b} = g_{d,b} \implies c = d).$$

Proof. Assume that a is a left inverse of b and $g_{b,c} = g_{b,d}$. Then $g_{a,(b,c)}(z) = \sum\limits_{u \in A} g_{a,u}(z) g_{b,c}(u) = \sum\limits_{u \in A} g_{a,u}(z) g_{b,d}(u) = g_{a,(b,d)}(z)$, and by associativity we have $g_{(a,b),c}(z) = g_{(a,b),d}(z)$. So, $\sum\limits_{u \in A} g_{a,b}(u) g_{u,c}(z) = \sum\limits_{u \in A} g_{a,b}(u) g_{u,d}(z)$ and, since $g_{a,b}(u) = 0$ when $u \neq e$, we obtain $g_{e,c}(z) = g_{e,d}(z)$. This means that $\epsilon_c = \epsilon_d$, i.e., $c = d$. □

As a corollary of Proposition 6.18 we have the following.

Theorem 6.19. *If each element of a probabilistic semigroup has inverse, then the semigroup is cancellative.*

The next simple lemma will be used in the next section.

Lemma 6.20. *If a and b are mutually inverse elements in a probabilistic groupoid (A, g) with unit e, then for each $c \in A$ we have $g_{c,(a,b)} = g_{c,e} = \epsilon_c$ and $g_{(a,b),c} = g_{e,c} = \epsilon_c$.*

Proof. We have $g_{c,(a,b)}(z) = \sum\limits_{u \in A} g_{c,u}(z) g_{a,b}(u) = g_{c,e}(z) g_{a,b}(e) = g_{c,e}(z) = \epsilon_c(z)$, since $g_{a,b}(u) = 0$ when $u \neq e$. □

7. Probabilistic groups

A probabilistic semigroup is said to be a *p-probabilistic group* if it has a unit and each element has a p-inverse. In what follows we will consider several examples in order to support our opinion that there are not finite essential p-groups. In fact, we found (without proofs) that there are no finite p-groups when $p < 1$, and that for $p = 1$ the probabilistic 1-groups are ordinary groups. Further on, we will say a probabilistic group instead of a probabilistic 1-group.

Example 7.1. We are asking for all p-groups on the set $\{e,a,b\}$, where $0<p<1$, e is the unit and b is a p-inverse of a. We have the distributions

	$g_{e,e}$	$g_{a,e}=g_{e,a}$	$g_{b,e}=g_{e,b}$	$g_{a,b}$	$g_{b,a}$	$g_{a,a}$	$g_{b,b}$
e	1	0	0	p	p	γ_1	γ_2
a	0	1	0	α_1	α_2	α_3	α_4
b	0	0	1	β_1	β_2	β_3	β_4

for some $\alpha_i,\beta_i,\gamma_i\in[0,1]$, $p+\alpha_1+\beta_1=1$, $p+\alpha_2+\beta_2=1, \gamma_1+\alpha_3+\beta_3=1, \gamma_2+\alpha_4+\beta_4=1$.

By the associativity, the following 8 equations have to be satisfied for $z\in\{e,a,b\}$: $g_{a,(a,a)}(z)=g_{(a,a),a}(z)$, $g_{a,(a,b)}(z)=g_{(a,a),b}(z),\dots,$ $g_{b,(b,b)}(z)=g_{(b,b),b}(z)$. We can infer several equations in unknowns $\alpha_i,\beta_i,\gamma_i$.

From $g_{a,(a,a)}(z)=g_{(a,a),a}(z)$, for $z=a$ we have

$$(\alpha_1-\alpha_2)\beta_3=0, \tag{1}$$

and for $z=b$ we have

$$(\beta_1-\beta_2)\beta_3=0. \tag{2}$$

From $g_{a,(a,b)}(z)=g_{(a,a),b}(z)$, for $z=e$ we have

$$\gamma_1\alpha_1+p\beta_1=p\alpha_3+\beta_3\gamma_2, \tag{3}$$

for $z=a$ we have

$$p+\alpha_1\beta_1=\beta_3\alpha_4, \tag{4}$$

and for $z=b$ we have

$$\beta_1\alpha_1+\beta_1\beta_1=\gamma_1+\alpha_3\beta_1+\beta_3\beta_4. \tag{5}$$

From $g_{a,(b,b)}(z)=g_{(a,b),b}(z)$, for $z=e$ we have

$$\gamma_1\alpha_4+p\beta_4=p\alpha_1+\beta_1\gamma_2, \tag{6}$$

and for $z=a$ we have

$$\alpha_4+\alpha_3\alpha_4+\alpha_1\beta_4=\alpha_1\alpha_1+\beta_1\alpha_4. \tag{7}$$

From $g_{b,(a,a)}(z) = g_{(b,a),a}(z)$, for $z = e$ we have

$$\gamma_2\beta_3 + p\alpha_3 = p\beta_2 + \alpha_2\gamma_1, \tag{8}$$

and for $z = b$ we have

$$\beta_3 + \beta_2\alpha_3 + \beta_3\beta_4 = \alpha_2\beta_3 + \beta_2\beta_2. \tag{9}$$

Finally, from $g_{b,(b,a)}(z) = g_{(b,b),a}(z)$, for $z = e$ we have

$$p\alpha_2 + \gamma_2\beta_2 = \alpha_4\gamma_1 + p\beta_4, \tag{10}$$

and for $z = a$ we have

$$\alpha_2\alpha_2 + \alpha_4\beta_2 = \gamma_2 + \alpha_4\alpha_3 + \beta_4\alpha_2. \tag{11}$$

The equation (4), since $0 < p < 1$, implies $\alpha_4 > 0, \beta_3 > 0$, and then by (1) and (2) we conclude that $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$. Now, from (5) and (11) we have

$$\gamma_1 = \beta\alpha + \beta\beta - \alpha_3\beta - \beta_3\beta_4 \tag{12}$$

and

$$\gamma_2 = \alpha\alpha + \alpha_4\beta - \alpha_4\alpha_3 - \beta_4\alpha. \tag{13}$$

We replace γ_1 and γ_2 in (3) and we obtain the equation

$$\beta\alpha\alpha + \beta\beta\alpha - \alpha_3\beta\alpha - \beta_3\beta_4\alpha + p\beta = p\alpha_3 + \alpha\alpha\beta_3 + \alpha_4\beta\beta_3 - \alpha_4\alpha_3\beta_3 - \beta_4\alpha\beta_3. \tag{14}$$

After replacing $\alpha_4\beta_3$ by $p + \alpha\beta$ (according (4)) and after simplifying, we obtain the equation $\beta\alpha\alpha = \alpha\alpha\beta_3$. The last equation implies $\alpha = 0$ or $\beta = \beta_3$. We have to consider three cases.

Case $\alpha = 0$ *and* $\beta = \beta_3$.

We replace $\alpha = 0$ and $\beta = \beta_3$ in the equation (9) and we get $\beta + \beta\alpha_3 + \beta\beta_4 = \beta\beta$. Since $\beta = \beta_3 > 0$, it follows that $1 + \alpha_3 + \beta_4 = \beta$, i.e. $\beta = 1$. This is a contradiction with $p + \alpha + \beta = 1,\ p > 0$.

Case $\alpha = 0$ *and* $\beta \neq \beta_3$.

We replace $\alpha = 0$ in the equation (7) and we get $\alpha_4 + \alpha_3\alpha_4 = \beta\alpha_4$. Since $\alpha_4 > 0$, it follows that $1 + \alpha_3 = \beta$, that leads to a contradiction again.

Case $\alpha > 0$ *and* $\beta = \beta_3$.

We replace $\beta = \beta_3$ in the equation (9) and we get $\beta + \beta\alpha_3 + \beta\beta_4 = \alpha\beta + \beta\beta$. Since $\beta = \beta_3 > 0$, it follows that $1 + \alpha_3 + \beta_4 = \alpha + \beta$, implying $\alpha + \beta = 1$. This is a contradiction with $p + \alpha + \beta = 1,\ 0 < p < 1$.

The obtained contradictions shows that there are no probabilistic p-groups on the set $\{e, a, b\}$, where $0 < p < 1$, e is the unit and b is a p-inverse of a. In a similar way one can show that there are no probabilistic p-groups on the set $\{e, a, b\}$, where $0 < p < 1$, e is the unit and a (b) is a p-inverse of a (b).

Example 7.2. Let (A, g), where $A = \{e, a, b\}$, be a probabilistic group with unit e. Let us first assume that $a^{-1} = a$, and then $b^{-1} = b$. Then we have the distributions

	$g_{e,e}, g_{a,a}, g_{b,b}$	$g_{a,e}, g_{e,a}$	$g_{b,e}, g_{e,b}$	$g_{a,b}$	$g_{b,a}$
e	1	0	0	α	α_1
a	0	1	0	β	β_1
b	0	0	1	γ	γ_1

for some $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1 \in [0, 1]$, $\alpha + \beta + \gamma = \alpha_1 + \beta_1 + \gamma_1 = 1$.

By associativity we have $g_{(a,a),b} = g_{a,(a,b)}$, where (according to Lemma 6.20) $g_{(a,a),b}(b) = \epsilon_b(b) = 1$, and $g_{a,(a,b)}(b) = g_{a,e}(b)g_{a,b}(e) + g_{a,a}(b)g_{a,b}(a) + g_{a,b}(b)g_{a,b}(b) = \gamma\gamma$.

So we get the equation $\gamma\gamma = 1$, i.e., $\gamma = 1$. This means that $g_{e,b} = g_{a,b}$, i.e., $e = a$. The obtained contradiction implies that $a \neq a^{-1}$.

Now, let $a^{-1} = b$. Then, by Example 7.1, for $p = 1$ we have $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_4 = \gamma_1 = \gamma_2 = 0$ and $\alpha_4 = \beta_3 = 1$. Hence, this probability group is in fact the cyclic group

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

.

Example 7.3. Let (A, g), where $A = \{e, a, b, c\}$, be a probabilistic group with unit e. We have to consider two cases, case *I* and case *II*.

I. Let first assume that $a^{-1} = a$, $b^{-1} = b$, $c^{-1} = c$. Then we have the following distributions, presented in more compact way,

	$g_{e,e}, g_{a,a}$ $g_{b,b}, g_{c,c}$	$g_{a,e}$ $g_{e,a}$	$g_{b,e}$ $g_{e,b}$	$g_{c,e}$ $g_{e,c}$	$g_{a,b}$	$g_{a,c}$	$g_{b,a}$	$g_{b,c}$	$g_{c,a}$	$g_{c,b}$
e	1	0	0	0	α_1	α_2	α_3	α_4	α_5	α_6
a	0	1	0	0	β_1	β_2	β_3	β_4	β_5	β_6
b	0	0	1	0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
c	0	0	0	1	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geqslant 0,\quad \alpha_i+\beta_i+\gamma_i+\delta_i=1$, for $i=1,2,\dots,6$.

By associativity we have the following equalities.

Case $g_{a,(a,b)} = g_{(a,a),b}$. By Lemma 6.20 we have $g_{(a,a),b}(z) = g_{e,b}(z) = \epsilon_b(z) = \begin{pmatrix} e & a & b & c \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and we compute the distribution $g_{a,(a,b)}$.

$g_{a,(a,b)}(z) = g_{a,e}(z)g_{a,b}(e)+g_{a,a}(z)g_{a,b}(a)+g_{a,b}(z)g_{a,b}(b)+g_{a,c}(z)g_{a,b}(c) =$

$$= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\alpha_1 + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\beta_1 + \begin{pmatrix}\alpha_1\\\beta_1\\\gamma_1\\\delta_1\end{pmatrix}\gamma_1 + \begin{pmatrix}\alpha_2\\\beta_2\\\gamma_2\\\delta_2\end{pmatrix}\delta_1 = \begin{pmatrix}\beta_1+\alpha_1\gamma_1+\alpha_2\delta_1\\ \alpha_1+\beta_1\gamma_1+\beta_2\delta_1\\ \gamma_1\gamma_1+\gamma_2\delta_1\\ \delta_1\gamma_1+\delta_2\delta_1\end{pmatrix}.$$

Hence, we have the following system of equations

$$\begin{cases} \beta_1+\alpha_1\gamma_1+\alpha_2\delta_1 &= 0,\\ \alpha_1+\beta_1\gamma_1+\beta_2\delta_1 &= 0,\\ \gamma_1\gamma_1+\gamma_2\delta_1 &= 1,\\ \delta_1\gamma_1+\delta_2\delta_1 &= 0, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \alpha_1=\beta_1 &= 0,\\ \alpha_1\gamma_1=\beta_1\gamma_1=\delta_1\gamma_1 &= 0,\\ \alpha_2\delta_1=\beta_2\delta_1=\delta_2\delta_1 &= 0,\\ \gamma_1\gamma_1+\gamma_2\delta_1 &= 1. \end{cases}$$

We consider two possibilities.

$\gamma_1 \neq 0$. Then we have $\alpha_1=\beta_1=\delta_1=0,\ \gamma_1=1$, and this implies $g_{a,b}=g_{e,b}$. After cancellation we get the contradiction $a=e$.

$\gamma_1 = 0$. Then, from $\gamma_2\delta_1 = 1$ we have $\gamma_2=1,\ \delta_1=1$. Hence, we have $\alpha_1=\beta_1=\gamma_1=0,\ \delta_1=1$, and this implies $\mathbf{g_{a,b}}=\mathbf{g_{e,c}}$, and also $\alpha_2=\beta_2=\delta_2=0,\ \gamma_2=1$, implying $\mathbf{g_{a,c}}=\mathbf{g_{e,b}}$.

Case $g_{b,(b,c)} = g_{(b,b),c}$. By Lemma 6.20 we have $g_{(b,b),c}(z) = g_{e,c}(z) = \epsilon_c(z) = \begin{pmatrix} e & a & b & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and we compute the distribution $g_{b,(b,c)}$.

$g_{b,(b,c)}(z) = g_{b,e}(z)g_{b,c}(e)+g_{b,a}(z)g_{b,c}(a)+g_{b,b}(z)g_{b,c}(b)+g_{b,c}(z)g_{b,c}(c) =$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\alpha_4 + \begin{pmatrix}\alpha_3\\\beta_3\\\gamma_3\\\delta_3\end{pmatrix}\beta_4 + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\gamma_4 + \begin{pmatrix}\alpha_4\\\beta_4\\\gamma_4\\\delta_4\end{pmatrix}\delta_4 = \begin{pmatrix}\alpha_3\beta_4+\gamma_4+\alpha_4\delta_4\\ \beta_3\beta_4+\beta_4\delta_4\\ \alpha_4+\gamma_3\beta_4+\gamma_4\delta_4\\ \delta_3\beta_4+\delta_4\delta_4\end{pmatrix}.$$

Hence, we have the following system of equations

$$\begin{cases} \alpha_3\beta_4+\gamma_4+\alpha_4\delta_4 &= 0,\\ \beta_3\beta_4+\beta_4\delta_4 &= 0,\\ \alpha_4+\gamma_3\beta_4+\gamma_4\delta_4 &= 0,\\ \delta_3\beta_4+\delta_4\delta_4 &= 1, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \alpha_4=\gamma_4 &= 0,\\ \alpha_3\beta_4=\beta_3\beta_4=\gamma_3\beta_4 &= 0,\\ \alpha_4\delta_4=\beta_4\delta_4=\gamma_4\delta_4 &= 0,\\ \delta_3\beta_4+\delta_4\delta_4 &= 1. \end{cases}$$

We consider two possibilities.

$\beta_4 \neq 0$. Then from $\beta_4\delta_4 = 0$ we get $\delta_4 = 0$, and so $\alpha_4 = \gamma_4 = \delta_4 = 0,\ \beta_4 = 1$, which implies $\mathbf{g_{b,c}} = \mathbf{g_{e,a}}$. On the other side, we have also $\alpha_3 = \beta_3 = \gamma_3 = 0,\ \delta_3 = 1$, implying $\mathbf{g_{b,a}} = \mathbf{g_{e,c}}$.

$\beta_4 = 0$. Then $\alpha_4 = \beta_4 = \gamma_4 = 0,\ \delta_4 = 1$, implying $g_{b,c} = g_{e,c}$, leading to the contradiction $b = e$.

Case $g_{c,(c,a)} = g_{(c,c),a}$. By Lemma 6.20 we have $g_{(c,c),a}(z) = g_{e,a}(z) = \epsilon_a(z) = \begin{pmatrix} e & a & b & c \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and we compute the distribution $g_{c,(c,a)}$.

$$g_{c,(c,a)}(z) = g_{c,e}(z)g_{c,a}(e) + g_{c,a}(z)g_{c,a}(a) + g_{c,b}(z)g_{c,a}(b) + g_{c,c}(z)g_{c,a}(c) =$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \alpha_5 + \begin{pmatrix} \alpha_5 \\ \beta_5 \\ \gamma_5 \\ \delta_5 \end{pmatrix} \beta_5 + \begin{pmatrix} \alpha_6 \\ \beta_6 \\ \gamma_6 \\ \delta_6 \end{pmatrix} \gamma_5 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \delta_5 = \begin{pmatrix} \alpha_5\beta_5 + \alpha_6\gamma_5 + \delta_5 \\ \beta_5\beta_5 + \beta_6\gamma_5 \\ \gamma_5\beta_5 + \gamma_6\gamma_5 \\ \delta_5\beta_5 + \delta_6\gamma_5 + \alpha_5 \end{pmatrix}.$$

Hence, we have the following system of equations

$$\left\{ \begin{array}{rcl} \alpha_5\beta_5 + \alpha_6\gamma_5 + \delta_5 &=& 0, \\ \beta_5\beta_5 + \beta_6\gamma_5 &=& 1, \\ \gamma_5\beta_5 + \gamma_6\gamma_5 &=& 0, \\ \delta_5\beta_5 + \delta_6\gamma_5 + \alpha_5 &=& 0, \end{array} \right. \quad \text{i.e.,} \quad \left\{ \begin{array}{rcl} \alpha_5 = \delta_5 &=& 0, \\ \alpha_5\beta_5 = \gamma_5\beta_5 = \delta_5\beta_5 &=& 0, \\ \alpha_6\gamma_5 = \gamma_6\gamma_5 = \delta_6\gamma_5 &=& 0, \\ \beta_5\beta_5 + \beta_6\gamma_5 &=& 1. \end{array} \right.$$

We consider two possibilities.

$\beta_5 = 0$. Then from $\beta_6\gamma_5 = 1$ we get $\beta_6 = 1,\ \gamma_5 = 1$, that implies $\alpha_6 = \gamma_6 = \delta_6 = 0, \beta_6 = 1$, and we infer that $\mathbf{g_{c,b}} = \mathbf{g_{e,a}}$. On other side, we also have $\alpha_5 = \beta_5 = \delta_5 = 0, \gamma_5 = 1$, implying $\mathbf{g_{c,a}} = \mathbf{g_{e,b}}$.

$\beta_5 \neq 0$. Then $\alpha_5 = \gamma_5 = \delta_5 = 0, \beta_5 = 1$, and this implies $g_{c,a} = g_{e,a}$, leading to the contradiction $c = e$.

Altogether, we get $\mathbf{g_{a,b}} = \mathbf{g_{e,c}}$, $\mathbf{g_{a,c}} = \mathbf{g_{e,b}}$, $\mathbf{g_{b,c}} = \mathbf{g_{e,a}}$, $\mathbf{g_{b,a}} = \mathbf{g_{e,c}}$, $\mathbf{g_{c,b}} = \mathbf{g_{e,a}}$, $\mathbf{g_{c,a}} = \mathbf{g_{e,b}}$. This means that the probability group is in fact the ordinary Klein group

$$\begin{array}{c|cccc} & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array}.$$

II. The other case is $c^{-1} = a,\ b^{-1} = b$ (or $c^{-1} = b, a^{-1} = a$, or $b^{-1} = a, c^{-1} = c$, these lead to isomorphic results). Then we have the following distributions,

	$g_{e,e}, g_{b,b}$ $g_{a,c}, g_{c,a}$	$g_{a,e}$ $g_{e,a}$	$g_{b,e}$ $g_{e,b}$	$g_{c,e}$ $g_{e,c}$	$g_{a,a}$	$g_{a,b}$	$g_{b,a}$	$g_{b,c}$	$g_{c,b}$	$g_{c,c}$
e	1	0	0	0	α_1	α_2	α_3	α_4	α_5	α_6
a	0	1	0	0	β_1	β_2	β_3	β_4	β_5	β_6
b	0	0	1	0	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
c	0	0	0	1	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6

where $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geqslant 0,\quad \alpha_i+\beta_i+\gamma_i+\delta_i=1$, for $i=1,2,\ldots,6$.

By associativity we have the following equalities.

Case $g_{(b,b),c} = g_{b,(b,c)}$. By Lemma 6.20 we have $g_{g_{(b,b),c}}(z) = g_{e,c}(z) = \epsilon_c(z) = \begin{pmatrix} e & a & b & c \\ 0 & 0 & 0 \end{pmatrix}$, and we compute the distribution $g_{b,(b,c)}$.

$$g_{b,(b,c)}(z) = g_{b,e}(z)g_{b,c}(e) + g_{b,a}(z)g_{b,c}(a) + g_{b,b}(z)g_{b,c}(b) + g_{b,c}(z)g_{b,c}(c) =$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\alpha_4 + \begin{pmatrix}\alpha_3\\\beta_3\\\gamma_3\\\delta_3\end{pmatrix}\beta_4 + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\gamma_4 + \begin{pmatrix}\alpha_4\\\beta_4\\\gamma_4\\\delta_4\end{pmatrix}\delta_4 = \begin{pmatrix}\alpha_3\beta_4+\gamma_4+\alpha_4\delta_4\\ \beta_3\beta_4+\beta_4\delta_4\\ \alpha_4+\gamma_3\beta_4+\gamma_4\delta_4\\ \gamma_3\beta_4+\delta_4\delta_4\end{pmatrix}.$$

Hence, we have the following system of equations

$$\begin{cases} \alpha_3\beta_4+\gamma_4+\alpha_4\delta_4 &= 0,\\ \beta_3\beta_4+\beta_4\delta_4 &= 0,\\ \alpha_4+\gamma_3\beta_4+\gamma_4\delta_4 &= 0,\\ \gamma_3\beta_4+\delta_4\delta_4 &= 1, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \alpha_4=\gamma_4 &= 0,\\ \alpha_3\beta_4=\beta_3\beta_4=\gamma_3\beta_4 &= 0,\\ \alpha_4\delta_4=\beta_4\delta_4=\gamma_4\delta_4 &= 0,\\ \gamma_3\beta_4+\delta_4\delta_4 &= 1. \end{cases}$$

We consider two possibilities.

$\beta_4 \neq 0$. Then we have $\alpha_3=\beta_3=\gamma_3=0, \delta_3=1$, implying $\mathbf{g_{b,a}} = \mathbf{g_{e,c}}$. It follows from $\beta_4\delta_4=0$ that $\delta_4=0$, i.e., we have $\alpha_4=\delta_4=\gamma_4=0, \beta_4=1$, and so we have $\mathbf{g_{b,c}} = \mathbf{g_{e,a}}$.

$\beta_4 = 0$. Then from $\delta_4\delta_4=1$ we have $\alpha_4=\beta_4=\gamma_4=0, \delta_4=1$, leading to the contradiction $g_{b,c}=g_{e,c}$.

Case $g_{(a,a),c} = g_{a,(a,c)}$. By Lemma 6.20 we have $g_{a,(a,c)}(z) = g_{a,e}(z) = \epsilon_a(z) = \begin{pmatrix} e & a & b & c \\ 0 & 1 & 0 & 0 \end{pmatrix}$, and we compute the distribution $g_{(a,a),c}$.

$$g_{(a,a),c}(z) = g_{a,a}(e)g_{e,c}(z) + g_{a,a}(a)g_{a,c}(z) + g_{a,a}(b)g_{b,c}(z) + g_{a,a}(c)g_{c,c}(z) =$$

$$= \alpha_1\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + \beta_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \gamma_1\begin{pmatrix}\alpha_4\\\beta_4\\\gamma_4\\\delta_4\end{pmatrix} + \delta_1\begin{pmatrix}\alpha_6\\\beta_6\\\gamma_6\\\delta_6\end{pmatrix} = \begin{pmatrix}\beta_1+\gamma_1\alpha_4+\delta_1\alpha_6\\ \gamma_1\beta_4+\delta_1\beta_6\\ \gamma_1\gamma_4+\delta_1\gamma_6\\ \alpha_1+\gamma_1\delta_4+\delta_1\delta_6\end{pmatrix}.$$

Hence, we have the following system of equations

$$\begin{cases} \beta_1+\gamma_1\alpha_4+\delta_1\alpha_6 &= 0,\\ \gamma_1\beta_4+\delta_1\beta_6 &= 1,\\ \gamma_1\gamma_4+\delta_1\gamma_6 &= 0,\\ \alpha_1+\gamma_1\delta_4+\delta_1\delta_6 &= 0, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \alpha_1=\beta_1 &= 0,\\ \gamma_1\alpha_4=\gamma_1\gamma_4=\gamma_1\delta_4 &= 0,\\ \delta_1\alpha_6=\delta_1\gamma_6=\delta_1\delta_6 &= 0,\\ \gamma_1\beta_4+\delta_1\beta_6 &= 1. \end{cases}$$

We consider two possibilities.

$\delta_1 \neq 0$. Then we have $\alpha_6=\gamma_6=\delta_6=0, \beta_6=1$, leading to a contradiction $g_{c,c}=g_{a,e}$, since we have shown in the previous case that $g_{b,c}=g_{a,e}$.

$\delta_1 = 0$. Then we have $\alpha_1=\beta_1=\delta_1=0, \gamma_1=1$, and this gives $\mathbf{g_{a,a}}=\mathbf{g_{b,e}}$.

Case $g_{(a,b),b}=g_{a,(b,b)}$. By Lemma 6.20 we have $g_{a,(b,b)}(z)=g_{a,e}(z)=\epsilon_a(z)=\begin{pmatrix} e & a & b & c\\ 0 & 1 & 0 & 0\end{pmatrix}$, and we compute the distribution $g_{(a,b),b}$.

$$g_{(a,b),b}(z)=g_{a,b}(e)g_{e,b}(z)+g_{a,b}(a)g_{a,b}(z)+g_{a,b}(b)g_{b,b}(z)+g_{a,b}(c)g_{c,b}(z)=$$

$$=\alpha_2\begin{pmatrix}0\\0\\1\\0\end{pmatrix}+\beta_2\begin{pmatrix}\alpha_2\\\beta_2\\\gamma_2\\\delta_2\end{pmatrix}+\gamma_2\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\delta_2\begin{pmatrix}\alpha_5\\\beta_5\\\gamma_5\\\delta_5\end{pmatrix}=\begin{pmatrix}\beta_2\alpha_2+\gamma_2+\delta_2\alpha_5\\ \beta_2\beta_2+\delta_2\beta_5\\ \alpha_2+\beta_2\gamma_2+\delta_2\gamma_5\\ \beta_2\delta_2+\delta_2\delta_5\end{pmatrix}.$$

Hence, we have the following system of equations

$$\begin{cases} \beta_2\alpha_2+\gamma_2+\delta_2\alpha_5 &= 0,\\ \beta_2\beta_2+\delta_2\beta_5 &= 1,\\ \alpha_2+\beta_2\gamma_2+\delta_2\gamma_5 &= 0,\\ \beta_2\delta_2+\delta_2\delta_5 &= 0, \end{cases} \quad \text{i.e.,} \quad \begin{cases} \alpha_2=\gamma_2 &= 0,\\ \beta_2\alpha_2=\beta_2\gamma_2=\beta_2\delta_2 &= 0,\\ \delta_2\alpha_5=\delta_2\gamma_5=\delta_2\delta_5 &= 0,\\ \beta_2\beta_2+\delta_2\beta_5 &= 1. \end{cases}$$

We consider two possibilities.

$\delta_2 \neq 0$. Then we have $\alpha_5=\gamma_5=\delta_5=0, \beta_5=1$, implying $\mathbf{g_{c,b}}=\mathbf{g_{e,a}}$. It follows from $\beta_2\delta_2=0$ that $\beta_2=0$, i.e., we have $\alpha_2=\beta_2=\gamma_2=0, \delta_2=1$, and so we have $\mathbf{g_{a,b}}=\mathbf{g_{e,c}}$.

$\delta_2=0$. Then from $\beta_2\beta_2=1$ we have $\alpha_2=\gamma_2=\delta_2=0, \beta_2=1,$, leading to the contradiction $g_{a,b}=g_{a,e}$.

Until now he have proved that $\mathbf{g_{c,b}}=\mathbf{g_{e,a}}$, $\mathbf{g_{a,b}}=\mathbf{g_{e,c}}$, $\mathbf{g_{b,a}}=\mathbf{g_{e,c}}$, $\mathbf{g_{b,c}}=\mathbf{g_{e,a}}$, $\mathbf{g_{a,a}}=\mathbf{g_{b,e}}$. We will show that the equality $\mathbf{g_{c,c}}=\mathbf{g_{e,b}}$ is also true. Namely, from $g_{c,b}=g_{a,e}$, we have $g_{c,(c,b)}=g_{c,(a,e)}=g_{(c,a),e}=g_{e,e}=\epsilon_e$, and hence $g_{(c,c),b}=g_{c,(c,b)}=\epsilon_e$. Now, $g_{((c,c),b),b}=g_{e,b}$, i.e., $g_{(c,c),(b,b)}=g_{(c,c),e}=g_{c,c}=g_{e,b}$. The obtained equalities show that this probabilistic group is in fact the cyclic group

$$\begin{array}{c|cccc} & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & b & c & e \\ b & b & c & e & a \\ c & c & e & a & b \end{array}.$$

The careful analyses of the Examples 7.2 and 7.3 can give us a hint for proving the following Hypothesis.

Hypothesis. *Each finite probabilistic group is a group.*

We are not going to give a complete proof here, mainly because of technical reasons. We only show how a proof can be inferred for finite groups.

Let (A, g) be a probabilistic group with unit e, where $A = \{e, a_1, a_2, a_n\}$. Let suppose that $a_2^{-1} = a_1$, i.e., $g_{a_1,a_2} = g_{a_2,a_1} = \epsilon_e$. Take an element a_k, $k > 2$, and consider the associativity $g_{(a_1,a_2),a_k} = g_{a_1,(a_2,a_k)}$. By Lemma 6.20 we have $g_{(a_1,a_2),a_k} = \epsilon_{a_k} = \begin{pmatrix} e & a_1 & a_2 & \ldots & a_k & \ldots & a_n \\ 0 & 0 & 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}$, and we compute the distribution $g_{a_1,(a_2,a_k)}(z) = \sum_{u \in A} g_{a_1,u}(z) g_{a_2,a_k}(u)$. (Note that $g_{a_2,a_k}(u) \in A$ and $g_{a_1,u}(z)$ are distributions.) The same way as in Examples 7.2 and 7.3 we will get a system of equations of type $\alpha = 0$, $\alpha\beta = 0$ for many unknowns α, β, γ, $\ldots$ and only one equation of type $\alpha\beta + \gamma\delta = 1$. From these equations one can infer equalities of type $g_{a_i,a_j} = g_{a_r,e}$. Note that, for the inverses a_1, a_2, we have $4(n-2)$ equalities of types $g_{(a_1,a_2),a_k} = g_{a_1,(a_2,a_k)}$, $g_{a_k,(a_1,a_2)} = g_{(a_k,a_1),a_2)}$, $g_{(a_2,a_1),a_k} = g_{a_2,(a_1,a_k)}$ and $g_{a_k,(a_2,a_1)} = g_{(a_k,a_2),a_1)}$ $(k = 3, 4, \ldots, n)$. Totally, since there are altogether $n-1$ pairs of inverses of types (a_i, a_j), (a_j, a_i) or (a_i, a_i), we can produce $4(n-1)(n-2)$ system of equations of previous type. Since the probabilistic group (A, g) have $(n-1)(n-2)$ distributions of type g_{a_i,a_j}, where $a_i \neq e$ or $a_j \neq e$ or a_i, a_j are not mutually inverse, it is reasonable to assume that for each i, j one can find an r such that $g_{a_i,a_j} = g_{a_r,e}$.

8. Conclusion

We have introduced the concept of probabilistic algebras, and our attention was given mainly to some types of probabilistic groupoids, and we had considered only the finite case.

The ideas of this paper are, in best of our knowledge, quite new. By retrieving the literature we could not find any notion or concept for probabilistic algebras.

The future work on probabilistic algebras can include (and are not restricted to) the following problems:

1. Define and investigate probabilistic groupoids on arbitrary universe (finite or infinite).
2. Define and investigate other types of probabilistic algebras (rings, lattices, modules, ...).
3. Prove the Hypothesis from Section 7 for finite groups.
4. Prove the Hypothesis from Section 7 for infinite groups (if it is true in the infinite cases).
5. Is it true that there are no finite p-groups when $0 < p < 1$? What about the infinite case?
6. Define probabilistic varieties of algebras.
7. Is it true that the distribution of g_T, when the length of the term T goes to infinity, is uniform? Can be characterized the class of probabilistic groupoids with this property?
8. How it can be defined quotient operations for probabilistic quasigroups? Can we apply them in cryptography and coding theory?

Remark for **References**: We could not find any reliable reference, except standard college algebra textbooks.

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